Real closed exponential fields

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Abstract

Ressayre considered real closed exponential fields and "exponential" integer parts; i.e., integer parts that respect the exponential function. In [23], he outlined a proof that every real closed exponential field has an exponential integer part. In the present paper, we give a detailed account of Ressayre's construction and then analyze the complexity. Ressayre's construction is canonical once we fix the real closed exponential field R, a residue field section, and a well ordering $\prec$ on R. The construction is clearly constructible over these objects. Each step looks effective, but there may be many steps. We produce an example of an exponential field R with a residue field section k and a well ordering $\prec$ on R such that $D^c(R)$ is low and k and $\prec$ are Δ^0_3, and Ressayre's construction cannot be completed in $L_{\omega_1^{CK}}$.

1 Introduction

Definition 1 (Real closed field). *A* real closed field *is an ordered field in which every non-negative element is a square, and every odd degree polynomial has a root.*

Tarski's celebrated elimination of quantifiers [27] shows that the axioms for real closed fields generate the complete theory of the ordered field of reals, so this theory is decidable.

Definition 2 (Integer part). *An* integer part *for an ordered field R is a discretely ordered subring Z such that for each $r \in R$, there exists some $z \in Z$ with $z \leq r < z + 1$.*

If R is Archimedean, then $\mathbb{Z}$ is the unique integer part. In general, the integer part for R is not unique. Shepherdson [26] showed that a discrete ordered ring Z is an integer part for some real closed field if and only if Z satisfies *Open Induction*, a weak fragment of first order Peano Arithmetic[1] (PA) with induction axioms just for open (i.e., quantifier-free) formulas. In [20], Mourgues and Ressayre proved the following.

[1] First order Peano Arithmetic has the axioms for discretely ordered commutative rings with unity, plus an induction axiom for each elementary first order formula $\varphi(\overline{u}, x)$ saying that for all tuples of parameters $\overline{u}$, if the set of non-negative elements satisfying $\varphi(\overline{u}, x)$ contains 0 and is closed under successor, then it includes all non-negative elements.

Theorem A. *Every real closed field has an integer part.*

By a deep result of Wilkie [28], the theory of the ordered field of reals with the exponential function e^x is model complete. Macintyre and Wilkie [19] showed that this theory is decidable assuming that Schanuel's Conjecture holds for the reals. Ressayre isolated a set of axioms for real closed fields with an exponential function, such that each model has an integer part "respecting" the exponential function. In this setting, 2^x is a more natural exponential function to use than e^x (although 2^x and e^x are interdefinable).

Definition 3 (Real closed exponential field)**.** *A* real closed exponential field *is a real closed field R with a function $x \mapsto 2^x$ satisfying the following axioms:*

1. $2^{x+y} = 2^x 2^y$*,*

2. $x < y$ *implies* $2^x < 2^y$*,*

3. for all $x > 0$*, there exists some* y *such that* $2^y = x$*; i.e.,* $\log(x)$ *is defined.*

4. $2^1 = 2$*,*

5. for $n \geq 1$*, for all* $x \in R$*,* $x > n^2$ *implies* $2^x > x^n$*.*

We say what it means for an integer part to respect the exponential function.

Definition 4 (Exponential integer part)**.** *Let R be a real closed exponential field. An* exponential integer part *is an integer part Z such that for all positive $z \in Z$, we have $2^z \in Z$.*

The result below is the natural analogue of Theorem A for exponential real closed fields. Ressayre outlined the proof of Theorem B below in an extended abstract [23].

Theorem B (Ressayre)**.** *If R is a real closed exponential field, then R has an exponential integer part.*

There is further work using the ideas from [20] and [23]. In [6] the authors show that every real closed field can be embedded in an "initial" substructure of the surreal numbers. In [15] and [8] embeddings of ordered fields in fields of power series are further analysed, giving a valuation theoretic interpretation of the results in [20].

In this paper, we first revisit Ressayre's extended abstract, filling in the details. We then analyze the complexity of the construction. Typically, in computable structure theory, we locate objects at various finite levels in the "arithmetical" hierarchy. Occasionally, we pass to some infinite level in the "hyperarithmetical" hierarchy. It turns out that Ressayre's construction cannot be located in the hyperarithmetical hierarchy. We say just a little about these hierarchies here. For more information, see [24], [1], or [13].

