Lengths of developments in $K((G))$

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Abstract

In [9], Mourgues and Ressayre showed that every real closed field F has an integer part, where this is an ordered subring with the properties appropriate for the range of a floor function. The Mourgues and Ressayre construction is canonical once we fix a residue field section K and a well ordering $\prec$ of F. The construction produces a value group section G, and a *development function* d mapping F isomorphically onto a truncation closed subfield R of the Hahn field $K((G))$. In [6], the authors conjectured that if $\prec$ has order type ω, then all elements of R have length less than ω^{ω^ω}, and they gave examples showing that the conjectured bound would be sharp. The current paper has two theorems bounding the lengths of elements of a truncation closed subfield R of a Hahn field $K((G))$ in terms of the length of a "*tc*-basis". Here K is a field that is either real closed or algebraically closed of characteristic 0, and G is a divisible ordered Abelian group. One theorem says that if R has a *tc*-basis of length at most ω, then the elements have length less than ω^{ω^ω}. This theorem yields the conjecture from [6]. The other theorem says that if the group G is Archimedean, and R has a *tc*-basis of length γ, where $\omega \leq \gamma < \omega_1$, then the elements of R have length at most ω^{ω^γ}.

1 Introduction

The goal of this paper is to give bounds on lengths of elements in certain subfields of a field of generalized power series. For a field K, the *Puiseux series over K*

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are the formal sums $\sum_{k\geq z} a_k t^{\frac{1}{n}}$, for k an integer and n a positive integer, where $a_k \in K$. We add and multiply Puiseux series as we do ordinary power series.

Theorem 1.1 (Newton-Puiseux Theorem)**.** *If K is a field that is real closed, or algebraically closed of characteristic* 0*, then the Puiseux series over K form a field that is also real closed, or algebraically closed.*

Newton [12] discovered this result in 1676, and Puiseux [14], [15] rediscovered it in 1850. For an ordered Abelian group G and a field K, the elements of the Hahn field $K((G))$ are formal sums $\sum_{g\in S} a_g t^g$, where S is a well ordered subset of G, and $a_g \in K$. Maclane [7] extended the result of Newton and Puiseux to Hahn fields, showing that if G is a divisible ordered Abelian group, and K is real closed, or algebraically closed of characteristic 0, then $K((G))$ is real closed, or algebraically closed.

For an element $s = \sum_{g\in S} a_g t^g$ of $K((G))$, a *truncation* of s has the form $s' = \sum_{g\in S'} a_g t^g$, where S' is an initial segment of S, i.e., $S' = \{h \in S \mid h < g'\}$ for some $g' \in G$. A subfield R of $K((G))$ is *truncation closed* if for all $s \in R$, all truncations of s are in R. For a truncation closed subfield R of $K((G))$, we consider a transcendence basis for R over K with special properties.

Definition 1.2 (tc-basis)**.**

1. *We call a sequence $(r_\alpha)_{\alpha<\gamma}$ in $K((G))$* tc-independent *if, for each $\alpha < \gamma$,*
   - *(a) r_α is not algebraic over $K \cup \{r_\beta : \beta < \alpha\}$, and*
   - *(b) either $r_\alpha = t^g$ for some $g \in G$, or else r_α has limit length, and all proper truncations of r_α are algebraic over $K \cup \{r_\beta : \beta < \alpha\}$.*
2. *Given a tc-independent sequence $(r_\alpha)_{\alpha<\gamma}$, we define a corresponding* canonical sequence $(R_\alpha)_{\alpha\leq\gamma}$ *of subfields of $K((G))$ by taking R_α to consist of the elements of $K((G))$ that are algebraic over $K \cup \{r_\beta : \beta < \alpha\}$.*
3. *We call $(r_\alpha)_{\alpha<\gamma}$ a* tc-basis for R *if it is a tc-independent sequence and R is the last term of the canonical sequence $(R_\alpha)_{\alpha\leq\gamma}$.*

For example, consider the subfield R of $K((\mathbb{Q}))$ with tc-basis consisting just of t. This R is the set of Puiseux series over K.

Mourgues and Ressayre [9] proved that every real closed field has an integer part. To do this, they produced an isomorphism d, the *development function*, from the given real closed field F onto a truncation closed subfield R of a certain Hahn field. The Mourgues and Ressayre construction is canonical once we specify a well ordering of F and a *residue field section* k. The construction yields a value group section G, and the development function d maps F into the Hahn field $k((G))$. While Mourgues and Ressayre [9] did not use this terminology, their construction specifies a tc-basis for R. Moreover, they proved that if $(r_\alpha)_{\alpha<\gamma}$ is a tc-independent sequence in $K((G))$, then the corresponding canonical sequence $(R_\alpha)_{\alpha \leq \gamma}$ consists of truncation closed subfields of $K((G))$.

If the universe of the field F is a subset of ω, then F inherits a well ordering of type ω from the standard ordering on ω. In this case, the procedure of Mourgues and Ressayre yields a tc-basis for R of length at most ω. In [6], the authors began studying the complexity of the Mourgues and Ressayre construction. They showed that F has a residue field section that is Π^0_2 relative to F. To bound the complexity of the development function d, they needed bounds on the lengths of the elements of R. The result below, the main result of the current paper, was essentially stated as a conjecture in [6] (see Conjecture 1.16 in §1.2 below).

Theorem 1.3. *Let $(r_n)_{n<\omega}$ be a tc-independent sequence in $K((G))$. Then for the corresponding canonical sequence $(R_n)_{n\leq\omega}$,*

1. *the elements of R_n have length at most $\omega^{\omega^{(n-1)}}$, and*
2. *the elements of R_ω all have length less than ω^{ω^ω}.*

In [6], it is shown that these bounds are sharp. To prove Theorem 1.3, we prove a second result, putting special assumptions on the group G, but allowing tc-bases of length greater than ω. The group G is *Archimedean* if for any positive elements g, h, there are natural numbers m, n such that $mg > h$ and $nh > g$. Note that a divisible ordered Abelian group G is Archimedean iff it is isomorphic to a subgroup of $(\mathbb{R}, +, <)$.

Theorem 1.4. *Let K be a real closed or algebraically closed field of characteristic zero, and let G be a divisible ordered Abelian group that is Archimedean. Let $(r_\alpha)_{\alpha<\gamma}$ be a tc-independent sequence in $K((G))$, where $\omega \leq \gamma < \omega_1$, and let $(R_\alpha)_{\alpha\leq\gamma}$ be the associated canonical sequence. Then for $\omega \leq \alpha \leq \gamma$, the elements of R_α have length at most ω^{ω^α}.*

Remark 1.5. *Every well ordered subset of $\mathbb{R}$, under the usual reverse ordering, has countable order type. If G is Archimedean, then G can be embedded in $(\mathbb{R}, +, >)$. The increasing sequences in such a group are all countable, so all elements of $K((G))$ have countable length. Thus, in Theorem 1.4, it is natural to consider only tc-independent sequences of countable length.*

In [6], there are examples showing that the bounds given in Theorem 1.3 are sharp. In Theorem 5.1, we give examples showing that the bounds in Theorem 1.4 are also sharp. Theorem 1.4 follows from a technical result bounding the lengths of roots of polynomials over $K((G))$ in terms of the lengths of the coefficients. This result, given as Theorem 3.2, involves looking closely at the process of finding roots of polynomials in a Hahn field.

The remainder of §1 contains further background material, first on Hahn fields and valuations, and then on the construction of Mourgues and Ressayre. In

§2, we give some background on ordinals and well ordered subsets of of an ordered group. In §3, we reduce the proof of Theorem 1.4 and Theorem 1.3 to Theorem 3.2. In §4, we prove Theorem 3.2. In §5, we give new examples showing that the bounds in Theorem 1.4 are sharp.

1.1 Hahn fields

Here we precisely describe the Hahn field $K((G))$, defining the operations, the ordering (in case K is ordered), and the natural valuation. Let G be an ordered Abelian group, and let K be a field. The elements of $K((G))$ are the formal sums $s = \sum_{g \in S} a_g t^g$ in an indeterminate t, where $S \subset G$ is well ordered and $a_g \in K^{\neq 0}$. We write 0 for the empty sum. For an element $s = \sum_{g \in S} a_g t^g$, the *support*, denoted by $Supp(s)$, is S. The *length* of s is the order type of S under the ordering of G.

1.1.1 Operations on $K((G))$

The operations on $K((G))$ are defined as follows.

Definition 1.6. *Let $s, s' \in K((G))$.*

1. *For the sum $s + s'$, the coefficient of t^g is the sum of the coefficients of t^g in s and s'. Then $Supp(s + s')$ is the set of elements $g \in Supp(s) \cup Supp(s')$ such that the coefficient in the sum is nonzero.*

2. *For the product $s \cdot s'$, the coefficient of t^g is the sum of the products $a \cdot b$ such that $a, b \in K$ and for some $h, h' \in G$,*

   - *a is the coefficient of t^h in s,*
   - *b is the coefficient of $t^{h'}$ in s',*
   - *and $h + h' = g$.*

   (Note that for $g, g' \in G$, $t^g t^{g'} = t^{g+g'}$.)

The fact that $Supp(s)$ and $Supp(s')$ are well ordered lets us see that the product is well-defined, and that the sum and product both have well ordered support (see Lemma 2.2).

1.1.2 Ordering and valuation on $K((G))$

There is a natural valuation on $K((G))$. We first recall the definition of a valuation. For an ordered Abelian group G with operation $+$, we add ∞ at the end, and we let $a + \infty = \infty + a = \infty$ for all $a \in G$. For a field F, a function $w : F \to G \cup \{\infty\}$ is a *valuation* on F if, for all $r, r' \in F$,

1. $w(r \cdot r') = w(r) + w(r')$ and

2. $w(r + r') \geq \min\{w(r), w(r')\}$.

We call the group G associated with the valuation w the *value group of* F. For $r \in F$ such that $r \neq 0$, we say that r is *infinitesimal* if $w(r) > 0$ and r is *infinite* if $w(r) < 0$.

Definition 1.7. *The* natural valuation *on the Hahn field* $K((G))$ *is the function* $w : K((G)) \to G \cup \{\infty\}$ *such that* $w(s)$ *is the least element of* $Supp(s)$ *if* $s \neq 0$, *or* ∞ *if* $s = 0$.

It is useful to think of the indeterminate t as infinitesimal. Then

t^g is infinite	if $g < 0$;
$t^g = 1$	if $g = 0$;
t^g is infinitesimal	if $g > 0$.

If K is an ordered field, then $K((G))$ is also ordered.

Definition 1.8. *Suppose that* K *is an ordered field. An element* s *of* $K((G))$ *is* positive *if the leading term is positive; that is, for the least* $g \in Supp(s)$, *the coefficient of* t^g *is positive.*

1.1.3 Archimedean equivalence

The natural valuation defined on $K((G))$ in §1.1.2 is based on Archimedean equivalence, and such a valuation can be defined for any ordered field.

Definition 1.9 (Archimedean equivalence)**.** *Let* F *be an ordered field. For* $x, y \in F^{>0}$, *we say* x *and* y *are* Archidemean equivalent, *denoted* $x \sim y$, *if there exist natural numbers* m, n *such that* $mx > y$ *and* $ny > x$.

Definition 1.10 (Natural value group and natural valuation)**.** *Let* F *be an ordered field.*

- *The* natural value group G *is the collection of* $\sim$*-classes, under the operation inherited from* $(F^{>0}, \cdot)$. *The ordering on* G *is the reverse of the ordering inherited from* F.
- *The* natural valuation $w : F \to G \cup \{\infty\}$ *is the function that takes each* $r \in F^{\neq 0}$ *to the* $\sim$*-class of* $|r|$. *We let* $w(0) = \infty$.

We emphasize that it is traditional to write $+$ for the value group operation, instead of $\cdot$, and to write 0 for the value group identity, instead of 1, because value groups are Abelian. For a real closed field F, the fact that the elements of $F^{>0}$ have n^{th} roots means that the natural value group G is divisible.

Recall that an ordered field F is *Archimedean* if there is just one $\sim$-class, which means that its natural value group is $\{0\}$. Of course, the notion of Archimedean equivalence makes sense in any ordered group G. For $a, b \in G^{>0}$, we write $a \approx b$

if there exist m, n such that $ma > b$ and $nb > a$. Thus, G is *Archimedean* iff there is just one $\approx$-class. If G is the natural value group for an ordered field F, then G is Archimedean just in case for any infinitesimal elements $x, y \in F^{>0}$, there exist natural numbers m, n such that $x^m < y$ and $y^n < x$.

1.1.4 Residue fields and sections

Given an arbitrary valuation (not necessarily a natural one), we define its associated residue field. Let F be a field equipped with a valuation w.

Definition 1.11 (Residue field)**.**

- *We let F_{fin} denote the subring of F consisting of the elements r such that $w(r) \geq 0$. We let μ_F denote the unique maximal ideal consisting of the elements r such that $w(r) > 0$.*
- *The* residue field *of F is the quotient of F_{fin} by μ_F.*

If F is a real closed field, the residue field is an Archimedean real closed field, representing the reals present in F.

We will use *sections* of the value group and residue field of F, copies of these structures that reside in F.

Definition 1.12 (Sections)**.**

1. *A* value group section *of F is a subgroup of $(F^{>0}, \cdot)$ that is isomorphic to the value group under the valuation map w.*
2. *A* residue field section *is a subfield of F_{fin} that is isomorphic to the residue field under the quotient map $R_{fin} \to R_{fin}/\mu_R$.*

1.1.5 Algebraic closure and real closure

We already mentioned the following generalization of Theorem 1.20, from [7].

Theorem 1.13 (Generalized Newton-Puiseux Theorem)**.** *If G is a divisible ordered Abelian group and K is field that is real closed, or algebraically closed of characteristic 0, then $K((G))$ is also a field that is real closed, or algebraically closed.*

There are various accounts of the proof of Theorem 1.13. In §4, we use the proof found in some notes of Starchenko [17], [18], which he kindly shared with the authors. We will make frequent use of the following version of Taylor's Formula.

