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CALCULUS OF FUNCTORS, OPERAD FORMALITY, AND EMBEDDING SPACES

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JOINT WITH:

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PASCAL LAMBRECHTS

M = smooth manifold

V = vector space

- Study rational homology of

$$\overline{\mathrm{Emb}}(M,V) := \mathrm{hofiber}\left(\mathrm{Emb}(M,V) \to \mathrm{Imm}(M,V)\right)$$

- Do this by studying

$$H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V)$$

- Use Goodwillie-Weiss <u>embedding</u> and <u>orthogonal</u> calculus of funtors.

<u>Main ingredient:</u>

Formality of the little balls operad.

Embedding calculus (Goodwillie-Weiss)

$\mathcal{O}(M)$ = cat. of open subsets of M

$\mathcal{O}_k(M)$ = cat. of open subsets of M diffeo to at most k disjoint balls

$$F : \mathcal{O}(M)^{op} \longrightarrow \begin{matrix} Top_* \\ Spectra \end{matrix}$$

Get tower of fibrations

$$F(-) \to \big(T_\infty F(-) \to \cdots \to T_k F(-) \to \cdots \to T_1 F(-)\big)$$

where

$$T_k F(W) = \operatorname*{holim}_{U \in \mathcal{O}_k(W)} F(U)$$

let

$$F(W) = H\mathbb{Q} \wedge \overline{Emb}(W, V)$$

$$\big(\text{or } F(W) = \overline{Emb}(W, V)\big)$$

and evaluate at $W = M$. Get

<u>Thm (Goodwillie-Weiss)</u>: The tower converges for $H\mathbb{Q} \wedge \overline{Emb}(M,V)$ if $2 \dim M + 1 < V$ (and for $\overline{Emb}(M,V)$ if $\dim M + 2 < V$).

$$T_k H\mathbb{Q} \wedge \overline{Emb}(M,V)$$

$$:= \operatorname*{holim}_{U \in \mathcal{O}_k(M)} H\mathbb{Q} \wedge \overline{Emb}(U,V)$$

$$\simeq \operatorname*{holim}_{U \in \mathcal{O}_k(M)} H\mathbb{Q} \wedge \underbrace{B(k_U, V)}_{k_U \text{ balls in } V}$$

Maps in diagrams defining the stages $T_k H\mathbb{Q} \wedge \overline{Emb}(M,V)$ (or $T_k \overline{Emb}(M,V)$) are closely related to structure maps in the <u>little balls operad</u>

$$\underline{\{B(n,V)\}}_{n \geq 0}^{\infty}.$$

<u>Thm (Kontsevich)</u>: little balls operad is <u>formal</u>, i.e. there are quasi-isomorphisms of CDGA's connecting $(\Omega^* B(n,V), d)$ and $(H^* B(n,V), 0)$ which commute with operad structure maps.

<u>Thm (Arone-Lambrechts-V.)</u>:

The diagram defining $T_k H\mathbb{Q} \wedge \overline{Emb}(M,V)$ (or $T_k \overline{Emb}(M,V)$) is formal.

- Formality often good for showing collapse of a spectral sequence
- In this case, better to apply orthogonal calculus to $H\mathbb{Q} \wedge \overline{Emb}(M,V)$ first.

Orthogonal calculus (Weiss)

$\mathcal{V}$ = cat. of vector spaces with linear isometric inclusions

$$F : \mathcal{V} \longrightarrow \begin{matrix} \text{Top}_* \\ \text{Spectra} \end{matrix}$$

ex: $V \longmapsto \Omega^V S^V,\ BO(V),\ \overline{\text{Emb}}(M,V)$

Get another Taylor tower

$$F(-) \to \left(P_\infty F(-) \to \cdots \to P_n F(-) \to \cdots \to P_1 F(-)\right)$$

Idea: Understand F for high-dimensional V and extrapolate to lower dimensions.

Let $\quad D_n F = \text{hofiber}(P_n F \longrightarrow P_{n-1} F)$

Thm (Arone-Lambrechts-V.):

The orthogonal tower for $H\mathbb{Q} \wedge \overline{Emb}(M,V)$ splits, i.e.

$$P_n H\mathbb{Q} \wedge \overline{Emb}(M,V) \simeq \prod_{i=1}^{n} D_i\, H\mathbb{Q} \wedge \overline{Emb}(M,V)$$

pf outline:

$$P_n H\mathbb{Q} \wedge \overline{Emb}(M,V)$$

$$(\text{large } k) \simeq \operatorname*{holim}_{U \in O_k(M)} P_n H\mathbb{Q} \wedge \overline{Emb}(U,V)$$

Ortogonal tower for configuration spaces splits stably rationally, so

$$\simeq \operatorname*{holim}_{U \in O_k(M)} \prod_{i=1}^{n} D_i\, H\mathbb{Q} \wedge \overline{Emb}(U,V)$$

$$(\text{formality}) \simeq \prod_{i=1}^{n} \operatorname*{holim}_{U \in O_k(M)} D_i\, H\mathbb{Q} \wedge \overline{Emb}(U,V)$$

$$(\text{large } k) \simeq \prod_{i=1}^{n} D_i\, H\mathbb{Q} \wedge \overline{Emb}(M,V).$$

Corollaries:

① Taylor tower gives rise to a spectral sequence which converges to

$$\pi_*(H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V)) = H_*(\overline{\mathrm{Emb}}(M,V))$$

for $2\dim M + 1 < \dim V$. This spectral sequence collapses at E^1.

② It can be seen that the D_i's are stable homotopy functors of M. It follows that the rational homology of $\overline{\mathrm{Emb}}(M,V)$ depends only on rational homology of M.

Special case: $\overline{\mathrm{Emb}}(S^1, V)$ (knots)

Stages of the embedding tower simplify to homotopy limits of finite diagrams. In fact, one has a cosimplicial model for the tower.

$$X^\bullet = \left\{ F\langle 0, V\rangle \rightrightarrows F\langle 1, V\rangle \underset{s^i}{\overset{d^i}{\rightrightarrows}} \cdots \rightrightarrows F\langle k, V\rangle \rightrightarrows \cdots \right\}$$

where

$F\langle k, V\rangle$ = Fulton-MacPherson-like compactification of the conf. space of k pts. in V.

d^i = doubling

s^i = forgetting

Thm (Sinha):

$$\mathrm{Tot}^k X^\bullet \simeq T_k \overline{\mathrm{Emb}}(S^1, V)$$

Thm (Lambrechts - V.):

$X^\bullet$ is formal.

$\exists$ Bousfield - Kan H^* spectral sequence converging to $H^* \operatorname{Tot} X^\bullet \simeq H^*(\overline{\operatorname{Emb}}(S^1, V))$ with

for $\dim V > 3$.

$$E_1^{-p,q} = H^q(F\langle p, V\rangle)$$

$$d_1 = \Sigma(-1)^i (d^i)^* : E_1^{-p,q} \longrightarrow E_1^{-p+1,q}$$

Thm (Lambrechts - V.):

This s.s. collapses at E_2.

Cor:

① Vassiliev's SS for $H^*(\overline{\operatorname{Emb}}(S^1, V))$ collapses at E_1.

② The CDGA

$$\left(\bigoplus_{p=0}^{\infty} s^{-p} H^*(F\langle p, V\rangle) \, , \, d_1 \right)$$

is a rational model for $\overline{\operatorname{Emb}}(S^1, V)$.