We say that Y is *computable relative* to X for $X, Y \in \mathcal{P}(\omega)$, and we write $Y \leq_T X$, if there is an interactive program that computes the characteristic function for Y given answers to questions about membership in X. The set X is referred to as the oracle in this computation. A set X is *computable* if $X \leq_T \emptyset$, i.e., no oracle is needed to compute X. We have an effective list of the interactive programs. We write φ_e^X for the partial function computed using program number e with oracle X. We write $\varphi_e^X(n) \downarrow$ if interactive program e eventually halts, given oracle X and input n. We let φ_e denote $\varphi_e^{\emptyset}$.

The effective lists of programs immediately suggest a way to build more complicated sets. For an arbitrary set $X \subset \omega$, the *jump* is $X' = \{e : \varphi_e^X(e) \downarrow\}$. This set is computably enumerable (c.e.) but not computable relative to X. A set X is *low* if $X' \leq_T \emptyset'$. We can iterate the jump to get $X^{(n)}$, for $n \in \omega$. We let $X^{(0)} = X$, and $X^{(n+1)} = (X^{(n)})'$. We may continue the iteration process through the "computable" ordinals.[2] Let $X^{(\omega)} = \{< n, x >: x \in X^{(n)}\}$, and $X^{(\alpha+1)} = (X^{(\alpha)})'$. In general, for limit α, the set $X^{(\alpha)}$ represents $\{< \beta, x >: x \in X^{(\beta)}\}$. (To make this last definition precise, we code the ordinal β by a natural number, using Kleene's system of ordinal notation.)

Stephen Kleene and Andrzej Mostowski independently defined what is now called the *arithmetical hierarchy.* Martin Davis and Andrzei Mostowski independently defined what is now called the *hyperarithmetical hierarchy*, extending the arithmetical hierarchy through the "computable" ordinals. We give the definition in a uniform way.

- **Arithmetical Hierarchy**. For $1 \leq n < \omega$, a set is Σ^0_n if it is computably enumerable relative to $\emptyset^{(n-1)}$. A set is Π^0_n if the complement is Σ^0_n, and it is Δ^0_n if it is both Σ^0_n and Π^0_n. A set is *arithmetical* if it is Δ^0_n for some $n \in \omega$.
- **Hyperarithmetical hierarchy**. For some computable $\alpha \geq \omega$, a set is Σ^0_α if it is c.e. relative to $\emptyset^{(\alpha)}$. A set is Π^0_α if the complement is Σ^0_α, and it is Δ^0_α if it is both Σ^0_α and Π^0_α. A set is *hyperarithmetical* if it is Δ^0_α for some computable ordinal α.

Ressayre's construction is canonical with respect to a given real closed field, a residue field section, and a well ordering of the elements of the real closed field. We produce an example of a real closed exponential field R with a residue field section k and a well ordering $\prec$ of R, all arithmetical, such that when we apply Ressayre's construction, it is not completed in $L_{\omega_1^{CK}}$, where ω_1^{CK} ("Church-Kleene ω_1") is the first non-computable ordinal.[3] We note that the subsets of ω in $L_{\omega_1^{CK}}$ are exactly the hyperarithmetical ones.

Here is our main new result. In it, the notation $D^c(R)$ stands for the *complete* (or *elementary*) diagram of R.

Theorem C (Main new result)**.** *There is a countable real closed exponential field R with a residue field section k and a well ordering $<$ (of order type $\omega+\omega$) such that $D^c(R)$ is low, k and $<$ are Δ^0_3, and Ressayre's construction applied to R, k, and $<$ is not completed in $L_{\omega_1^{CK}}$.*

[2] An ordinal α is *computable* if there is a linear ordering $\mathcal{L}$ of order type α such that its atomic diagram $D^a(\mathcal{L})$ is computable.

[3] The *constructible hierarchy* is defined as follows: $L_0 = \emptyset$, $L_{\alpha+1}$ is the family of sets definable (with parameters) in L_α, and for limit α, we have $L_\alpha = \cup_{\beta<\alpha} L_\beta$.

We state one more result below. The result is an easy consequence of older work of Ressayre. To state the result, we need just a little more background. An "admissible" set is a transitive set satisfying the axioms of Kripke-Platek set theory. This is a weak version of ZF, in which we drop the power set axiom and restrict the comprehension and separation axioms to formulas with bounded quantifiers. For an admissible set A, a *fattening* is an extension B to a larger admissible set with no new ordinals. We note that $L_{\omega_1^{CK}}$ is the least admissible set (containing ω). A fattening of $L_{\omega_1^{CK}}$ may have non-constructible subsets of ω, but the ordinals are just the computable ones.