Lemma 1.14. *Let $p(x) = A_0 + A_1x + \cdots + A_nx^n \in K((G))[x]$. If $a, b \in K((G))$, then $p(a+bx) = B_0 + B_1x + \cdots + B_nx^n$, where for each i, $B_i = \sum_{i \leq j \leq n} A_j \begin{pmatrix} j \\ i \end{pmatrix} a^{j-i}b^j$. Hence, $p(a+x) = p(a) + p'(a)x + H(x,a)x^2$, where $H(x,a) = B_2 + B_3x + \cdots + B_nx^{n-2}$ for $B_i = \sum_{i \leq j \leq n} A_j \begin{pmatrix} j \\ i \end{pmatrix} a^{j-i}$.*

Proof. The result holds by direct calculation. Note that $i!B_i = \frac{d^{(i)}}{dx^i} p(a+bx)|_{x=0}$. □

1.2 Construction of Mourgues and Ressayre

We describe Mourgues and Ressayre's result and give some details of the construction. We will say enough to make clear how the main result of the present paper, Theorem 1.3, actually proves the conjecture that was stated in [6] (see Conjecture 1.16 below). Theorem 1.3 is stated in terms of *tc*-bases, and these are not mentioned in [6]. We note that many of the details in this subsection (§1.2.2 in particular) are not needed to understand the main results. Rather, this subsection provides context and motivation for the results.

1.2.1 Constructing an integer part

Mourgues and Ressayre proved that every real closed field F has an *integer part*, where this is a discrete ordered subring I such that, for each $r \in F$, there is some $i \in I$ with $i \leq r < i + 1$. They showed the following.

Theorem 1.15 (Mourgues-Ressayre, with help from Marker and Delon [9])**.** *Let F be a real closed field with residue field section k. Then there exists a value group section G and an isomorphism d from F onto a truncation closed subfield R of $k((G))$.*

Mourgues and Ressayre noted that R has a simple integer part I, consisting of the elements of the form $s + zt^0$, where $s \in R$ with $Supp(s) \subseteq G^{<0}$, and $z \in \mathbb{Z}$. Then $d^{-1}(I)$ is an integer part for F.

Mourgues and Ressayre gave an explicit procedure for passing from a given real closed field F, with a well ordering $\prec$ and a residue field section k, to a value group section G and an isomorphism d from F to a truncation closed subfield R of $k((G))$. The procedure yields a sequence of elements $(b_\alpha)_{\alpha<\gamma}$ forming a transcendence basis for F over k. The order type of $\prec$ bounds the length of this sequence. For $\alpha \leq \gamma$, let F_α be the set of elements of F that are algebraic over $k \cup \{b_\beta : \beta < \alpha\}$. We get a chain of groups $(G_\alpha)_{\alpha\leq\gamma}$ and a chain of functions $(d_\alpha)_{\alpha\leq\gamma}$, where G_α is a value group section of F_α, and d_α is a truncation closed embedding of F_α into $k((G_\alpha))$. Let $R_\alpha = d(F_\alpha)$.

As we will see below, the procedure also yields a *tc*-basis $(r_\alpha)_{\alpha<\gamma}$ for R, where for each $\alpha < \gamma$, the elements r_α and $d(b_\alpha)$ are inter-algebraic over R_α. From this, it follows that $(R_\alpha)_{\alpha\leq\gamma}$ is the canonical sequence corresponding to the *tc*-basis $(r_\alpha)_{\alpha<\gamma}$. The *tc*-basis $(r_\alpha)_{\alpha<\gamma}$ has the same length as the sequence $(b_\alpha)_{\alpha<\gamma}$ forming a transcendence basis for F over k.

In [6], the authors stated the following conjecture.

Conjecture 1.16 ([6]). *Let F be a real closed field with a well ordering $\prec$ of order type ω, so the sequence (b_n) forming a transcendence basis for F over k has length at most ω. Let F_n be the set of elements of F algebraic over $k \cup \{b_i : i < n\}$. For $n \geq 1$, the elements of $R_n = d(F_n)$ have length at most $\omega^{\omega^{n-1}}$, so the elements of $R = d(F)$ have length less than $\omega^{\omega^{\omega}}$.*

The conjecture follows immediately from Theorem 1.3 once we know that there is a tc-basis $r_0, r_1, \ldots$ for R for which the corresponding canonical sequence consists of the subfields $R_n = d(F_n)$.

1.2.2 Details of the construction

Fix F a real closed field, a well-ordering $\prec$, and a residue field section k. For the Mourgues and Ressayre construction, we first say how to determine the sequence $(b_\alpha)_{\alpha<\gamma}$ forming a transcendence basis for F over k, and the corresponding chain of subfields $(F_\alpha)_{\alpha\leq\gamma}$. This is where we use the well ordering $\prec$. Let $F_0 = k$. If $F_\alpha \neq F$, then b_α is the $\prec$-first element of $F - F_\alpha$, and $F_{\alpha+1}$ is the set of elements of F algebraic over $F_\alpha(b_\alpha)$. For limit α, $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$. Note that $F = F_\gamma$. The length of the sequence $(b_\alpha)_{\alpha<\gamma}$ is obviously bounded by the order type of $\prec$.

Below, we say exactly how Mourgues and Ressayre define a chain of groups $(G_\alpha)_{\alpha\leq\gamma}$ and a chain of functions $(d_\alpha)_{\alpha\leq\gamma}$, where G_α is a value group section of F_α and d_α is an isomorphism from F_α onto a truncation closed subfield R_α of $k((G_\alpha))$. Then $G = G_\gamma$ is a value group section for F, and $d = d_\gamma$ is an isomorphism from F to a truncation closed subfield $R = R_\gamma$ of $k((G))$. We will see that the construction also yields a sequence $(r_\alpha)_{\alpha<\gamma}$ such that for each $\alpha < \gamma$, the elements r_α and $d(b_\alpha)$ are inter-algebraic over R_α, and either $r_\alpha = t^g$ for some $g \in G_{\alpha+1} - G_\alpha$, or else r_α has limit length, and all proper truncations are in R_α. Thus, $(r_\alpha)_{\alpha<\gamma}$ is a tc-basis for R in $k((G))$, and $(R_\alpha)_{\alpha\leq\gamma}$ is the corresponding canonical sequence.

Keep in mind that in the value group sections G_α, the operation is $\cdot$ (inherited from F) and the identity is $1 \in F$. Nonetheless, we use additive notation for G_α, and we write 0 for the identity since this notation is more natural once we begin working in Hahn fields.

Base Case: G_0 and d_0. By definition, $F_0 = k$. The value group section is just $\{1\}$. We write $G_0 = \{0\}$. We let $d_0 : k \to k((G_0))$ be the "identity" function; that is, $d(a) = a \cdot t^0$ for $a \in k$.

Successors. Given G_α and d_α, we must define $G_{\alpha+1}$, $d_{\alpha+1}$, and r_α. We suppose that d_α is an isomorphism from F_α onto R_α, where this is a truncation closed subfield of $k((G_\alpha))$. Using d_α, we define an initial segment of $d_{\alpha+1}(b_\alpha)$ of form $\sum_{j<\zeta} a_j t^{g_j}$ in $k((G_\alpha))$. (This initial segment may or may not be proper.) For each $j < \zeta$, we find $c_j \in F_\alpha$ such that $d_\alpha(c_j) = \sum_{i<j} a_i t^{g_i}$. The sequence $(c_j, g_j, a_j)_{j<\zeta}$ satisfies the following conditions:

1. $c_0 = 0$,
2. if c_j is defined, with $c_j \neq b_\alpha$, and $w(b_\alpha - c_j) \in G_\alpha$, then $g_j = w(b_\alpha - c_j)$,
3. if c_j and g_j are defined, then a_j is the unique element of k such that $w(b_\alpha - (c_j + a_j g_j)) > g_j$,
4. if for all $i < j$, elements c_i, g_i, and a_i are defined, and there is some $c \in F_\beta$ such that $d_\alpha(c)$ is $\sum_{i<j} a_i t^{g_i}$, then $c_j = c$.

We will come to ζ such that one of the following happens:

Case 1: For $j < \zeta$, we have defined a_j and g_j, so that $\sum_{j<\zeta} a_j t^{g_j}$ is defined, but this is not in $R_\alpha = d(F_\alpha)$. In this case, ζ must be a limit ordinal. We let $G_{\alpha+1} = G_\alpha$ and $r_\alpha = \sum_{j<\zeta} a_j t^{g_j}$, and we let $d_{\alpha+1}$ be the unique extension of d_α taking b_α to $\sum_{j<\zeta} a_j t^{g_j}$. In this case, $r_\alpha = d_{\alpha+1}(b_\alpha)$. All elements of $F_{\alpha+1}$ are algebraic over $F_\alpha(b_\alpha)$. Then $d_{\alpha+1}$ is determined on all of $F_{\alpha+1}$. It must map the m^{th} root of a polynomial over $F_\alpha(b_\alpha)$ to the m^{th} root of the corresponding polynomial over $R_\alpha(r_\alpha)$.

Case 2: We have defined c_ζ, but G_α has no element appropriate for $w(b_\alpha - c_\zeta)$. The absolute value $|b_\alpha - c_\zeta|$ is a positive element of $F_{\alpha+1}$ not $\sim$-equivalent to any element of F_α. We let g be either this element or its reciprocal, whichever is infinitesimal. We let the value group section $G_{\alpha+1}$ be the set of elements of the form $h \cdot g^q$, where $h \in G_\alpha$, $q \in \mathbb{Q}$. Thinking additively, $G_{\alpha+1}$ consists of the elements $h + qg$, where $h \in G_\alpha$ and q is rational. Note that g and b_α are inter-algebraic over F_α. Then all elements of $F_{\alpha+1}$ are algebraic over $F_\alpha(g)$. We set $r_{\alpha_1} = t^g$, and let $d_{\alpha+1}$ be the unique extension of d_α mapping g to t^g and taking the m^{th} root of a polynomial over $F_\alpha(g)$ to the m^{th} root of the corresponding polynomial over $R_\alpha(t^g)$.

This completes the description of the Mourgues and Ressayre construction.

1.2.3 Truncation closed subfields and tc-bases

Mourgues and Ressayre showed that, for each $\alpha \leq \gamma$, R_α is a truncation closed subfield of $k((G_\alpha))$ (and, hence, of $k((G)))$. For this, they proved the following.

Lemma 1.17 (Mourgues-Ressayre)**.** *Suppose K is a field that is either real closed or algebraically closed of characteristic 0, and G is a divisible ordered Abelian group. Let R be a truncation closed, relatively algebraically closed subfield of $K((G))$, and let r be an element of $K((G))$ such that either $r = t^g$ for some $g \in G$, or else r has limit length, and all proper truncations are in R. Then $R(r)$ is truncation closed, and so is the relative algebraic closure of $R(r)$.*

We have seen how the procedure of Mourgues and Ressayre yields a *tc*-independent sequence $(r_\alpha)_{\alpha<\gamma}$ in $k((G))$ forming a transcendence basis for $R = d(F)$ over k. The corresponding canonical sequence is $(R_\alpha)_{\alpha\leq\gamma}$, where R_α is the set of elements of R algebraic over $k \cup \{r_\beta : \beta < \alpha\}$. Lemma 1.17 shows that all R_α are truncation closed in $k((G))$, so R is truncation closed. More generally, we have the following.

Proposition 1.18. *Suppose K is a field that is either real closed or algebraically closed of characteristic* 0*, and G is a divisible ordered Abelian group. Let $(r_\alpha)_{\alpha<\gamma}$ be a tc-independent sequence in $K((G))$, and let $(R_\alpha)_{\alpha\leq\gamma}$ be the corresponding canonical sequence. Then all R_α are truncation closed and relatively algebraically closed.*

The converse is also true.

Proposition 1.19. *Suppose K is a field that is either real closed or algebraically closed of characteristic* 0*, and G is a divisible ordered Abelian group. If R is a subfield of $K((G))$ that is truncation closed and relatively algebraically closed, then R has a tc-basis.*

Proof. Let $\prec$ be a well ordering of R. We define r_α inductively. Let R_α be the set of elements of R algebraic over $K \cup \{r_\beta : \beta < \alpha\}$. If there is some $r \in R - R_\alpha$, take the $\prec$-first such r. Say $r = \sum_{g\in S} a_g t^g$. Let r' be the shortest truncation of r that is not in R_α. If r' has limit length, then set $r_\alpha = r'$. If r' has successor length, with last term at^g, then set $r_\alpha = t^g$. Eventually, we come to γ such that $R_\gamma = R$. Then $(r_\alpha)_{\alpha<\gamma}$ is the required *tc*-basis for R. □

We conclude this section with comments on the lengths of elements in R_0 and R_1 under the construction given in §1.2.2. Since $R_0 = K$, all elements of R_0 have length one. Recall that Theorem 1.13 states that $K((G))$ is a real closed field. In the case where $G = \mathbb{Q}$, the methods of Newton and Puiseux say more.

Theorem 1.20 (Newton-Puiseux (see [1], §2.6))**.** *Suppose $b \in R_1$ is a root of*

$$p(x) = A_0 + A_1 x + \cdots + A_n x^n$$

where $A_i \in K[g]$. Then $d_1(b) = \sum_{i\geq I} a_i t^{\frac{i}{n!}g}$, where $I \in \mathbb{Z}$ and $a_i \in K$.

Since every element of R_1 is a root of some polynomial of the form given in Theorem 1.20, all elements of R_1 have length at most ω.

2 Ordinals and well ordered subsets of G

Let K be a field, and let G be an ordered Abelian group. We want to bound the length of a root of a polynomial over $K((G))$ in terms of the lengths of the coefficients. First, we discuss bounds under simpler operations. Given $s, s' \in K((G))$, where s has length α and s' has length β, we examine the possible lengths of $s+s'$ and $s \cdot s'$. Since supports are contained in G, we use some notation and results on well ordered subsets of ordered Abelian groups.

Definition 2.1. *Let G be an ordered Abelian group.*

1. *For $A, B \subseteq G$, let $A + B = \{g + h : g \in A \ \& \ h \in B\}$.*
2. *For $A \in G$, let $S_m(A) = \underbrace{A + \cdots + A}_{m}$.*
3. *For $A \subseteq G$, let $[A] = \bigcup_{m \in \omega} S_m(A)$; i.e., $[A]$ is the semi-group generated by elements of A.*

Lemma 2.2 (Neumann [11], §3)**.** *Let G be an ordered Abelian group.*

1. *If $A, B \subseteq G$ are well ordered, then so is $A \cup B$.*
2. *If $A, B \subseteq G$ are well ordered, then so are the sets $A + B$ and $S_m(A)$ for all $m \in \omega$.*
3. *If $A \subseteq G$ consists of nonnegative elements and A is well ordered, then $[A]$ is well ordered.*

To bound the order types of the sets described in Lemma 2.2, we use the operations of commutative sum and product.

