CALCULUS OF FUNCTORS,
OPERAD FORMALITY, AND
EMBEDDING SPACES

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JOINT WITH:

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$M =$ smooth manifold
$V =$ vector space

- Study rational homology of

\[ \text{Emb} (M, V) = \text{hofiber} (\text{Emb} (M, V) \rightarrow \text{Imm} (M, V)) \]

- Do this by studying

\[ H^Q \wedge \text{Emb} (M, V) \]

- Use Goodwillie-Weiss embedding and orthogonal calculus of functors.

Main ingredient:

Formality of the little balls operad.
Embedding calculus (Goodwillie-Weiss)

\[ O(M) = \text{cat. of open subsets of } M \]

\[ O_k(M) = \text{cat. of open subsets of } M \text{ differing to at most } k \text{ disjoint balls} \]

\[ F : O(M)^o \to \text{Top}^* \to \text{Spectra} \]

Get tower of fibrations

\[ F(-) \to (T_0 F(-)) \to \cdots \to T_k F(-) \to \cdots \to T_n F(-) \]

where

\[ T_k F(W) = \holim_{U \in O_k(W)} F(U) \]

let

\[ F(W) = H\Omega \wedge \overline{\text{Emb}}(W, V) \]

(or \( F(W) = \overline{\text{Emb}}(W, V) \))

and evaluate at \( W = M \). Get
Thm (Goodwillie–Weiss): The tower converges for $HQ \wedge \overline{\text{Emb}}(M, V)$ if $2 \dim M + 1 < V$ (and for $\overline{\text{Emb}}(M, V)$ if $\dim M + 2 < V$).

\[ T_k HQ \wedge \overline{\text{Emb}}(M, V) = \text{holim}_{u \in O_k(M)} HQ \wedge \overline{\text{Emb}}(u_\ast V) \]

Maps in diagrams defining the stages $T_k HQ \wedge \overline{\text{Emb}}(M, V)$ (or $T_k \overline{\text{Emb}}(M, V)$) are closely related to structure maps in the little balls operad

\[ \{ \mathbf{E}(n, V) \} \quad \text{for} \quad n \geq 0. \]
Thm (Kontsevich): little balls operad is formal, i.e. there are quasi-isomorphisms of CDGA's connecting $(\Omega^* B(n,V), d)$ and $(H^* B(n,V), 0)$ which commute with operad structure maps.

Thm (Arone-Lambrechts-V.):
The diagram defining $T_k HQ \land \overline{Emb}(n,V)$ (or $T_k \overline{Emb}(n,V)$) is formal.

- Formality often good for showing collapse of a spectral sequence
- In this case, better to apply orthogonal calculus to $HQ \land \overline{Emb}(n,V)$ first.
Orthogonal calculus (Weiss)

$V$ = cat. of vector spaces with linear isometric inclusions

$F : V \rightarrow \text{Top}^*$

Spectra

$\text{Ex: } V \rightarrow \Omega^n V, BO(V), \text{Emb}(M, V)$

Get another Taylor tower

$F(-) \rightarrow (p_\alpha F(-) \rightarrow \cdots \rightarrow p_n F(-) \rightarrow \cdots \rightarrow p_1 F(-))$

Idea: Understand $F$ for high-dimensional $V$ and extrapolate to lower dimensions.

Let $D_n F =hofiber(p_n F \rightarrow p_{n-1} F)$
Thm (Arone--Lambrechts--V.):

The orthogonal tower for \( HQ \wedge \overline{Emb}(\mathbb{R}^n, V) \) splits, i.e.

\[
P_n HQ \wedge \overline{Emb}(\mathbb{R}^n, V) = \prod_{i=1}^n D_i HQ \wedge \overline{Emb}(\mathbb{R}^n, V)
\]

Proof outline:

\( P_n HQ \wedge \overline{Emb}(\mathbb{R}^n, V) \) (large \( \mathbb{R} \)) \( \cong \) \( \operatorname{holim}_{u \in O_k(n)} P_n HQ \wedge \overline{Emb}(u, V) \)

Orthogonal tower for configuration spaces splits stably rationally, so

\[
\cong \operatorname{holim}_{u \in O_k(n)} \prod_{i=1}^n D_i HQ \wedge \overline{Emb}(u, V)
\]

(formality) \( \cong \prod_{i=1}^n \operatorname{holim}_{u \in O_k(n)} D_i HQ \wedge \overline{Emb}(u, V) \)

(large \( \mathbb{R} \)) \( \cong \prod_{i=1}^n D_i HQ \wedge \overline{Emb}(\mathbb{R}^n, V) \).
Corollaries:

1. Taylor tower gives rise to a spectral sequence which converges to

$$\pi_*(\text{H}_p \cap \overline{\text{Emb}}(M,V)) = H_*(\overline{\text{Emb}}(M,V))$$

for $2 \dim M + 1 < \dim V$. This spectral sequence collapses at $E^1$.

2. It can be seen that the $D_i$'s are stable homotopy functors of $M$. It follows that the rational homology of $\overline{\text{Emb}}(M,V)$ depends only on rational homology of $M$. 

Special case: $\overline{\text{Emb}}(S', V)$ (knots)

Stages of the embedding tower simplify to homotopy limits of finite diagrams. In fact, one has a cosimplicial model for the tower.

$$X^* = \left\{ F(0, V) \equiv F(1, V) \equiv \cdots \right\}$$

where

$F(k, V) = \text{Fulton-MacPherson-like compactification of the conf. space of } k \text{ pts. in } V.$

$d^i = \text{doubling} \quad s^i = \text{forgetting}$

Thus (Sinha):

$$\text{Tot}^k X^* = T^k \overline{\text{Emb}}(S', V)$$
Thm (Lambrechts - V.):

\( x^* \) is formal.

\[ E_1^{p,1} = H^q(F < p, V>) \]

\[ d_1 = E^{-1,0}: E_1^{p,1} \rightarrow E_1^{p+1,1} \]

Thm (Lambrechts - V.):

This S.S. collapses at \( E_2 \).

Cor:

1. Vassiliev's S.S for \( H^*(\text{Emb}(S^1, V)) \)
   collapses at \( E_1 \).
2. The CDA
   \[ (\oplus H^p(F < p, V>), d_1) \]
   is a rational model for \( \text{Emb}(S^1, V) \).