Theorem D. *For any countable real closed exponential field R, there is an exponential integer part Z such that (R, Z) lives in a fattening of the least admissible set over (i.e., containing) R.*

In §2, we give some algebraic preliminaries. In §3, we briefly outline Mourgues and Ressayre's construction of an integer part for a real closed field. In §4, we give Ressayre's construction of an exponential integer part for a real closed exponential field. In §5, we describe some background needed for Theorems C and D, and in §6, we prove these results.

2 Algebraic preliminaries

In this section, we give some algebraic background for the construction of Mourgues and Ressayre. We recall the natural valuation on an ordered field R.

Definition 5 (Archimedean equivalence)**.** *For $x, y \in R^\times := R - \{0\}$, $x \sim y$ iff there exists $n \in \mathbb{N}$ such that $n|x| \geq |y|$ and $n|y| \geq |x|$, where $|x| := \max\{x, -x\}$. We denote the equivalence class of $x \in R$ by $v(x)$.*

Definition 6 (Value group)**.** *The* value group *of R is the set of Archimedean equivalence classes $v(R^\times) = \{v(x) \mid x \in R^\times\}$ with multiplication on $v(R^\times)$ defined to be $v(x)v(y) := v(xy)$. We endow $v(R^\times)$ with the order*

$$v(x) < v(y) \text{ if } (\forall n \in \mathbb{N})[\,n|x| < |y|\,].$$

By convention, we let $v(0) < v(R^\times)$.

Under the given operation and ordering, $v(R^\times)$ is an ordered Abelian group with identity $v(1)$. Moreover, the map $x \mapsto v(x)$ is a *valuation*, i.e. it satisfies the axioms $v(xy) = v(x)v(y)$ and $v(x+y) \leq \max\{v(x), v(y)\}$.

If R is a real closed field, then the value group $v(R^\times)$ is divisible [7, Theorem 4.3.7]. An Abelian group $(G, \cdot)$ is *divisible* if for all $g \in G$ and $0 \neq n \in \mathbb{N}$, $g^{\frac{1}{n}} \in G$. Note that a divisible Abelian group $(G, \cdot)$ is a $\mathbb{Q}$-vector space when scalar multiplication by $q \in \mathbb{Q}$ is defined to be g^q. This observation motivates the following definition.

Definition 7 (Generating set)**.** *Let $(G, \cdot)$ is a divisible Abelian group. We say that B is a* generating set *if each element of G can be expressed as a finite product of rational powers of elements of B. We denote the Abelian group generated by a set $B \subseteq R$ by $\langle B\rangle_{\mathbb{Q}}$.*

Definition 8 (Value group section)**.** *A* value group section *for R is the image of an embedding of ordered groups $t : v(R^\times) \hookrightarrow R^{>0}$ such that $v(t(g)) = g$ for all $g \in v(R^\times)$.*

If R is a real closed field, there are subgroups of $(R^{>0}, \cdot)$ that are value group sections for R (see [10, Theorem 8]). Note that we use the term "value group section" to refer to the image of the described embedding, not the embedding itself. In [14], it is shown that for a countable real closed field R, there is a value group section G that is $\Delta_2^0(R)$. Moreover, this is sharp, in the sense that there is a computable real closed field R such that the halting set $\emptyset'$ is computable relative to every value group section.

Definition 9 (Valuation ring)**.** *The* valuation ring *is the ordered ring*

$$\mathcal{O}_v := \{x \in R : v(x) \leq 1\}$$

of finite elements.

The valuation ring has a unique maximal ideal

$$\mathcal{M}_v := \{x \in R : v(x) < 1\}$$

of infinitesimal elements.

Definition 10 (Residue field)**.** *The* residue field *is the quotient $\mathcal{O}_v/_{\mathcal{M}_v}$.*

The residue field k is an ordered field under the order induced by R. It is Archimedean, so it is isomorphic to a subfield of $\mathbb{R}$.

Definition 11 (Residue field section)**.** *A* residue field section *is the image of an embedding of ordered fields $\iota : k \hookrightarrow R$ such that $\iota(c) + \mathcal{M}_v = c$ for all $c \in k$.*

If R is a real closed field, then k is real closed [7, Theorem 4.3.7] and residue field sections exist [10, Theorem 8]. To construct a residue field section, we look for a maximal real closed Archimedean subfield. In [14], the second and fourth authors proved the following result on the complexity of residue field sections.

