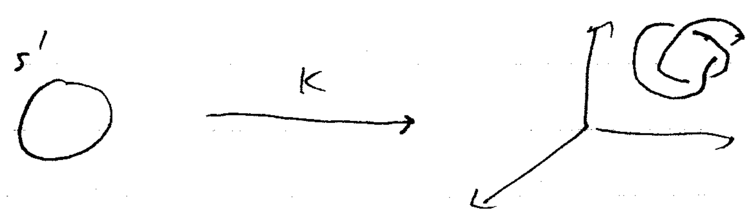
11/15/05 ①

VIGRE Seminar, University of Georgia

Intro to Vassiliev knot invariants

(I'll just talk for a while about some generalities in knot theory - what are knots and knot invariants - and then say something about a special class of invariants.)

A knot K is an embedding (smooth, 1-1 map) of S^1 in $\mathbb{R}^3$.



Two knots are considered to be the same if one can be deformed into another through embeddings, i.e. isotoped.

Each isotopy class is a knot type.

Open question: what are all the knot types?

(we know ∃ ∞ many, but we don't have a nice way to enumerate them)

Related question: How can we tell two knots apart (up to isotopy)? (I.e. tell knot types)

A knot invariant I is a function

$$I : \mathcal{K} \longrightarrow \text{abelian grp.}$$

$\mathcal{K}$ = space of knots (set)

which takes the same value on isotopic knots.

(Another way to say this:

knot types $= H_0(\mathcal{K})$ = connected components of $\mathcal{K}$

invariants $= H^0(\mathcal{K})$ = fctns which are constant on conn. components of $\mathcal{K}$)

(But here's the problem: This def. does not say that I should take on different values for different knots!

ex: let $I(K) = \text{const.} \quad \forall K$

(This is a legitimate invariant.)

(This is clearly not very useful, so it would be nice to have invariants that take different values for different knots. But we don't know an invariant that will *always* do this, for *all* different knots!)

Open question: what invariant, or a set of invariants, can tell all knots apart?

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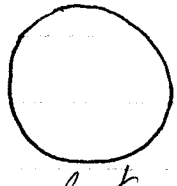
ex: *Tricolorability*: Try to color a knot in 3 colors s.t. 3 or 1 color meet at a crossing. If this can be done, knot is *tricolorable*. All tricolorable knots are *not* isotopic to nontricolorable knots.



unknot
tricolorable

trefoil
tricolorable

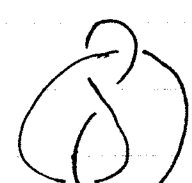


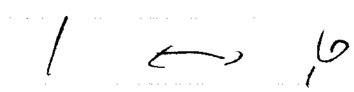
figure-8
not tricolorable

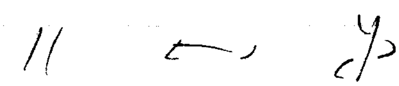
$\Rightarrow$

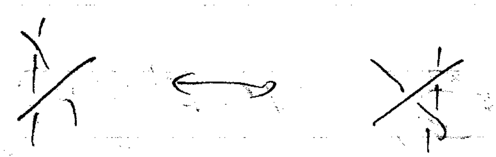
trefoil
$?\,\|$ $\quad\not\simeq$ figure-8
unknot $\quad\not\simeq$

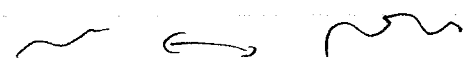
<u>Aside</u>: There's an easy way to tell if you have an invariant.

<u>Reidemeister</u>: Two knots are isotopic if they differ by a sequence comprised of moves:

1. 

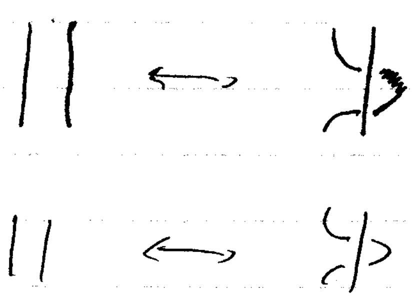
2. 

3. 

(+ "planar" isotopies, )

So <u>I invariant</u> $\Longleftrightarrow$ gives <u>same value before and after a Reidemeister move</u>.

<u>ex</u>:



(Tricolorability is not a very exciting invariant either. Better invariants which people have studied and used extensively are <u>polynomial invariants</u> — I just want to mention this since you might hear it in various contexts. These are very popular — Alexander, HOMFLY, Jones, Conway, etc.)

Ex: <u>Conway polynomial</u> is defined by

- $C(O) = 1$

- $C(\nearrow\!\!\!\!\nwarrow) = (t^{1/2} - t^{-1/2})\, C(\uparrow\uparrow) + C(\nwarrow\!\!\!\!\nearrow)$

local picture of an <u>oriented</u> knot

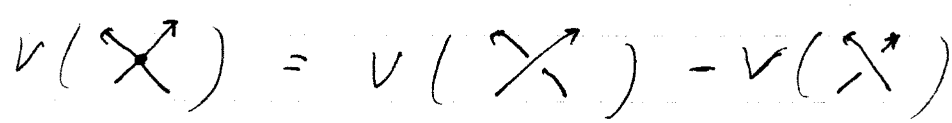
(The formula lets you change a crossing and pay some price. But by changing crossings can always get down to unknot.)

(also need $\bullet\ C(L \cup O) = C(L)$)

Finite type (Vassiliev) invariants

(I'm not going to define any new invariants here; I'll just be interested in some existing ones which satisfy some property)

Let V be an invariant. Extend it to <u>singular knots</u> (finite # of self-intersections) via <u>Vassiliev skein relation</u>



n-singular knot $\rightsquigarrow$ 2^n ~~singularities~~ resolutions

<u>Def</u>: V is finite (Vassiliev) type n if it vanishes on $(n+1)$-singular knots.