Definition 2.3 (Commutative Sum)**.** *For ordinals α and β, the* commutative sum*, denoted by $\alpha \oplus \beta$, is the ordinal whose Cantor normal form is obtained by expressing α and β in Cantor normal form and summing the coefficients of like terms.*

Definition 2.4 (Commutative Product)**.** *For ordinals γ and β, the* commutative product*, denoted by $\gamma \otimes \beta$, is the ordinal whose Cantor normal form is obtained by expressing γ and β in Cantor normal form, multiplying as for polynomials, using commutative sum in the exponents, and then combining like terms.*

Note that if $\gamma = \sum_i \omega^{\gamma_i} m_i$ and $\beta = \sum_j \omega^{\beta_j} n_j$, then the commutative product of the terms $\omega^{\gamma_i} m_i$ and $\omega^{\beta_j} n_j$ is $\omega^{\gamma_i \oplus \beta_j}(m_i \cdot n_j)$.

The following result can be obtained using work of De Jongh and Parikh [4], [10], and Schmidt [16] on the maximal order type for a linearization of a well partial ordering. The ideas go back to Carruth [3], and they are also found in [5]. One can find a proof using only Definitions 2.3 and 2.4 in [2, Lemmas 4.1, 4.5].

Lemma 2.5. *If A, B are subsets of G, where A has order type α and B has order type β, then*

1. *$A \cup B$ has order type at most $\alpha \oplus \beta$, and*

2. *$A+B$ has order type at most $\alpha \otimes \beta$. So, $S_m(A)$ has order type at most $\underbrace{\alpha \otimes \ldots \otimes \alpha}_{m}$.*

Corollary 2.6. *If $s, s' \in K((G))$, where s has length α and s' has length β, then*

1. *$s + s'$ has length at most $\alpha \oplus \beta$, and*

2. *$s \cdot s'$ has length at most $\alpha \otimes \beta$. Thus, s^m has length at most $\underbrace{\alpha \otimes \ldots \otimes \alpha}_{m}$.*

The bounds on lengths of developments in Theorems 1.3 and 1.4 turn out to be the infinite "multiplicatively indecomposable" ordinals.

Definition 2.7 (Indecomposable ordinals)**.** *An ordinal α is* additively (respectively, muliplicatively) indecomposable *if there do not exist $\beta, \gamma < \alpha$ such that $\beta + \gamma = \alpha$ (respectively, $\beta \cdot \gamma = \alpha$).*

Fact 2.8 (See Pohlers [13], §7)**.**

1. *The infinite additively (respectively, multiplicatively) indecomposable ordinals are exactly those of the form ω^α (respectively, ω^{ω^α}).*

2. *Suppose $\beta, \gamma < \alpha$. If α is additively (respectively, multiplicatively) indecomposable, then $\beta \oplus \gamma < \alpha$ (respectively, $\beta \otimes \gamma < \alpha$).*

Fact 2.9. *If $\alpha = \omega^{\omega^\delta m}$ and $\beta = \omega^{\omega^\gamma n}$, where $\delta \geq \gamma$, then $\alpha \otimes \beta = \alpha \cdot \beta$. Hence, if α has the form $\omega^{\omega^\delta m}$, then $\underbrace{\alpha \otimes \cdots \otimes \alpha}_{n} = \alpha^n$.*

We use the following statements repeatedly in the proofs of Theorems 1.3 and 1.4. Lemma 2.10 (2a) below plays a particularly crucial role.

Lemma 2.10. *Let A, B be well ordered subsets of $G^{\geq 0}$, and let α be an infinite multiplicatively indecomposable ordinal.*

1. *If A, B have order type less than α, then $A \cup B$, $S_m(A)$, and $\bigcup_{k \leq m} S_k(A)$ have order type less than α.*

2. *Suppose G is Archimedean.*

   (a) *If the order type of pred$(b) \cap A$ is less than α for all $b \in [A]$, then the order type of $[A]$ is at most α.*

   (b) *Suppose $A \neq \{0\}$. For any $g \in G$, the order types of pred$(g) \cap [A]$ and pred$(g) \cap A$ are less than the order type of $[A]$.*

Proof.

(1) These statements follow immediately from Lemma 2.5 and Fact 2.8 (2).

(2a) The result is clear if $A = \{0\}$, so suppose otherwise. Let d be the least positive element of A. Let $b \in [A]$. Since G is Archimedean, there is some m such that $md \geq b$. Hence, all predecessors of b in $[A]$ are in $S_m(pred(b) \cap A)$. By (1), the order type of $S_m(pred(b) \cap A)$ is less than α. So, the order type of $[A]$ is at most α.

(2b) Let $g \in G$. Since G is Archimedean and A contains positive elements, $[A]$ is cofinal in G. Thus, $pred(g) \cap [A]$ is a proper initial segment of $[A]$. The statements follow since $pred(g) \cap A \subseteq pred(g) \cap [A]$. □

We note that both (2a) and (2b) of Lemma 2.10 are false without the assumption that G is Archimedean. Let G be the divisible ordered Abelian group generated by two positive elements $g_1 < g_2$ that are not Archimedean equivalent. We claim $A = \{g_1, g_2\}$ gives a counterexample to (2a) in the non-Archimedean setting for $\alpha = \omega$. The order type of A is 2, but the order type of $[A]$ is $\omega^2 > \alpha$. The set $A' = \{g_1\}$ gives a counterexample to (1); the order type of $[A']$ is ω but so is the order type of $pred(g_2) \cap [A']$.

3 Bounds on lengths of roots

In this section, we reduce the proofs of Theorems 1.3 and 1.4 to a result on lengths of roots of polynomials (Theorem 3.2), which we will prove in §4. To state this result, we need some notation.

Notation 3.1. *Let $p(x) = A_0 + A_1x + \cdots + A_nx^n \in K((G))[x]$. The* support of *p, denoted by Supp(p), is the set $\bigcup_{i \leq n} Supp(A_i)$.*

Lemma 2.5 provides bounds on the lengths of $s + s'$ and $s \cdot s'$, for $s, s' \in K((G))$. Given a polynomial $p(x) \in K((G))[x]$, we can bound the length of a root r of $p(x)$ in terms of the order type of $[Supp(p)]$, at least in the case where G is Archimedean.

Theorem 3.2 (Length of supports of roots)**.** *Let K be a field that is real closed or algebraically closed of characteristic 0. Let G be an Archimedean divisible ordered Abelian group, and let α be an infinite multiplicatively indecomposable ordinal. Let r be a root of $p(x)$, where $p(x) \in K((G))[x]$, and suppose that $Supp(p) \subseteq G^{\geq 0}$ and $w(r) > 0$. If $[Supp(p)]$ has order type at most α, then $[Supp(r)]$ has order type at most α.*

We first use Theorem 3.2 to prove Theorem 1.4 (where G is Archimedean). We then use Theorem 1.4 to prove Theorem 1.3.

We recall the setting of Theorems 1.3 and 1.4. We have a real closed or algebraically closed field of characteristic zero K and a divisible ordered Abelian group

G. Let $(r_\beta)_{\beta<\alpha}$ be a tc-independent sequence in $K((G))$, i.e., it is algebraically independent over K and satisfies, for each $\beta < \alpha$,

- either $r_\beta = t^g$, for some $g \in G$, or else
- r_β has limit length and all proper truncations of r_β are algebraic over $K \cup \{r_\gamma : \gamma < \beta\}$.

We let $(R_\beta)_{\beta\leq\alpha}$ be the canonical sequence associated with $(r_\beta)_{\beta<\alpha}$, so R_β consists of the elements of $K((G))$ that are algebraic over $K \cup \{r_\gamma : \gamma < \beta\}$. Recall that by Proposition 1.17, each R_β is truncation closed and relatively algebraically closed. For each $\beta \leq \alpha$, we let G_β be the subgroup of G generated by rational multiples of the elements g such that $r_\gamma = t^g$ for some $\gamma < \beta$. The next lemma says that R_β is contained in $K((G_\beta))$. Hence, $\{t^g : g \in G_\beta\}$ is a value group section for R_β.

Lemma 3.3. *Let $(r_\beta)_{\beta<\alpha}$ be a tc-independent sequence in $K((G))$, with corresponding canonical sequence $(R_\beta)_{\beta\leq\alpha}$. Then, for all $\beta \leq \alpha$, the field R_β is contained in $K((G_\beta))$. Furthermore, if $R_{\beta+1} - R_\beta$ has an element of form t^g, then r_β has this form.*

Proof. The proof goes by induction on β. The base case holds, since $R_0 = K$ and G_0 is the trivial group, so r_0 cannot have limit length. At limit β, the first statement is immediate. We suppose that $R_\beta \subset K((G_\beta))$. Now, by the Generalized Newton-Puiseux Theorem (Theorem 1.13), both $K((G_\beta))$ and $K((G_{\beta+1}))$ are algebraically closed, or real closed (whichever K is). If $r_\beta = t^g$ for some $g \in G$, then it immediately follows that $R_{\beta+1} \subset K((G_{\beta+1}))$. Otherwise, by definition of a tc-independent sequence, all proper truncations of r_β are in $K((G_\beta))$. Since r_β has limit length, $r_\beta \in K((G_\beta))$ as well. Again we can conclude that $R_{\beta+1} \subset K((G_\beta))$. This last argument also proves the contrapositive of the second statement. □

We would like to assume that all elements of a given tc-independent sequence are positive, with positive valuation. The next lemma justifies this assumption.

Lemma 3.4. *Suppose $(r_\beta)_{\beta<\alpha}$ is a tc-independent sequence in $K((G))$, with corresponding canonical sequence $(R_\beta)_{\beta\leq\alpha}$. There is a tc-independent sequence $(r'_\beta)_{\beta<\alpha}$, such that $r'_\beta > 0$ and $w(r'_\beta) > 0$ for all β, and the canonical sequence is unchanged.*

Proof. If $r_\beta = t^g$, where $g < 0$, we may take r'_β to be t^{-g}. Suppose r_β has limit length. If $w(r_\beta) = g < 0$, we may take r'_β to be $|t^{-2g} r_\beta|$. If $w(r_\beta) = 0$ and a is the coefficient of t^0 in r_β, we may take $r'_\beta = |r_\beta - at^0|$. □

Starting now, we adopt the following convention.

Convention. If $(r_\beta)_{\beta<\alpha}$ is a tc-independent sequence, then for all $\beta < \alpha$, the element r_β is positive and $w(r_\beta) > 0$.

We now turn to proving Theorems 1.3 and 1.4, which bound the lengths of elements in the R_β.

3.1 Bounds when G is Archimedean

Theorem 1.4 says that if G is Archimedean, then for all $n < \omega$, the elements of R_{n+1} have length at most ω^{ω^n}, and for any β satisfying $\omega \leq \beta \leq \alpha$, the elements of R_β have length at most ω^{ω^β}.

Proof of Theorem 1.4. Suppose G is Archimedean. The elements of R_1 have length at most ω by the Newton-Puiseux Theorem 1.20. If β is a limit ordinal, and the statement holds for all $\gamma < \beta$, then it holds for β; in fact, since each element of R_β is in R_γ for some $\gamma < \beta$, all elements of R_β have length less than ω^{ω^β}. It remains to show that if the statement holds for a given ordinal β, then it holds for the successor $\beta + 1$.

Let β^* be the bound on the lengths of elements of R_β given in the statement of Theorem 1.4. This ordinal has the form ω or ω^{ω^γ}, so it is multiplicatively indecomposable. By definition of a tc-independent sequence, if $G_{\beta+1} \neq G_\beta$, then $r_\beta = t^g$ for some positive $g \in G$, a length 1 element. If $G_{\beta+1} = G_\beta$, the element r_β has limit length and all its proper truncations are in R_β. By the induction hypothesis, elements of R_β have length at most β^*. Elements of R_β can only have length β^* if the support is cofinal in G_β (otherwise, there would be elements of length longer than β^*). Therefore, each proper truncation of r_β has length less than β^*, and so r_β has length at most β^*.

Each element $r \in R_{\beta+1}$ is a root of a polynomial $p(x)$ over $R_\beta[r_\beta]$. We may suppose that $w(r) > 0$ and $Supp(p) \subseteq G^{\geq 0}$. If either of these conditions fails to hold, then we can choose $g, h \in G$ such that $g + w(r) > 0$ and $Supp(t^h p(xt^{-g})) \subseteq G^{\geq 0}$. We then consider the root $r' = rt^g$ of the polynomial $t^h p(xt^{-g})$ instead of the root r of $p(x)$. Clearly, r' and r have the same length.

Since β^* has form ω or ω^{ω^γ}, the ordinal $(\beta^*)^\omega$ has the form ω^ω or $\omega^{\omega^{\gamma+1}}$. This ordinal is the next multiplicatively indecomposable one after β^*. We must show that this ordinal bounds the length of the root r of the polynomial $p(x)$. Suppose that $p(x) = A_0 + A_1 x + \cdots + A_n x^n$, where the coefficients A_i are in $R_\beta[r_\beta]$. Since β^* bounds the lengths of elements in $R_\beta \cup \{r_\beta\}$, each A_i has length at most $(\beta^*)^{k_i}$ for some $k_i \in \omega$, by Lemma 2.5 and Fact 2.9. Similarly, $Supp(p)$ has order type at most $(\beta^*)^m$ for some $m \in \omega$. Then, $[Supp(p)]$ has order type at most $(\beta^*)^\omega$ by Lemma 2.10 (2a). Applying Theorem 3.2, we get that $[Supp(r)]$, and hence $Supp(r)$, have order type at most $(\beta^*)^\omega$. □

3.2 Bounds when G is non-Archimedean

We now consider the case where G is non-Archimedean. We will use Theorem 1.4 to prove Theorem 1.3 by considering convex subgroups.

3.2.1 Convex subgroups

Definition 3.5. *For an ordered group G, a subgroup H is* convex *if for any positive $g, h \in G$, if $h \in H$ and $g < h$, then $g \in H$.*

Note that if H is a convex subgroup of G, then $H^{>0}$ is closed under $\approx$; i.e., if $g, h \in G^{>0}$, where $g \in H$ and $h \approx g$, then $h \in H$.

Since the group G we are considering is non-Archimdean, it has non-trivial proper convex subgroups.

Definition 3.6. *Let H be a convex subgroup of G. For $x \in K((G))$ with support in $G^{\geq 0}$, we let $o_H(x)$ be the truncation of x with support in H. If $Supp(x) \cap H = \emptyset$, then $o_H(x) = 0$.*

Proposition 3.7. *Suppose H is a convex subgroup of G. For $x, y \in K((G))$, both with support in $G^{\geq 0}$, we have the following:*

1. $o_H(x + y) = o_H(x) + o_H(y)$,
2. $o_H(x \cdot y) = o_H(x) \cdot o_H(y)$.

Proof. Each term in $o_H(x + y)$ has the form $(a + b)t^h$, where $h \in H$ and at^h and bt^h are terms in x and y respectively (here we allow the possibility that a or b is 0). Each term in $x \cdot y$ has the form $\sum_{(h,h') \in S_g} (ab)t^g$, where S_g is the set of pairs (h, h') such that at^h and $bt^{h'}$ are terms in x and y respectively and $g = h + h'$. For a term ct^g in $o_H(x \cdot y)$, we must have $g \in H$. Finally, the convexity of H implies that if $h, h' \geq 0$ and $h + h' = g$, both h and h' must be in H. □

We apply Proposition 3.7 to say something about roots of polynomials.

Notation 3.8. *For a polynomial $p(x) = A_0 + A_1x + \cdots + A_nx^n$ over $K((G))$, where $Supp(p) \subseteq G^{\geq 0}$, we write $o_H(p)(x)$ for the polynomial*

$$o_H(A_0) + o_H(A_1)x + \cdots + o_H(A_n)x^n \ .$$

Lemma 3.9. *Let H be a convex subgroup of G, and let $p(x)$ be a polynomial over $K((G))$ with $Supp(p) \subseteq G^{\geq 0}$. If r is a root of p with $w(r) \geq 0$, then $o_H(r)$ is a root of $o_H(p)(x)$.*

Proof. We have $p(r) = 0$. By Proposition 3.7, $0 = o_H(p(r)) = (o_H(p))(o_H(r))$. □

3.2.2 Reordering tc-bases with respect to convex subgroups

Let H be a convex subgroup of G. We first show that we can modify a tc-independent sequence, without changing its associated canonical sequence, so that the supports of elements are contained in H or disjoint from H in a specific way. Our eventual goal is to say when it is possible to re-order a tc-independent sequence, putting the elements with support in H in front of the rest.

Definition 3.10 (respecting a convex subgroup)**.** *Let H be a convex subgroup of G. Let $(r_\beta)_{\beta<\alpha}$ be a tc-independent sequence in $K((G))$, with corresponding canonical sequence $(R_\beta)_{\beta\leq\alpha}$. We say that $(r_\beta)_{\beta<\alpha}$* respects H *if it satisfies the following conditions:*

1. *if $R_{\beta+1} - R_\beta$ has an element of the form t^h for some $h \in H$, then r_β has this form.*

2. *if r_β has limit length, then the support of r_β is entirely in H or entirely outside H.*

Lemma 3.3 implies that if $G_{\beta+1} \neq G_\beta$, then $r_\beta = t^g$ for some positive $g \in G$, so every tc-independent sequence in $K((G))$ respects G. The next lemma says that we can adjust a given tc-independent sequence to make it respect a particular convex subgroup, without changing the corresponding canonical sequence.

Lemma 3.11. *Let $(r_\beta)_{\beta<\alpha}$ be a tc-independent sequence in $K((G))$. Let H be a convex subgroup of G. Then there is a tc-independent sequence $(r'_\beta)_{\beta<\alpha}$ that respects H and has the same canonical sequence as $(r_\beta)_{\beta<\alpha}$. Moreover, If G is generated (as a rational vector space) by those g such that some r_β has the form t^g, then $G = H \oplus \tilde{G}$, where H is generated by those $g \in H$ such that some r'_β has form t^g and $\tilde{G}$ is generated by the remaining $g \in G$ such that some r'_β has form t^g.*

Proof. Let $(R_\beta)_{\beta\leq\alpha}$ be the canonical sequence corresponding to $(r_\beta)_{\beta<\alpha}$. We define $(r'_\beta)_{\beta<\alpha}$. First, suppose that $R_{\beta+1} - R_\beta$ has an element of the form t^g for some $g \in G$. Then r_β is such an element by Lemma 3.3. If $R_{\beta+1} - R_\beta$ has an element of form t^h, for $h \in H$, then we take r'_β to be some such element. If r_β has form t^g for $g \in G$, and $R_{\beta+1} - R_\beta$ has no element of form t^h, for $h \in H$, then set $r'_\beta = r_\beta$. Our choice of these r'_β guarantees that the elements h such that some $r'_\beta = t^h$ form a basis for H, as a rational vector space. Consider the elements r_β not of form t^g for $g \in G$. Then r_β has limit length, with proper initial segments in R_β. If r_β has support entirely in H, then we let $r'_\beta = r_\beta$. Suppose $r_\beta = s + s'$, where s has support in H and s' has support entirely outside H. Then r_β and s' are inter-algebraic over R_β. We let $r'_\beta = s'$. It is clear that $(r'_\beta)_{\beta<\alpha}$ is a new tc-basis that respects H and has the same canonical sequence as $(r_\beta)_{\beta<\alpha}$. Finally, let $\tilde{G}$ be the group generated, as a rational vector space, by the set of $g \in G$ such that some $r'_\beta = t^g$ and $g \notin H$. We have $G = H \oplus \tilde{G}$.

Claim: $G = H \oplus \tilde{G}$.

Proof of Claim. First, we note that if $t_1^g, \ldots, t^{g_n}$ are algebraically independent in $K((G))$, and G' is the subgroup of G generated (as a $\mathbb{Q}$-vector space, by $g_1, \ldots, g_n$, then $K((G'))$ is relatively algebraically closed. In particular, if $g \in G$ and t^g is algebraic over K and the elements $t^{g_1}, \ldots, t^{g_n}$, then t^g must be in $K((G'))$. Hence, g is in G'.

The tc-basis $(r'_\beta)_{\beta<\alpha}$ was chosen carefully. The set B consisting of $g \in G$ such that some r'_β has form t^g is a basis for G as a $\mathbb{Q}$-vector space. The fact that B generates G follows from the fact that if $R_{\beta+1} - R_\beta$ contains an element of form

t^g, for $g \in G$, then r'_β has this form. Since the set $\{t^g \mid g \in B\}$ is algebraically independent (by definition of a *tc*-independent sequence), B is certainly linearly independent. The set $B_H \subseteq B$ consisting of those $h \in H$ such that some r'_β has form t^h forms a basis for H as a $\mathbb{Q}$-vector space. The fact that B_H generates H follows from the fact that if $R_{\beta+1} - R_\beta$ contains an element of form t^h, for $h \in H$, then r'_β has this form. Now, $B_{\tilde{G}} = B - B_H$ generates $\tilde{G}$, as a $\mathbb{Q}$-vector space, and $G = H \oplus \tilde{G}$. □

□

We want to re-order a *tc*-independent sequence, putting first the elements with support in a convex subgroup. The next lemma will help us show that the new sequence is still *tc*-independent.

Lemma 3.12. *Let H be a convex subgroup of G, and let $(r_\beta)_{\beta<\alpha}$ be a tc-independent sequence in $K((G))$ that respects H. Let $(R_\beta)_{\beta\leq\alpha}$ be the canonical sequence corresponding to $(r_\beta)_{\beta<\alpha}$. For all $\beta \leq \alpha$, all elements of R_β with support in H are algebraic over $K \cup \{o_H(r_{\beta'}) : \beta' < \beta\}$.*

Proof. Take $r \in R_\beta$ with support in H. Now, r is a root of a polynomial $p(x)$ with coefficients in $K[r_{\beta_1}, \ldots, r_{\beta_k}]$ for some $\beta_1, \ldots, \beta_k < \beta$. Say

$$p(x) = A_0 + A_1 x + \cdots + A_n x^n \ ,$$

where $A_i \in K[r_{\beta_1}, \ldots, r_{\beta_k}]$. By our convention, $w(r_{\beta_j}) > 0$. By Proposition 3.7, $o_H(A_i)$ is in the ring $K[o_H(r_{\beta_1}), \ldots, o_H(r_{\beta_k})]$. So, $o_H(p)(x)$ is a polynomial over this same ring. By Lemma 3.9, $o_H(r) = r$ is a root of $o_H(p)(x)$. □

Proposition 3.13. *Let H be a convex subgroup of G, and let $(r_\beta)_{\beta<\alpha}$ be a tc-independent sequence in $K((G))$ that respects H. Let $(\hat{r}_\gamma)_{\gamma<\alpha^*}$ be the sequence obtained by putting first those r_β's with support in H, and then those with support outside H, leaving the ordering otherwise unchanged. Then $(\hat{r}_\gamma)_{\gamma<\alpha^*}$ is tc-independent.*

Proof. If all r_β's have support in H, then the new sequence is the same as the original. So, we suppose that some r_β has support outside H. Let $(R_\beta)_{\beta\leq\alpha}$ be the canonical sequence corresponding to $(r_\beta)_{\beta<\alpha}$, and let $(\hat{R}_\gamma)_{\gamma\leq\alpha^*}$ be the canonical sequence corresponding to $(\hat{r}_\gamma)_{\gamma<\alpha^*}$. For each $\gamma < \alpha^*$, let $\beta(\gamma)$ be the location of $\hat{r}_\gamma$ in the original sequence; i.e., $\hat{r}_\gamma = r_{\beta(\gamma)}$. Since the sequence $(\hat{r}_\gamma)_{\gamma<\alpha^*}$ is just a re-ordering of the sequence $(r_\beta)_{\beta<\alpha}$, it is algebraically independent over K. Also, for each $\gamma < \alpha^*$, either $\hat{r}_\gamma = r_{\beta(\gamma)}$ has the form t^g for some $g \in G$, or else it has limit length, and then all proper truncations are in $R_{\beta(\gamma)}$.

To prove the proposition, it is enough to show that for $\gamma < \alpha^*$ such that $\hat{r}_\gamma$ has limit length, the proper truncations are all in $\hat{R}_\gamma$. There is some first γ, say γ^*, such that $\hat{r}_\gamma$ has support outside H. There are two cases to consider.

Case 1: Suppose $\gamma \geq \gamma^*$. In this case, all predecessors of $r_{\beta(\gamma)}$ in the original sequence are predecessors of $\hat{r}_\gamma$ in the new sequence. Then $R_{\beta(\gamma)} \subseteq \hat{R}_\gamma$. It follows that if $\hat{r}_\gamma$ has limit length, then all proper truncations of $\hat{r}_\gamma$ are in $\hat{R}_\gamma$.

Case 2: Suppose $\gamma < \gamma^*$. The predecessors of $\hat{r}_\gamma$ in the new sequence are just the predecessors of $r_{\beta(\gamma)}$ in the old sequence that have support in H. In this case, we cannot say that $R_{\beta(\gamma)} \subseteq \hat{R}_\gamma$. Nonetheless, we can show that if $\hat{r}_\gamma$ has limit length, then the proper truncations are in $\hat{R}_\gamma$. Consider the sequence $(o_H(r_\beta))_{\beta<\alpha}$. Since the sequence $(r_\beta)_{\beta<\alpha}$ respects H, each r_β has support entirely in H or entirely outside of H. Then $o_H(r_\beta)$ is either r_β or 0. If we drop from the sequence those $o_H(r_\beta)$'s that are 0, then what remains is exactly the sequence $(\hat{r}_\gamma)_{\gamma<\gamma^*}$. We are considering $\hat{r}_\gamma$ of limit length, where $\gamma < \gamma^*$, so $\hat{r}_\gamma$ has support in H. Now, $\hat{R}_\gamma$ is the set of elements algebraic over K and the set $\{\hat{r}_{\gamma'} : \gamma' < \gamma\} = \{o_H(r_{\beta'}) : \beta' < \beta(\gamma)\}$. By Lemma 3.12, all elements of $R_{\beta(\gamma)}$ with support in H are in $\hat{R}_\gamma$. The proper truncations of $\hat{r}_\gamma$ are all in $R_{\beta(\gamma)}$, and they have support in H, so they are in $\hat{R}_\gamma$.

In both cases, we have seen that if $\hat{r}_\gamma$ has limit length, then the proper truncations of $\hat{r}_\gamma$ are in $\hat{R}_\gamma$. This completes the proof that $(\hat{r}_\gamma)_{\gamma<\gamma^*}$ is tc-independent. □

We end with a lemma saying that we can re-order a tc-basis by moving a term corresponding to a group element to an earlier place. The proof is obvious.

Lemma 3.14. *Let $(r_\beta)_{\beta\leq\alpha}$ be a tc-independent sequence in $K((G))$. If the element r_α has form t^g, for $g \in G$, and we form a new sequence, putting r_α in an earlier position, and leaving the rest of the sequence unchanged, then the resulting sequence is also a tc-basis.*

3.2.3 Proof of Theorem 1.3

We recall the statement and setting of Theorem 1.3. Let $(r_n)_{n<\omega}$ be a tc-independent sequence. Let $(R_n)_{n\leq\omega}$ be the associated canonical sequence of truncation closed and relatively algebraically closed subfields of $K((G))$. We set G_n equal to the divisible subgroup of G generated by rational multiples of the elements $g \in G$ such that for some $i < n$, $r_i = t^g$. Theorem 1.3 states that for any tc-independent sequence $(r_n)_{n\in\omega}$ in $K((G))$, with corresponding canonical sequence $(R_n)_{n\in\omega}$, the elements of R_n have length at most $\omega^{\omega^{(n-1)}}$, for $1 \leq n < \omega$, regardless of whether G is Archimedean.

Proof of Theorem 1.3. The positive elements of G_n break into only finitely many $\approx$-classes. These $\approx$-classes are ordered by the ordering on G. We prove the following, by induction on m.

Claim: Let G be a divisible Abelian group such that $G^{>0}$ has m $\approx$-classes. Let K be a real closed or algebraically closed field of characteristic 0. Let R be a truncation-closed subfield of $K((G))$ with a tc-basis of length n. Then the elements of R all have length at most $\omega^{\omega^{(n-1)}}$.

For $m = 1$, the group G is Archimedean, so the Claim holds, by Theorem 1.4. Supposing that the Claim holds for m, we prove it for $m+1$. Let G be a divisible Abelian group such that $G^{>0}$ has m $\approx$-classes, let K be an appropriate field, and let R be a truncation-closed subfield of $K((G))$ with a finite tc-basis, say of length n. We must show that the elements of R all have length at most $\omega^{\omega^{(n-1)}}$. Let H be the subgroup of G generated by the positive elements in the first m $\approx$-classes of G. It is not difficult to see that H is convex.

By Lemma 3.11, we have a tc-basis $(r_i)_{i<n}$ for R over K that respects H. Furthermore, G has a basis (as a rational vector space) consisting of the positive elements g such that some r_i has the form t^g. Now, H is generated by the basis elements g in the first m $\approx$-classes, and we let $\tilde{G}$ be the group generated by those basis elements in the last $\approx$-class. So, G is the direct sum $H \oplus \tilde{G}$. Our choice of the tc-basis guarantees that rational linear combinations of the basis elements not in H are not in H. Moreover, the positive elements of $\tilde{G}$ are all in the last $\approx$-class. Hence, $\tilde{G}$ is Archimedean.