Let $V_n = \{$ type n invariants $\}$

<u>Note</u>: $V_{n-1} \subseteq V_n$

$$\Big(V_{n-1}(\underbrace{\times - - \times}_{n+1}) = V(\underbrace{\times \text{---} \times}_{n} \times) - V(\underbrace{\times \text{---} \times}_{n} \times) = 0 - 0 = 0 \Big)$$

ex: $v \in V_0$

vassiliev skein rel'n

$$0 = v(\times) = v(\nearrow\!\!\!\nwarrow) - v(\nwarrow\!\!\!\nearrow)$$

$$\Rightarrow \quad v(\nearrow\!\!\!\nwarrow) = v(\nwarrow\!\!\!\nearrow)$$

$\Rightarrow$ v constant on $\mathcal{K}$

(Turns out nothing new happens for V_1. V_2 is the first place where something nontrivial happens.)

(Most invariants we care about are finite type, in particular the polynomial ones are — <u>the coefficients of poly invariants</u> are finite type.)

ex: <u>Conway</u>. Let $t^{1/2} - t^{-1/2} = z$.

$$c(\times) = c(\nearrow\!\!\!\nwarrow) - c(\nwarrow\!\!\!\nearrow) = z\, c(\uparrow\uparrow)$$

skein rel'n. def. of Conway

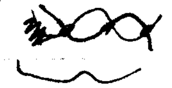
So

$$c(\underbrace{\times \text{---} \times}_{n}) = z\, c(\times \text{---} \times \, ||)$$

$$= z^n c(|| \text{---} ||)$$

$\Rightarrow$ coef. of z^{n-1} is 0, so this coef is type $n-1$ invariant)

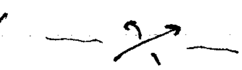
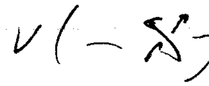
Value of $v \in V_n$ on an n-singular knot only depends on the placement of singularities and not on the embedding.

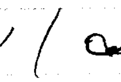
ex: $v\Big($  $\Big) - v\Big($  $\Big) \underset{\text{def}}{=} v\Big($  $\Big) = 0$

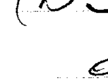
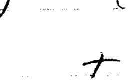
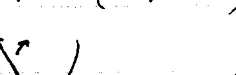
n-sing   n-sing   (n+1)-sing

(stupid, cause you can just rotate one into the other, but you know what I mean...)

(With this observation can see that $\exists$ no interesting type 1 invariants:

$v\Big($  $\Big) = v\Big($  $\Big) + v\Big($  $\Big)$

$= v\Big($  $\Big) + v\Big($  $\Big)$

free to choose knot $= v\Big($  $\Big) - v\Big($  $\Big) + v\Big($  $\Big)$

since isotopic $= 0 + v\Big($  $\Big)$.)

But placement of singularities can be encoded by a <u>chord diagram</u>.

Let $CD_n = \{$ chord diagrams w/ n chords $\}$

$CD_2 = \{$  ,  $\}$

(I.e. have oriented circle w/ 2n distinct, paired-off points, regarded up to diffeo of the circle. Think of this as the prescription for where you put the singularities — Embed the circle any way you want, as long as you identify the points at the ends of each of the chords.)

(So what I've just told you is really that there is a map:)

Given $D \in CD_n$, let K_D be any n-singular knot corresponding to D. Then there is a map

$$V_n / V_{n-1} \hookrightarrow \{ f : \mathbb{Z}[CD_n] \longrightarrow \mathbb{R} \}$$

usually, things are produced in this story

$$f(D) = V_n(K_D) \quad \text{(well-def.)}$$

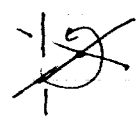
By def, kernel is V_{n-1}, so really

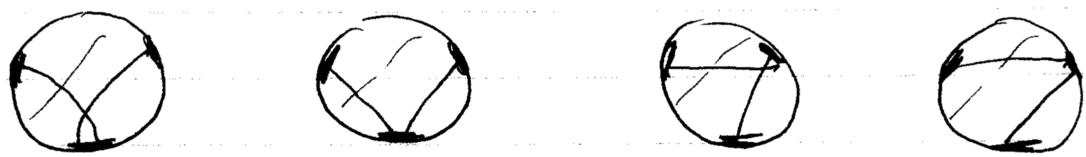
Extend linearly.

(After you stare at this for a while, you see that:)

Image is at most those f which satisfy the <u>four-term (4T) relation</u>:

(Imagine evaluating a type n invariant on an $(n-1)$-singular knot and passing a strand around one of the singularities. Get 4 n-singular knots, each of which corresponds to some chord diagrams, which are





(diagrams differ only in indicated segments.)

(Now, since you want to get to where you started, get)

$$f(\bigcirc) + f(\bigcirc) = f(\bigcirc) + f(\bigcirc)$$

(get signs according to ~~[illegible]~~ Vassiliev relation)

Let $W_n = \{ f : \mathbb{Z}[CD_n]/4T \longrightarrow \mathbb{Q} \}$

Thm (Kontsevich):

$$\frac{V_n}{V_{n-1}} \underset{\longleftarrow}{\overset{\cong}{\longrightarrow}} W_n$$

map I described

Kontsevich Integral (one of reasons he got Fields medal)

(So this puts finite type invariants in a combinatorial setting, which is nice. However, the fact that there is some strange integration here is a little mysterious.)

(Alternative pf. by Bott-Taubes Integrals, which I studied.)

(Finite type invariants very important;
- separation of knots
- 3-mflds
- physics connections)