By Proposition 3.13, the re-ordering of $(r_i)_{i<n}$ in which the elements with support in H appear first is still tc-independent (and still respects H). We re-order this new sequence once more so that elements of the form t^h for some $h \in H$ appear first and elements of the form t^g for some $g \in \tilde{G}$ appear first among elements with support in $\tilde{G}$. By Lemma 3.14, the resulting sequence is also tc-independent (and respects H). Say this final sequence is $(\tilde{r}_i)_{i<n}$, and let $(\tilde{R}_i)_{i\leq n}$ be the corresponding canonical sequence. Note that $\tilde{R}_n = R$. Let j be the greatest index such that the elements of $\tilde{R}_j$ all have support in H. Note that $0 < j < n$, since at least one r_i has form t^g for $g \in H$, and at least one r_i has form t^g for $g \notin H$).

Let $\tilde{K}$ be $\tilde{R}_j$, and note that $\tilde{K}$ is a truncation closed subfield of $K((H))$. Since $H^{>0}$ breaks up into m $\approx$-classes, the induction hypothesis says that elements of $\tilde{K}$ have length at most $\omega^{\omega^{(j-1)}}$. We must show that the elements of R have length at most $\omega^{\omega^{(n-1)}}$. To do this, we view the elements of $R = \tilde{R}_n$, a subfield of $K((G))$, also as elements of $\tilde{K}((\tilde{G}))$. Recall that $\tilde{K}((\tilde{G}))$ is equipped with the valuation $\tilde{w} : \tilde{K}((\tilde{G})) \to \tilde{G} \cup \{\infty\}$ defined in §1.1, which satisfies $\tilde{w}(\tilde{K}^{\neq 0}) = 0$.

We embed R into $\tilde{K}((\tilde{G}))$ as follows. Let $r \in \tilde{R}$. Since $G = H \oplus \tilde{G}$, any element of $Supp(r)$ can be written uniquely as a sum $h+g$, for $h \in H$ and $g \in \tilde{G}$. Let $r = \sum_{\gamma<\alpha} a_\gamma t^{h_\gamma+g_\gamma} \in \tilde{R}_{n+1}$, where $h_\gamma \in H$, $g_\gamma \in \tilde{G}$, and $a_\gamma \in K$. We define s_r, the intended image of r, as follows. Focusing on a single $g \in \tilde{G}$, we let $I_g = \{\gamma < \alpha : g_\gamma = g\}$. For the coefficient of t^g in s_r, we take $b_g = \sum_{\gamma\in I_g} a_\gamma t^{h_\gamma}$. Now, $b_g \in K((H))$. When s_r is viewed as an element of the Hahn field $[\,K((H))\,]((\tilde{G}))$, its support is contained in $\{g_\gamma \in \tilde{G} : \gamma < \alpha\}$.

Claim 1. The map that sends each r in R to s_r is an embedding of R into $\tilde{K}((\tilde{G}))$ that is the identity on $\tilde{K} \cup \{t^g : g \in G\}$.

Proof. To be sure that $s_r \in \tilde{K}((\tilde{G}))$, we must show that each coefficient b_g appearing in s_r is in $\tilde{K}$. We first show that $b_g \in R$. Observe that for all $g < g'$ in $\tilde{G}$, we have $g + H < g' + H$, since H is convex and $G = H \oplus \tilde{G}$. Thus, if $g < g'$ in $\tilde{G}$, all $\gamma \in I_g$ come before all $\gamma' \in I_{g'}$. Let $r_{<g}$ be the truncation of r defined by $\sum_{\{\gamma<\alpha:g_\gamma<g\}} a_\gamma t^{h_\gamma+g_\gamma}$, and let $r_{\leq g}$ be the truncation of r defined by $\sum_{\{\gamma<\alpha:g_\gamma\leq g\}} a_\gamma t^{h_\gamma+g_\gamma}$. Since $\tilde{R}_n$ is truncation closed, $r_{<g}$ and $r_{\leq g}$ are both in $\tilde{R}_n$. Then, $b_g = \frac{r_{\leq g}-r_{<g}}{t^g}$ is also in $\tilde{R}_n$. By Lemma 3.12, any element of R with support in H is in $\tilde{K}$. For $g \in \tilde{G}$, the coefficient b_g is in $K((H))$, and we can conclude that $b_g \in \tilde{K}$.

It is routine to check that the map sending r to s_r is an embedding with the desired properties. □

Claim 2. The sequence $(s_{\tilde{r}_i})_{i=j}^{n-1}$ is algebraically independent over $\tilde{K}$, and for each i such that $j \leq i < n$, the collection $\{s_r : r \in \tilde{R}_i\}$ is a truncation closed subfield of $\tilde{K}((\tilde{G}))$ that is the relative algebraic closure of $\tilde{K} \cup \{s_{\tilde{r}_{i'}} : j \leq i' < i\}$. Moreover, all truncations of $s_{\tilde{r}_i}$ are in $\{s_r : r \in \tilde{R}_i\}$.

Proof. The independence and algebraic closure statements follow from Claim 1 and the fact that R_i is the set of elements algebraic over $\tilde{K} \cup \{\tilde{r}_{i'} : j \leq i' < i\}$. The statements about truncations follow from the definition of the map and the fact that $(\tilde{r}_i)_{i=j}^{n-1}$ is a tc-independent sequence. □

Claim 3. The sequence $(s_{\tilde{r}_i})_{i=j}^{n-1}$ is a tc-basis in $\tilde{K}((\tilde{G}))$ for $\{s_r : r \in R\}$.

Proof. Recall that $(\tilde{r}_i)_{i=j}^{n-1}$ is a tc-independent sequence in $K((G))$ such that the support of $\tilde{r}_i$ is disjoint from H. Moreover, if $\tilde{r}_i$ has the form t^g for some $g \in G$, then we ensured that $g \in \tilde{G}$. In this case, $s_{\tilde{r}_i} = t^g$, as required. Now suppose that $\tilde{r}_i$ has limit length. If $s_{\tilde{r}_i}$ has limit length, then all proper truncations of $s_{\tilde{r}_i}$ are in $\{s_r : r \in \tilde{R}_i\}$. Hence, they are algebraic over $\tilde{K} \cup \{s_{\tilde{r}_{i'}} : j \leq i' < i\}$, by Claim 2. We claim that $s_{\tilde{r}_i}$ does, in fact, have limit length. Otherwise, $s_{\tilde{r}_i}$ has a last term $b_g t^g$, where $g \in \tilde{G}$ and $b_g \in \tilde{K}$. Then t^g is inter-algebraic with $s_{\tilde{r}_i}$ over $\tilde{K} \cup \{s_{\tilde{r}_{i'}} : j \leq i' < i\}$, so $t^g \in \{s_r : r \in \tilde{R}_{i+1} - \tilde{R}_i\}$. Hence, $t^g \in \tilde{R}_{i+1} - \tilde{R}_i$. By Lemma 3.3, $\tilde{r}_i$ has the form $t^{g'}$ for some $g' \in G$, contradicting that $\tilde{r}_i$ has limit length. Thus, $(s_{\tilde{r}_i})_{i=j}^{n-1}$ is tc-independent. By Claim 2, it is a tc-basis for $\{s_r : r \in R\}$. Note that the length of the tc-basis is $n - j$. □

By Claim 3 and the fact that $\tilde{G}$ is Archimedean, we are in a position to apply Theorem 1.4 to $(s_{\tilde{r}_i})_{i=j}^{n-1}$ and $\{s_r : r \in R\}$, to bound the lengths of elements in R. Let $r \in R$. By Theorem 1.4, the length of s_r in $\tilde{K}((\tilde{G}))$ is at most $\omega^{\omega^{n-j-1}}$. As mentioned above, $\omega^{\omega^{j-1}}$ bounds the length of the elements of $\tilde{K}$, thought of as elements of $K((H))$. Thus, for all $r \in R$, the length of r, as an element of

$K((G))$ is bounded by $\omega^{\omega^{j-1}} \cdot \omega^{\omega^{n-j-1}} = \omega^{(\omega^{j-1}+\omega^{n-j-1})} < \omega^{\omega^{n-1}}$. This completes the inductive proof. □

The proof above suggests a stronger (but more complicated) statement than Theorem 1.3. The calculations show that the lengths of developments of elements in R must be shorter when there are additional $\approx$-classes.

4 Supports of roots

In this section, we have two goals. Our first goal is to present a constructive proof of Theorem 1.13. The theorem says that if G is a divisible ordered Abelian group and K is a field that is real closed, or algebraically closed field of characteristic 0, then $K((G))$ is also real closed or algebraically closed field. We will prove the theorem for algebraically closed fields of characteristic 0 and then indicate briefly the changes needed for real closed fields. Our second goal is to prove Theorem 3.2. This is the result bounding the length of a root r of a polynomial $p(x)$ over $K((G))$ in the case where G is Archimedean. In the previous section, we saw that Theorem 3.2 implies Theorem 1.3 and Theorem 1.4.

4.1 Newton's method in $K((G))$

For the proof of Theorem 1.13, we follow unpublished notes of Starchenko [17], [18]. (A related account is given in [8], but this reference is not easily accessible.) The definition of "k-regularity", given below, is important. To show that a polynomial $p(x)$ has a root in $K((G))$, we transform $p(x)$ into a polynomial $q(x)$ that is k-regular for some k, with the same roots. Using further transformations, we show by induction on k that the k-regular polynomials have roots in $K((G))$.

Definition 4.1 (regular, semi-regular)**.** *Let $p(x) = A_0 + A_1x + \cdots + A_nx^n$ be a polynomial over $K((G))$.*

1. *$p(x)$ is* k-regular *if*
   (a) *$w(A_k) = 0$,*
   (b) *for all $i < k$, $w(A_i) > 0$, and*
   (c) *for all $i > k$, $w(A_i) \geq 0$.*
2. *$p(x)$ is* k-semi-regular *if*
   (a) *for all $i < k$, $w(A_i) > w(A_k)$, and*
   (b) *for all $i > k$, $w(A_i) \geq w(A_k)$.*

The proof of Theorem 1.13 will involve some lemmas about transformations of polynomials. In these lemmas, the group G is not assumed to be Archimedean, and the lemmas do not provide bounds on lengths. We prove some additional lemmas, for use in the proof of Theorem 3.2. These additional lemmas describe how the transformations affect the order types of the supports of the polynomials and the supports of the roots. The lemma below says that any polynomial is k-semi-regular for some k, and that we may replace a k-semi-regular polynomial $p(x)$ by one that is k-regular with the same roots.

Lemma 4.2. *Let $p(x) = A_0 + A_1x + \cdots + A_nx^n$ be a nonzero polynomial over $K((G))$. Then $p(x)$ is k-semi-regular for some $k \leq n$. If $g = w(A_k)$ for this k, then $q(x) = t^{-g}p(x)$ is k-regular, and $q(x)$ has the same roots as $p(x)$.*

Proof. Let g be the least element of $Supp(p)$. There is a first k such that $g = w(A_k)$. It is easy to see that $p(x)$ is k-semi-regular for this k. If $q(x) = t^{-g}p(x)$, then $q(x)$ is k-regular, with the same roots as $p(x)$. □

We note that if p and q are as in Lemma 4.2, then $Supp(q)$ has the same order type as $Supp(p)$. The lemma below says more, in the case where G is Archimedean.

Lemma 4.3. *Suppose G is Archimedean, and let α be an infinite multiplicatively indecomposable ordinal. Let p and q be as in Lemma 4.2; i.e., $q(x) = t^{-g}p(x)$, where g is least in $Supp(p)$. Suppose $Supp(p) \subset G^{\geq 0}$, so that $[Supp(p)]$ is well ordered. If $[Supp(p)]$ has order type at most α, then $[Supp(q)]$ also has order type at most α.*

Proof. If $g = 0$, then $q(x) = p(x)$. Suppose $g > 0$. By Lemma 2.10 (2a), it suffices to prove the following claim.

Claim: For any $b \in [Supp(q)]$, the order type of $pred(b) \cap Supp(q)$ is less than α.

Proof of Claim. Take $b \in [Supp(q)]$. For $d \in Supp(q)$, we have $d < b$ iff $d+g < b+g$ and $d + g \in Supp(p)$. Since $[Supp(p)]$ has order type at most α, the order type of $pred(b+g) \cap Supp(p)$ is less than α by Lemma 2.10 (2b), and the claim follows. □

□

The next lemma lets us transform a polynomial $p(x)$ with a root r satisfying $w(r) > 0$ into a k-regular polynomial $q(x)$ with a root s such that r and s have the same length and $w(s) = 0$.

Lemma 4.4. *Let $p(x) = A_0 + A_1x + \cdots + A_nx^n$ be a nonzero polynomial over $K((G))$ with a root r such that $w(r) = g > 0$. Let δ be the minimum of $w(A_i) + ig$, and let $q(x) = t^{-\delta}p(t^g x)$. For all $r' \in K((G))$, element r' is a root of $p(x)$ iff $t^{-g}r'$ is a root of $q(x)$. Moreover, $q(x)$ is k-regular for some $k \leq n$.*

Proof. It is clear that r' is a root of $p(x)$ iff $t^{-g}r'$ is a root of $q(x)$. We show that $q(x)$ is k-regular. We have $q(x) = B_0 + B_1 x + \cdots + B_n x^n$, where $B_i = A_i t^{ig-\delta}$. Then $w(B_i) = w(A_i) + ig - \delta$. By our choice of δ, we have $w(B_i) \geq 0$ for all i, and there is some least k such that $w(B_k) = 0$. Hence, $q(x)$ is k-regular. □

The lemma below says more, in the case where G is Archimedean.

Lemma 4.5. *Suppose G is Archimedean, and let α be an infinite multiplicatively indecomposable ordinal. Let p and q be as in Lemma 4.4; i.e., if A_i is the coefficient of x^i in $p(x)$, then $B_i = A_i t^{ig-\delta}$ is the coefficient of x^i in $q(x)$, where $g = w(r')$ for some $r' \in K((G))$ that is a root of $p(x)$ and δ is the least $w(A_i) + ig$. We suppose that $Supp(p) \subseteq G^{\geq 0}$, so that $[Supp(p)]$ is well ordered. If $[Supp(p)]$ has order type at most α, then $[Supp(q)]$ also has order type at most α.*

Proof. If $Supp(p) = \{0\}$, then $\delta = 0$, and $p(x)$ is just a polynomial over K. Then r' is in K, $g = 0$, and $q(x) = p(x)$. We suppose $Supp(p) \neq \{0\}$. By Lemma 2.10 (2a), it suffices to show that for any $b \in [Supp(q)]$, $pred(b) \cap Supp(q)$ has order type less than α. By Lemma 2.10 (1), it is enough to prove the following.

Claim: For any $b \in [Supp(q)]$, the order type of $pred(b) \cap Supp(B_i)$ is less than α.

Proof of Claim. For $d \in Supp(B_i)$, $d < b$ iff $d - ig + \delta < b - ig + \delta$; also, $d - ig + \delta \in Supp(A_i)$. Since $[Supp(p)]$ has order type at most α, the set of elements of $Supp(A_i)$ bounded by $b + ig - \delta$ must have order type less than α, by Lemma 2.10 (2b). The set of elements of B_i bounded by b has the same order type. □

□

We prove both Theorem 3.2 and Theorem 1.13 by considering k-regular polynomials, and proceeding by induction. In the preceding lemmas, we gave transformations taking a polynomial over $K((G))$ to one that is k-regular for some k with the same roots. We want to focus on infinitesimal roots. In the case where K is algebraically closed, we will show by induction that if $p(x)$ is k-regular for $k \geq 1$, then $p(x)$ has infinitesimal roots.

Remark 4.6. *Suppose $p(x)$ is 0-regular. Then $p(x)$ cannot have infinitesimal roots. To see this, note that if $w(r) > 0$, then $w(p(r)) = w(A_0)$, so $p(r) \neq 0$. However, given a root r of $p(x)$ with $w(r) \leq 0$, transformations involving an appropriate g, take $p(x)$ to a polynomial $q(x)$ that is k-regular for some $k \geq 1$ such that the element rt^g is an infinitesimal root of $q(x)$. Hence, the stated induction suffices to prove that, in the case where K is an algebraically closed field of characteristic 0, every polynomial over $K((G))$ has roots.*

The next lemma is the base case of Theorem 1.13.

Lemma 4.7 (Base Case of Theorem 1.13)**.** *If $p(x) = A_0 + A_1x + \cdots + A_nx^n$ is a 1-regular polynomial over $K((G))$, then $p(x)$ has a unique root r such that $w(r) > 0$. Moreover, $Supp(r) \subseteq [Supp(p)]$.*

Proof. We have $w(A_1) = 0$. Dividing by the coefficient of t^0 in A_1, we may also assume that the coefficient of t^0 in A_1 is 1. Then

$$p'(x) = A_1 + 2A_2x + \cdots + nA_nx^{n-1} \ .$$

If $w(r) > 0$, then $w(p'(r)) = 0$, and the coefficient of t^0 in $p'(r)$ is 1.

Uniqueness. Suppose $r_1 \leq r_2$ are roots of $p(x)$, where $w(r_1), w(r_2) > 0$. Let $r_2 - r_1 = \epsilon$. Then $0 = p(r_2) = p(r_1 + \epsilon)$. By Taylor's Formula, we have

$$0 = p(r_2) = p(r_1 + \epsilon) = p(r_1) + p'(r_1)\epsilon + H(r_1, \epsilon) \cdot \epsilon^2, \tag{1}$$

where $p(r_1) = 0$, $w(p'(r_1)\epsilon) = w(\epsilon) > 0$, and $w(H(r_1, \epsilon)) \geq 0$ by the 1-regularity of $p(x)$. Equation (1) is only possible if $\epsilon = 0$. Therefore, $r_1 = r_2$.

Existence. We show that $p(x)$ has a root r such that $Supp(r) \subseteq [Supp(p)]$ and $w(r) > 0$. Let $\Gamma = [Supp(p)]$. Lemma 2.2 (3) and the 1-regularity of $p(x)$ imply that Γ is well ordered, with least element 0. We inductively define coefficients $a_g \in K$, for each $g \in \Gamma$, such that if $r_g = \sum_{h\in\Gamma, h\leq g} a_h t^h$, then $w(p(r_g)) > g$. Then for all $h \leq g$, the coefficient of h in $p(r_g)$ is 0.

We allow $a_g = 0$. In particular, for $g = 0$, we let $a_0 = 0$. Then $r_0 = 0$, and $p(0) = A_0$, where $w(A_0) > 0$. Given $g > 0$, suppose we have defined a_h for $h \leq g$ so that $w(p(r_g)) > g$. Let g' be the successor of g in Γ. For all $b \in K$,

$$p(r_g + bt^{g'}) - p(r_g) = p'(r_g)bt^{g'} + H(r_g, bt^{g'})(bt^{g'})^2, \tag{2}$$

where $w(H(r_g, bt^{g'})) \geq 0$. We let $b = -c$, where c is the coefficient of $t^{g'}$ in $p(r_g)$. Then the coefficient of $t^{g'}$ in $p(r_g + bt^{g'})$ is 0, since the coefficient of $t^{g'}$ in $p'(r_g)bt^{g'}$ is b and the coefficient of t^0 in A_1 is 1 by our earlier assumption. We set $a_{g'} = b$. Hence, $w(p(r_{g'})) > g'$, since $w(p(r_g)) \geq g'$.

Suppose we have defined a_h for all $h < g$, where g is a limit of elements in Γ. Let $r_{<g} = \sum_{h\in\Gamma, h<g} a_h t^h$. Let $h < g$ and let $\epsilon_h = \sum_{h<h'<g, h'\in\Gamma} a_{h'} t^{h'}$. Then

$$p(r_{<g}) - p(r_h) = p'(r_h)\epsilon_h + H(r_h, \epsilon_h)\epsilon_h^2, \tag{3}$$

where $w(H(r_h, \epsilon_h)) \geq 0$. The right hand side has valuation $w(\epsilon_h)$, which is greater than h. It follows that for all $h < g$, the coefficient of t^h in $p(r_{<g})$ matches the coefficient of t^h in $p(r_h)$, which is 0, by induction. As in the successor case, we set $a_g = -c$ where c is the coefficient of t^g in $p(r_{<g})$. By examining $p(r_g) - p(r_{<g})$, we see that $w(p(r_g)) > g$.

Finally, let $r = \sum_{h\in\Gamma} a_h t^h$. We show that $p(r) = 0$. As in the limit case, for $h \in \Gamma$, we let $\epsilon_h = \sum_{h'\in\Gamma:h'>h} a_{h'} t^{h'}$. Then

$$p(r) - p(r_h) = p'(r_h)\epsilon_h + H(r_h, \epsilon_h)\epsilon_h^2 \ ,$$

where $w(H(r_h, \epsilon_h)) \geq 0$. Again, the righthand side has valuation $w(\epsilon_h)$, which is greater than h. Hence, for all $h' \leq h$, the coefficient of h' in $p(r)$ is 0. □

In Lemma 4.7, we showed that $Supp(r) \subseteq [Supp(p)]$. So, we get the base case of Theorem 3.2 even without the assumption that G is Archimedean.

Lemma 4.8 (Base Case of Theorem 3.2)**.** *Let $p(x)$ be a 1-regular polynomial over $K((G))$, and let r be the unique root such that $w(r) > 0$. Then the order type of $[Supp(r)]$ is at most the order type of $[Supp(p)]$.*

Proof. Since $Supp(r) \subseteq [Supp(p)]$, we have that $[Supp(r)] \subseteq [Supp(p)]$. □

With the exception of Definition 4.9, Lemma 4.16, and Theorem 4.18, the remaining material in this sub-section is given only for use in proving Theorem 1.13; it is not needed for Theorem 3.2. Definition 4.9 helps to organize the terms in $p(x)$ with respect to their valuations and the potential valuations of roots; in particular, the coefficients of monomials in a root will need to "cancel" when a root is substituted into the polynomial.

Definition 4.9 (ν-degrees, carriers, and ν-principal parts)**.** *Let $\nu \in G$, and let $p(x) = A_0 + A_1 x + \cdots + A_n x^n \in K((G))[x]$.*

(i) *For $A \in K((G))$, the* ν-degree *of Ax^i is $deg_\nu(Ax^i) = w(A) + i\nu$.*

(ii) *The* ν-degree *of $p(x)$, denoted by $deg_\nu(p(x))$, is the minimum of the ν-degrees of the monomials $A_i x^i$ in $p(x)$.*

(iii) *The* carrier *of ν in $p(x)$, denoted by Δ_ν, is the set of all $i \leq n$ such that $deg_\nu(A_i x^i) = deg_\nu(p(x))$.*

(iv) *The* ν-principal part *of $p(x)$ is $q(y) = \sum_{i\in\Delta_\nu} b_i y^i$, where b_i is the coefficient of t^ν in A_i and y is a variable.*

Since Δ_ν is finite for any $p(x) \in K((G))$, the ν-principal part of $p(x)$ is a polynomial over K.

Lemma 4.10. *Let $p(x) = A_0 + A_1 x + \cdots + A_n x^n \in K((G))[x]$. Let $r \in K((G))$ with $w(r) > 0$ and let $\nu = w(r)$.*

1. *$w(A_i r^i) = deg_\nu(A_i x^i)$ for any monomial $A_i x^i$.*
2. *If $p(x) \in K((G))[x]$, then $w(p(r)) \geq deg_\nu(p)$.*

Proof. The first statement is clear, and the second follows from the first. □

Lemma 4.11. *Let $\nu, \delta \in G$, where $\nu > 0$ and suppose $p(x) \in K((G))[x]$, where $deg_\nu(p(x)) > \delta$.*

1. *$p(t^\nu x) = t^\delta p_1(x)$ for some $p_1(x) \in K((G))[x]$ such that $Supp(p_1) \subset G^{>0}$.*
2. *If $a \in K((G))$ satisfies $w(a) \geq \nu > 0$, then $p(a + t^\nu x) = t^\delta p_1(x)$ for some $p_1(x) \in K((G))[x]$ such that $Supp(p_1) \subset G^{>0}$.*

Proof.
(1) Suppose $p(x) = A_0 + A_1 x + \cdots + A_n x^n$. We have

$$p(t^\nu x) = A_0 + A_1 t^\nu x + \cdots + A_n t^{n\nu} x^n \ .$$

Then $p_1(x) = t^{-\delta} p(t^\nu x)$ satisfies Statement (1).
(2) Let $p_2(x) = p(a + x)$. By Taylor's Formula, the coefficient of x^i in $p_2(x)$ is $\sum_{j \geq i} A_j \left(\begin{array}{c} j \\ i \end{array} \right) a^{j-i}$. This coefficient has valuation greater than $\delta - i\nu$. Then $deg_\nu(p_2(x)) > \delta$, and we can apply Statement (1) to get Statement (2). □

Lemma 4.12. *Let $p(x) = \sum_i A_i x^i \in K((G))[x]$, and let $\nu \in G$, where $\nu > 0$. If $\delta = deg_\nu(p(x))$, then $p(x) = t^\delta q(xt^{-\nu}) + s(x)$, where $q(x)$ is the ν-principal part of $p(x)$, and $s(x) \in K((G))[x]$ with $deg_\nu(s(x)) > \delta$.*

Proof. For $i \in \Delta_\nu$, let b_i be the leading coefficient of A_i, i.e., b_i is the coefficient of $t^{w(A_i)}$ in A_i. If $i \in \Delta_\nu$, then $w(A_i) = \delta - i\nu$ and $A_i = b_i t^{\delta - i\nu} + A'_i$ for some $A'_i \in K((G))[x]$ with $w(A'_i) + i\nu > \delta$. Let $p_\delta(x) = \sum_{i \in \Delta_\nu} b_i t^{\delta - i\nu} x^i$. Observe that $p_\delta(x) = t^\delta q(xt^{-\nu})$. Then $p(x) = p_\delta(x) + s(x)$, where $s(x) \in K((G))[x]$ and the ν-degree of $s(x)$ is strictly greater than δ. □

We consider how cancellation occurs when evaluating a polynomial at an infinitesimal root. Let $p(x) = \sum_i A_i x^i \in K((G))[x]$ and suppose $r \in K((G))$ with $0 < w(r) < \infty$. Let $\nu = w(r)$, let $\delta = deg_\nu(p(x))$, and let $q(y)$ be the ν-principal part of $p(x)$. By Lemmas 4.10 and 4.12, the coefficient of t^δ in $p(r)$ is $q(b)$, where b is the leading coefficient of r; i.e., b is the coefficient of t^ν in r. Then $w(p(r)) > \delta$ iff $q(b) = 0$. Since $r \neq 0$ and $b \neq 0$, $q(b)$ cannot equal 0 unless q contains at least two terms. Hence, the following definition is natural.

Definition 4.13 (Newton exponents and degrees)**.** *Suppose $\nu \in G$ and $p(x) \in K((G))[x]$.*

(i) *We say that ν is a* Newton exponent *of $p(x)$ if Δ_ν contains at least two elements.*

(ii) *Let ν be a Newton exponent of p. If i is the minimal element in Δ_ν and j is the maximal element in Δ_ν, then $j - i$ is called the* degree *of ν.*

We summarize our earlier observations about Newton exponents.

Lemma 4.14. *If $p(x) \in K((G))[x]$ and r is a nonzero root of p with $w(r) > 0$, then $\nu = w(r)$ is a Newton exponent of p, and the coefficient of t^ν in r is a root of the ν-principal part of p.*

Lemma 4.15. *Let ν, ν' be Newton exponents for p, where $\nu > \nu' > 0$. If $i \in \Delta_\nu$ and $j \in \Delta_{\nu'}$, then we must have $i \geq j$.*

Proof. Suppose $i < j$. Since $i \in \Delta_\nu$, we have $w(A_i) + i\nu \leq w(A_j) + j\nu$, and since $j \in \Delta_{\nu'}$, we have $w(A_j) + j\nu' \leq w(A_i) + i\nu'$. Then, $\nu' \leq \frac{w(A_i)-w(A_j)}{j-i} \leq \nu$, a contradiction. □

Lemma 4.16. *Let $p(x) = \sum_i A_i x^i$ be a k-semi-regular polynomial. Let $\nu \in G$, where $\nu > 0$. If $i > k$, then $deg_\nu(A_i x^i) > deg_\nu(A_k x^k)$. Hence, if ν is a positive Newton exponent of p, then $\Delta_\nu \subseteq \{0, \ldots, k\}$.*

Proof. Since $i > k$ and $p(x)$ is k-semi-regular, $w(A_k) \leq w(A_i)$. Since $\nu > 0$, $k\nu < i\nu$. Then $w(A_k) + k\nu < w(A_i) + i\nu$. □

Lemma 4.17. *Let $p(x) = \sum_i A_i x^i$ be k-semi-regular, where $A_0 \neq 0$ and $k > 0$. Let $\nu_0 > \cdots > \nu_l$ be the positive Newton exponents of $p(x)$. Then the least element of Δ_{ν_0} is 0, the greatest element of Δ_{ν_l} is k, and for $j = 1, \ldots, l$, the least element of Δ_{ν_j} is the greatest element of $\Delta_{\nu_{j-1}}$. Hence, the sum of the degrees of all positive Newton exponents of $p(x)$ is k.*

Proof. First, we show that 0 is the least element in Δ_{ν_0}. Let ν be the greatest among $\frac{w(A_0)-w(A_j)}{j}$, for $j = 1, \ldots, k$. Then, $\nu \geq \frac{w(A_0)-w(A_j)}{j}$ for all j, so $w(A_0) \leq w(A_j) + j\nu$. Hence, $deg_\nu(A_0) \leq deg_\nu(A_j x^j)$ for all j, with equality for at least one $j \leq k$ by Lemma 4.16. Since $w(A_k) < w(A_0)$, both $\frac{w(A_0)-w(A_k)}{k}$ and ν are positive. Thus, ν is a positive Newton exponent of $p(x)$. Since Δ_ν includes 0, we can apply Lemma 4.15 to see that ν is the greatest Newton exponent for $p(x)$; i.e., $\nu = \nu_0$.

To prove the lemma, it is enough to show, by Lemmas 4.15 and 4.16, that if ν_m is a positive Newton exponent for $p(x)$ and $i_m < k$ is the greatest element in Δ_{ν_m}, then i_m is the least element in Δ_ν for some Newton exponent ν. Let ν be the greatest element of form $\frac{w(A_{i_m})-w(A_j)}{j-i_m}$, for $i_{m+1} \leq j \leq k$. As above, we see that ν is positive, and $deg_\nu(A_{i_m} x^{i_m}) < deg_\nu(A_i x^i)$ for $i > k$. To show that ν is a Newton exponent, with i_m the least element of Δ_ν, we must show that $w(A_{i_m}) + i_m\nu < w(A_i) + i\nu$ for all $i = 0, \ldots, i_m - 1$.

Take $j \in \{i_m + 1, \ldots, k\}$ such that

$$\nu = \frac{w(A_{i_m}) - w(A_j)}{j - i_m}.$$

Now, i_m is maximal in Δ_{ν_m}. Therefore, $w(A_{i_m}) + i_m\nu_m < w(A_j) + j\nu_m$. Then,

$$\frac{w(A_{i_m}) - w(A_j)}{j - i_m} < \nu_m, \text{ so } \nu < \nu_m.$$

Let $i \in \{0, \ldots, i_m - 1\}$. Then $w(A_i) + i\nu_m \geq w(A_{i_m}) + i_m\nu_m$, so we have $w(A_i) - w(A_{i_m}) \geq (i_m - i)\nu_m$. Since $i_m - i > 0$ and $\nu_m > \nu > 0$, we get $w(A_i) - w(A_{i_m}) > (i_m - i)\nu$, and hence $w(A_i) + i\nu > w(A_{i_m}) + i_m\nu$, as required. □

Theorem 4.18. *Let K be an algebraically closed field of characteristic 0. Let $p(x) = \sum_i A_i x^i \in K((G))[x]$, with $k > 0$.*

- I_k *If $p(x)$ is k-regular, then $p(x)$ has exactly k roots with positive valuation, counted with multiplicity.*
- II_k *Let ν be a positive Newton exponent of $p(x)$.*
  - *(a) If k is the degree of the Newton exponent ν of p, then $p(x)$ has exactly k roots having valuation ν, counted with multiplicity.*
  - *(b) If $q(x) \in K[x]$ is the ν-principal part of $p(x)$, and $a \in K$ is a nonzero root of $q(x)$ of multiplicity k, then $p(x)$ has exactly k roots having valuation ν and leading coefficient a, counted with multiplicity.*

Proof. It is easy to see that statements II_1 (b),..., II_k (b) imply II_k (a). Below, we show that I_k implies II_k (b). We suppose the hypotheses of II_k (b). By Lemma 4.2, $p(x) \in K((G))[x]$ is k-semi-regular for some $k > 0$. Let ν be a positive Newton exponent of $p(x)$. Let $q(x) \in K[x]$ be the ν-principal part of $p(x)$, and let $a \in K$ be a nonzero root of $q(x)$ of multiplicity k. We show that $p(t^\nu(a+y))$ has exactly k infinitesimal roots counted with multiplicity. By Lemma 4.12, $p(t^\nu(a+y))$ can be expressed in the form $t^\delta q(a+y) + s(t^\nu(a+y))$, where $\delta = deg_\nu(p(t^\nu(a+y)))$ and $s(x) \in K((G))[x]$ with $deg_\nu(s) > \delta$. By Lemma 4.11, $s(t^\nu(a+y)) = t^\delta p_1(y)$ where $Supp(p_1) \subset G^{>0}$. Then $p(t^\nu(a+y)) = t^\delta(q(a+y) + p_1(y))$. It is enough to show that $q(a+y) + p_1(y)$ is k-regular. We can write $q(x)$ in the form $(x-a)^k q_1(x)$, where $q_1(a) \neq 0$. Then $q(a+y) = y^k q_2(y)$, where $q_2(x) \in K[x]$, with nonzero constant term. We can see that $y^k q_2(y) + p_1(y)$ is k-regular. Therefore, I_k implies II_k (b).

Finally, we show by induction on k that I_k holds. The case where $k = 1$ is Lemma 4.7. Suppose $k > 1$, so $I_1, \ldots, I_{k-1}$ hold and $II_1, \ldots, II_{k-1}$ hold. We show I_k. Let $p(x) \in K((G))[x]$, where $p(x)$ is k-regular. If the constant term is 0, then $p(x) = xq(x)$, where $q(x)$ is $(k-1)$-regular, and the induction hypothesis gives the conclusion. We suppose the constant term is not 0. By Lemma 4.17, k is the sum of the degrees of the positive Newton exponents. If $p(x)$ has more than one positive Newton exponent, then the conclusion follows from II_1 (a), ..., II_{k-1} (a).

We suppose that $p(x)$ has only one positive Newton exponent ν. Then the minimal element of Δ_ν is 0, the maximal element in Δ_ν is k, and the ν-principal part of $p(x)$ is a polynomial $q(x)$ of degree k, with nonzero constant coefficient. If $q(x)$ has more than one root, then all roots have multiplicity less than k, and the conclusion follows from II_1 (b), ..., II_{k-1} (b). We suppose that $q(x) = b(x-a)^k$, for some $b, a \in K$. We make a substitution to shift away from this case. Observe that $p'(x)$ is $(k-1)$-regular. Therefore, $p'(x)$ has a root r of positive valuation by I_{k-1}. (Note that r may be 0, which would simplify the argument below.) We consider the polynomial $p_1(x) = p(r+x)$. Now $p_1(x)$ is also k-regular, and $p_1'(0) = 0$. Therefore, the coefficient of x in $p_1(x)$ is 0. If $p_1(x)$ has more than one positive Newton exponent μ, we are done, by induction, as in the same case for $p(x)$. If $p_1(x)$ has only one positive Newton exponent μ, then the carrier Δ_μ does not contain 1, and the μ-principal part of $p_1(x)$ does not have the form $b(x-a)^k$. Then $p_1(x)$ has k roots of positive valuation, as argued earlier, and so does $p(x)$. □

Recall that every polynomial is k-semi-regular for some k, by Lemma 4.2. Hence, Theorem 4.18, Lemma 4.17, and Remark 4.6 together imply Theorem 1.13 in the case where K is algebraically closed of characteristic 0. We next consider the case where K is real closed.

Theorem 4.19. *Let K be a real closed field, let G be a divisible ordered Abelian group, and let $p(x) \in K((G))[x]$. Suppose $p(x)$ is k-semi-regular, where k is an odd integer.*

I_k *If $p(x)$ is k-regular, then $p(x)$ has a root with positive valuation in $K((G))$.*

II_k *Let ν be a positive Newton exponent of $p(x)$.*

- *(a) If k is the degree of the Newton exponent ν, then $p(x)$ has a root with valuation ν.*
- *(b) If $q(x) \in K[x]$ is the ν-principal part of $p(x)$ and $a \in K$ is a nonzero root of $q(x)$, then $p(x)$ has a root of valuation ν with leading coefficient a.*

Proof. The proof is like that for Theorem 4.18. We make three observations.

Observation (a). Since the sum of the degrees of the positive Newton exponents of $p(x)$ is k, and k is odd, there is at least one positive Newton exponent ν of odd degree.

Observation (b). Let ν be a Newton exponent of $p(x)$ of odd degree k_1. Then the ν-principal part of $p(x)$ has the form $x^m q(x)$, where the degree of $q(x)$ is k_1 and the constant coefficient of $q(x)$ is nonzero.

Observation (c). If $q(x) \in K[x]$ is a polynomial of odd degree, then there is at least one root in K. □

Theorem 4.19 and the following lemma, showing $K((G))^{>0}$ is closed under square roots, give Theorem 1.13 in the case where K is real closed.

Lemma 4.20. *Let K be a real closed field, let G be a divisible ordered Abelian group, and let $s \in K((G))^{>0}$. Then, there is some $r \in K((G))$ satisfying $r^2 = s$.*

Proof. Since K is real closed, $K(i)$ is algebraically closed. Let $s \in K((G))^{>0}$. Now, $K(i)$ is algebraically closed, and $s \in K(i)((G))$. By the version of Theorem 1.13 for algebraically closed fields, there exists some $r \in K(i)((G))$ satisfying $r^2 = s$. Suppose $r = \sum_{i<\alpha} b_i t^{g_i}$. We show that all coefficients b_i have imaginary part 0, so that $r \in K((G))$, as desired. Suppose otherwise, and let β be the least index such that b_β has nonzero imaginary part. First, observe that $\beta > 0$ since $s = r^2 = b_0^2 t^{2g_0} + \cdots$ with $b_0^2 > 0$. Similarly, the imaginary part of the coefficient of $t^{g_0+g_\beta}$ in $s = r^2$ must be 0. Consider the finite set $S = \{(i,j) \in \alpha\times\alpha \mid g_i+g_j = g_\beta\}$. As noted already, the imaginary part of $\sum_{(i,j)\in S} b_i b_j$ is 0. Since b_0 has imaginary part 0, the term $2b_0b_\beta$ has nonzero imaginary part. Hence, there is some $(i,j) \in S$ such that $0 < i, j < \beta$ and b_i or b_j must have nonzero imaginary part, contradicting the choice of β. □

4.2 Lengths of roots

We will prove Theorem 3.2. Recall the statement.

Theorem 3.2. Let G be an Archimedean divisible ordered Abelian group and let K be a field that is algebraically closed of characteristic 0 or real closed. Suppose α is an infinite multiplicatively indecomposable ordinal. Let $p(x)$ be a polynomial over $K((G))$ with $Supp(p) \subseteq G^{\geq 0}$, and let r be a root of $p(x)$ with $w(r) > 0$. If $[Supp(p)]$ has order type at most α, then $[Supp(r)]$ also has order type at most α.

Remark 4.21. *In the proof of Theorem 3.2, we may assume that K is algebraically closed. If K is real closed, not algebraically closed, we can view the given polynomial $p(x)$ and root r as elements of $K(i)((G))$.*

We need just a little more notation.

Definition 4.22 (Notation)**.**

1. *If $s \in K((G))$ satisfies $w(s) \geq 0$, the unique $\hat{s} \in K$ such that $w(s-\hat{s}) > w(s)$ is called the* residue *of s. (If $w(s) > 0$, then $\hat{s} = 0$.)*
2. *If $Supp(p) \subseteq G^{\geq 0}$, we write $\hat{p}(x)$ for the polynomial $\hat{A}_0 + \hat{A}_1 x + \cdots + \hat{A}_n x^n$, in $K[x]$.*

Proof of Theorem 3.2. The polynomial $p(x) = A_0 + A_1x + \cdots + A_nx^n$ is k-semiregular for some $k \leq n$. By Lemma 4.3, we may suppose that it is k-regular without changing the fact that $[Supp(p)]$ has order type at most α. The fact that $w(r) > 0$ implies that $k \geq 1$. We have the base case, where $k = 1$—this is Lemma 4.8. We continue by induction on k. We suppose that $k \geq 2$, where the statement holds for polynomials that are m-regular for $m < k$.

First, suppose $A_0 = 0$. Then $p(x) = xq(x)$, where r is a root of $q(x)$, $Supp(q) = Supp(p)$, and $q(x)$ is $(k-1)$-regular. By the induction hypothesis, $[Supp(r)]$ has order type at most α. Now, suppose $A_0 \neq 0$. Let $w(r) = \nu$. Let

Δ_ν be the carrier of ν in $p(x)$, and let δ be the ν-degree of $p(x)$, where this is the minimum of $w(A_i) + i\nu$, for $i \leq n$, and $\Delta_\nu = \{i : w(A_i) + i\nu = \delta\}$. By Lemma 4.16, $\Delta_\nu \subseteq \{0, \ldots, k\}$. Let $r_1 = t^{-\nu} r$ so $w(r_1) = 0$. Now, $p(t^\nu r_1) = 0$ and $Supp(r_1) \subset G^{\geq 0}$. Hence, r_1 is a root of the polynomial $p_1(x) = t^{-\delta} p(t^\nu x)$. Note that $p_1(x) = B_0 + B_1 x + \cdots + B_n x^n$, where $B_i = t^{-\delta} A_i t^{i\nu}$. In Lemma 4.5, we saw that $[Supp(p_1)]$ has order type at most α.

Claim 1: If $[Supp(r_1)]$ has order type at most α, then $[Supp(r)]$ also has order type at most α.

Proof of Claim 1. If $Supp(r) = \{\nu\}$ (so $Supp(r_1) = \{0\}$), then $[Supp(r)]$ has order type ω, and the claim holds. Otherwise, $Supp(r_1) \neq \{0\}$. Let $b \in [Supp(r)]$. By Lemma 2.10 (2a), it is enough to show that $pred(b) \cap Supp(r)$ has order type less than α. By Lemma 2.10 (2b), $pred(b - \nu) \cap Supp(r_1)$ is less than α. Since $Supp(r) = Supp(r_1) + \{\nu\}$, the order type of $pred(b) \cap Supp(r)$ is also less than α, as required. □

By the claim, we are done if we show that $[Supp(r_1)]$ has order type at most α. This condition is certainly satisfied if $Supp(r_1) = \{0\}$ or $Supp(p_1) = \{0\}$, so suppose otherwise. We express $p_1(x)$ as $q(x) + s(x)$, where $q(x) = \sum_{i \in \Delta_\nu} B_i x^i$ and $s(x) = \sum_{i \notin \Delta_\nu} B_i x^i$. Since the coefficients in $s(x)$ all have positive valuation, $\hat{s}(x) = 0$. Then $\hat{p}_1(x) = \hat{q}(x)$. Since $w(r_1) = 0$, we can write $r_1 = a + r_2$, where $a = \hat{r}_1 \in K$ and $0 < w(r_2) < \infty$. Then $Supp(r_1) = Supp(r_2) \cup \{0\}$. We have

$$0 = p_1(r_1) = p_1(a + r_2) = q(a + r_2) + s(a + r_2).$$

Taking residues, we get $0 = \hat{p}_1(a) = \hat{q}(a)$. So, a is a root of $\hat{q}(x)$. Since $\Delta_\nu \subseteq \{0, \ldots, k\}$, both $q(x)$ and $\hat{q}(x)$ have degree at most k. Let $m \leq k$ be the multiplicity of a as a root of $\hat{q}(x)$. We write $\hat{q}(x)$ as $(x - a)^m q_0(x)$, where $q_0(x) \in K[x]$ and $q_0(a) \neq 0$. Note that r_2 is a root of the polynomial $p_2(x) = p_1(a + x)$. By Taylor's Formula,

$$p_2(x) = C_0 + C_1 x + \cdots + C_n x^n \text{ , where } C_i = \frac{p_1^{(i)}(a)}{i!} \text{ .}$$

Then $\{0\} \subsetneq Supp(p_2) \subseteq Supp(p_1)$. We have $\hat{p}_2(x) = \hat{p}_1(a + x) = \hat{q}(a + x) = x^m q_0(a + x)$. Since $q_0(a) \neq 0$, it follows that $p_2(x)$ is m-regular.

Case 1. Suppose $m < k$. Now $Supp(p_2) \subseteq Supp(p_1)$, so $[Supp(p_2)]$ has order type at most α. By the induction hypothesis, $[Supp(r_2)]$ has order type at most α. We have $[Supp(r_1)] = \{0\} \cup [Supp(r_2)]$ since $r_1 = a + r_2$. Then, the order type of $[Supp(r_1)]$ is also at most (the limit ordinal) α, and the result holds in this case by Claim 1.

Case 2. Suppose $m = k$. Then $\hat{q}(x) = b(x - a)^k$ for some $b \in K$. Since p_2 is k-regular, its derivative $p_2'(x)$ is $(k-1)$-regular. Then $p_2'(x)$ has an infinitesimal root c by Theorem 4.18. Now, $Supp(p_2') \subseteq Supp(p_2)$, so the order type of $[Supp(p_2')]$

is at most α. By the induction hypothesis, $[Supp(c)]$ has order type at most α. Consider the polynomial $p_3(x) = p_2(c + x)$ with root $r_3 = r_2 - c$.

Claim 2: $[Supp(p_3)]$ has order type at most α.

Proof of Claim 2. First, observe that $Supp(p_3)$ and, hence, $[Supp(p_3)]$ are contained in $[Supp(c) \cup Supp(p_2)]$. So, it suffices to show that $[Supp(c) \cup Supp(p_2)]$ has order type at most α. Let $b \in [Supp(c) \cup Supp(p_2)]$. By Lemma 2.10 (2a), it suffices to show that $pred(b) \cap (Supp(c) \cup Supp(p_2))$ is less than α.

Since $[Supp(c)] \neq \{0\}$ has order type at most α, Lemma 2.10 (2b) gives that $pred(b) \cap Supp(c)$ has order type less than α. Similarly, since $[Supp(p_2)] \neq \{0\}$ has order type at most α, the order type of $pred(b) \cap Supp(p_2)$ is less than α. Then $pred(b) \cap (Supp(c) \cup Supp(p_2))$ has order type less than α by Lemma 2.10 (1). □

Claim 3: $[Supp(r_3)]$ has order type at most α.

Proof of Claim 3. Since $p_2(x)$ is k-regular and c is infinitesimal, $p_3(x) = p_2(c+x)$ is k-regular as well. We chose c such that $p_2'(c) = 0$ so the coefficient of x in p_3 is 0. For any positive Newton exponent μ of p_3, the carrier Δ_μ (relative to $p_3(x)$) does not contain 1. We now run the whole argument again with the polynomial $p_3(x)$ replacing $p(x)$ and root r_3 replacing r. Since $\Delta_{w(r_3)}$ (relative to the new p and the new p_1) does not contain 1, Case 2 cannot occur. In particular, the root a of polynomial $\hat{q}$ in this argument cannot have multiplicity k. If it did, then $\hat{q}(x)$ would equal $(x - a)^k b$ for some nonzero $a, b \in K$. Thus, $\hat{q}(x)$ and $q(x)$ would have a nonzero degree 1 term, contradicting the definition of $q(x)$. Hence, the argument demonstrates that $[Supp(r_3)]$ has order type at most α. □

Claim 4: $[Supp(r_2)]$ has order type at most α.

Proof of Claim 4. We have $Supp(r_2) \subseteq Supp(c) \cup Supp(r_3)$ since $r_2 = r_3 + c$. Let $b \in [Supp(r_3) \cup Supp(c)]$. As in Claim 2, we see that $pred(b) \cap Supp(c)$ and $pred(b) \cap Supp(r_3)$ have order type less than α. This suffices to prove the claim. □

Claim 4 implies that $[Supp(r)]$ has order type at most α, as in Case 1. □

5 Examples

In [6], the authors gave an example showing that for finite $n \geq 1$, the bounds given in Theorem 1.3 are best possible. Let K be an arbitrary field that is either algebraically closed of characteristic 0 or real closed (in [6], we took the field of real algebraic numbers, but the choice did not matter). Let G be $\mathbb{Q}$. There is a tc-independent sequence $(r_n)_{n<\omega}$ in $K((G))$ with associated canonical sequence $(R_n)_{n\in\omega}$ such that for each n, there is some $s_n \in R_{n+1}$ of length ω^{ω^n}.

In this section, we show that for all countable ordinals α, the bounds in Theorem 1.4 are sharp. Again we let K be an arbitrary field that is either algebraically closed of characteristic 0 or real closed, and we let $G = \mathbb{Q}$.

Theorem 5.1. *Let K be a field that is either algebraically closed of characteristic 0 or real closed, and let G be $\mathbb{Q}$. For each countable ordinal α, there is a tc-independent sequence $(r_\beta)_{\beta<\alpha}$ in $K((G))$ with associated canonical sequence $(R_\beta)_{\beta\leq\alpha}$ such that for each $\beta < \alpha$, there is some $s_\beta \in R_{\beta+1}$ of length equal to the upper bound (specified below).*

1. *For $1 \leq n < \omega$, the element $s_n \in R_{n+1}$ has length ω^{ω^n}.*

2. *For $\beta \geq \omega$, the element $s_\beta \in R_{\beta+1}$ has length $\omega^{\omega^{\beta+1}}$.*

Hence, for any limit ordinal β and any $\lambda < \omega^{\omega^\beta}$, the field R_β has an element of length λ.

We need some lemmas for the proof of Theorem 5.1.

Lemma 5.2. *Suppose A and B are sets of positive rationals, where A has order type α, additively indecomposable, and B has order type β. Suppose A has least upper bound $a^* \in \mathbb{R} - A$. Then $A + B$ has order type at least $\alpha \cdot \beta$.*

Proof. Let $b \in B$. All elements of $A + \{b\}$ are less than $a^* + b$. Suppose $b' > b$, and let $v \in A$ be such that $a^* - v < b' - b$. Then $a^* + b < v + b'$. We note that if A' is the "tail" of A consisting of elements greater than v, then A' has order type α, and $a^* + b$ is less than any element of $A' + \{b'\}$. Let the elements of B be b_i for $i < \beta$. For the lemma, we take $A_i \subseteq A$ such that

1. A_i has order type α,

2. $a^* + b_i$ is less than any element of $A_{i+1} + \{b_{i+1}\}$.

Then, $\bigcup_{i<\beta} A_{i+1} + \{b_{i+1}\}$ witnesses that $A + B$ has order type at least $\alpha \cdot \beta$. □

Corollary 5.3. *Let α be a countably infinite multiplicatively indecomposable ordinal. Suppose that A is a set of positive rationals of order type α with least upper bound $a^* \in \mathbb{R} - A$. Then $S_m(A)$ has order type at least α^m, and $[A]$ has order type α^ω.*

Proof. We apply Lemma 5.2 inductively. Since α is an infinite multiplicatively indecomposable ordinal, then α^m is an infinite additively indecomposable ordinal. By Lemma 2.5 and Fact 2.9, the order type of $S_m(A)$, and, hence, $[A]$, cannot be greater than α^ω. □

We need one more technical lemma.

Lemma 5.4. *Let K be a field that is either algebraically closed of characteristic 0 or real closed, and let $R \subseteq K((\mathbb{Q}))$ be a truncation closed and relatively algebraically closed subfield containing t. Suppose that $p < p'$ and $q < q'$ are all in $\mathbb{Q}^{>0}$. Given $s \in R$ such that $Supp(s)$ is contained in the rational interval (p, p') and the order type α of $Supp(s)$ is additively indecomposable, there is some $\tilde{s} \in R$ such that $Supp(\tilde{s})$ is contained in the rational interval (q, q') and $Supp(\tilde{s})$ also has order type α.*

Proof. The support of $st^{\frac{q'}{p'}}$ is contained in the rational interval $(\frac{pq'}{p'}, q')$. If $q < \frac{pq'}{p'}$, then we take $\tilde{s} = st^{\frac{q'}{p'}}$. Otherwise, we let $\tilde{s}$ equal the difference between $st^{\frac{q'}{p'}}$ and the truncation of $st^{\frac{q'}{p'}}$ having support in $(-\infty, q]$. Since the order type of $Supp(s)$ is additively indecomposable, $Supp(s)$ and $Supp(\tilde{s})$ have the same order type. □

Proof of Theorem 5.1. As mentioned above, Theorem 5.1 (1) is proved in §7 of [6]. We outline the proof for Theorem 5.1 (2). We construct a tc-independent sequence $(r_\beta)_{\beta<\alpha}$ in $K((\mathbb{Q}))$ (with associated canonical sequence $(R_\beta)_{\beta\leq\alpha}$) and we construct elements $(s_{\beta+1})_{\beta<\alpha}$ such that, for all $\beta \geq \omega$, $s_{\beta+1}$ is in $R_{\beta+1}$ and has length $\omega^{\omega^{\beta+1}}$.

We begin with the tc-independent sequence $(r_n)_{n<\omega}$, guaranteed by Theorem 5.1 (1), such that for each nonzero $n < \omega$, there is some $s_n \in R_{n+1}$ of length ω^{ω^n}. Given $\beta \geq \omega$ such that the statement is true for all infinite $\gamma \leq \beta$, we show that it holds for $\beta + 1$. We first construct an element $r_\beta \notin R_\beta$ of length ω^{ω^β} such that $Supp(r_\beta)$ is bounded in $\mathbb{Q}$. Let $(q_n)_{n\in\omega}$ be a strictly increasing bounded sequence of positive rationals. There are two cases to consider.

Case 1: First, suppose that β itself is a successor ordinal, i.e., $\beta = \nu + 1$. By induction, there is an element $s_\nu \in R_\beta$ of length $\omega^{\omega^\beta} = \omega^{\omega^{\nu+1}}$. Set $\gamma_n = \omega^{\omega^\nu \cdot n}$ for $n \in \omega$. Let s'_{γ_n} be the truncation of s_ν having length γ_n. Let $(p_n)_{n\in\omega}$ be a sequence of rationals such that $Supp(s'_{\gamma_{n+1}} - s'_{\gamma_n})$ is contained in the interval (p_n, p_{n+1}) for all $n \in \omega$. For all $n \in \omega$, let $r_{\beta,n+1} \in R_\beta$ be the result of applying Lemma 5.4 to $s'_{\gamma_{n+1}} - s'_{\gamma_n} \in R_\beta$ with pairs of rationals $p_n < p_{n+1}$ and $q_n < q_{n+1}$. (Let $r_{\beta,0}$ be the result of applying Lemma 5.4 to s'_{γ_0} with the pairs of rationals $0 < p_0$ and $0 < q_0$.) Since $s'_{\gamma_{n+1}} - s'_{\gamma_n}$ has length γ_{n+1} (as γ_{n+1} is additively indecomposable), $r_{\beta,n+1}$ has this length as well. Let $r_\beta = \sum_{n\in\omega} r_{\beta,n}$. By construction, $Supp(r_\beta)$ has order type $\bigcup_{n\in\omega} \gamma_n = \omega^{\omega^\beta}$ and is bounded by some $q^*_\beta \in \mathbb{Q}$. Note that all proper initial segments of r_β are in R_β, but $r_\beta \notin R_\beta$. If $r_\beta \in R_\beta$, then $r_\beta + t^{q^*_\beta}$, which has length $\omega^{\omega^\beta} + 1$, would be an element of R_β, contradicting Theorem 1.4.

Case 2: Second, suppose that β is a limit ordinal. Since β is countable, there is a strictly increasing sequence of infinite successor ordinals $(\beta_n)_{n\in\omega}$ such that $\beta = \bigcup_{n\in\omega} \beta_n$. (The argument for $\beta = \omega$ is similar.) By induction and the construction above, R_β contains the elements r_{β_n} so that $Supp(r_{\beta_n})$ has order type $\omega^{\omega^{\beta_n}}$ and is bounded in $\mathbb{Q}$. As above, using Lemma 5.4, we can obtain $r_{\beta,n}$ from r_{β_n} so that the support of $r_{\beta,n}$ is contained in the rational interval (q_n, q_{n+1}) and has length $\omega^{\omega^{\beta_n}}$. (Note that $\omega^{\omega^{\beta_n}}$ is additively indecomposable.) Now, we let $r_\beta = \sum_{n\in\omega} r_{\beta,n}$. As above, all proper initial segments of r_β are in R_β, but $r_\beta \notin R_\beta$. Moreover, $Supp(r_\beta)$ has order type $\bigcup_{n\in\omega} \omega^{\omega^{\beta_n}} = \omega^{\omega^\beta}$ and is bounded by some q^*_β.

In either case, let $s_\beta = \frac{1}{1-r_\beta}$. Note that $w(r_\beta) > 0$, since $Supp(r_\beta) \subset \mathbb{Q}^{>0}$. By Corollary 5.3, $1 + r_\beta + r_\beta^2 + r_\beta^3 + \ldots$ is an element of $K((G))$ having support

$[Supp(r_\beta)]$ of order type $(\omega^{\omega^\beta})^\omega$. Hence, by Taylor's Theorem,

$$s_\beta = 1 + r_\beta + r_\beta^2 + r_\beta^3 + \dots$$

and s_β has support of order type $\omega^{\omega^{\beta+1}}$, as desired. This completes the induction and the proof of Theorem 5.1. □

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