Proposition 2.1. *For a countable real closed field R, there is a residue field section that is $\Pi_2^0(R)$.*

Proposition 2.1 is sharp in the following sense.

Proposition 2.2. *There is a computable real closed field with no Σ_2^0 residue field section.*

Definition 12 ($k((G))$)**.** *Let k be an Archimedean ordered field and let G be an ordered Abelian group.*

1. *The field $k((G))$ of generalized series is the set of formal sums $s = \sum_{g \in G} a_g g$, where $a_g \in k$ and Supp$(s) := \{g \in G : s_g \neq 0\}$ is an anti-wellordered subset of G.*
2. *The* length *of s is the order type of Supp(s) under the reverse ordering. Later, we may write $s = \sum_{i<\alpha} a_i g_i$, where $g_i \in G$ with $g_i > g_j$ for $i < j < \alpha$, and $a_i \in k^\times$. Under this notation, the length of s is α.*

3. *For $s = \sum_{g\in S} a_g g$ and $t = \sum_{g\in T} b_g g$ in $k((G))$ where $Supp(s) \subseteq S$ and $Supp(t) \subseteq T$, the* sum $s+t$ *and the* product $s \cdot t$ *are defined as for ordinary power series.*
   (a) *In $s+t$, the coefficient of g is $a_g + b_g$.*
   (b) *In $s \cdot t$, the coefficient of g is the sum of the products $a_{g'} b_{g''}$, where $g = g' \cdot g''$.*
4. *$k((G))$ is ordered anti-lexicographically by setting $s > 0$ if $a_g > 0$ where $g =: max(Supp(s))$.*

For a proof that $k((G))$ is a totally ordered field, see [9, Chapter VIII, Theorem 10]. If k is real closed and $(G, \cdot)$ is an ordered divisible Abelian group, then $k((G))$ is real closed by [7, Theorem 4.3.7]. The field $k((G))$ carries a canonical valuation $v : k((G))^\times \to G$, defined by $s \mapsto \max(Supp(s))$, with value group G. Given a subset $X \subseteq G$, we set

$$k((X)) = \{s \in k((G)) \mid Supp(s) \subseteq X\}.$$

We let $G^{\leq 1} = \{g \in G \mid g \leq 1\}$, and similarly define $G^{<1}$ and $G^{>1}$. The valuation ring is the ring of finite elements $k((G^{\leq 1}))$, its valuation ideal is the ideal of infinitesimals $k((G^{<1}))$, and the residue field is k. The canonical additive complement to the valuation ring is $k((G^{>1}))$, the ring of purely infinite series. The group of positive units of $k((G^{\leq 1}))$ is denoted by $\mathcal{U}_v^{>0}$, and consists of series s in the valuation ring with coefficient $a_g > 0$ for $g = 1$. In this setting the following decompositions of the additive and multiplicative groups of $k((G))$ will be useful

$$(k((G)), +) = k((G^{\leq 1})) \oplus k((G^{>1})) \text{ and } (k((G))^{>0}, \cdot) = \mathcal{U}_v^{>0} \cdot G.$$

2.1 Truncation-closed embeddings

Definition 13 (Truncation-closed subfield)**.** *Let F be a subfield of $k((G))$. We say that F is truncation closed if whenever $s = \sum_{g\in G} a_g g \in F$ and $h \in G$, the restriction $s_{<h} = \sum_{g<h} a_g g$ also belongs to F. We call any such $s_{<h}$ a* truncation *of s.*

Definition 14. *Let R be a real closed field with value group section G and residue field section k. Let δ be order preserving embedding from R into $k((G))$. We say that δ is an embedding over k and G if its restriction to k and G is the identity map.*

In [20, Lemma 3.2], Mourgues and Ressayre observed the following.

Proposition 2.3 (Mourgues-Ressayre)**.** *If F is a truncation closed subfield of $k((G))$ and $Z_F = \{t + z \mid t \in F \cap k((G^{>1})) \ \&\ z \in \mathbb{Z}\}$, then Z_F is an integer part for F.*

Proof. If $s \in F$, we have $s = t + t'$, where $t \in k((G^{>1}))$ and $t' \in k((G^{\leq 1}))$. Take $z \in \mathbb{Z}$ such that $z \leq t' < z+1$. Then $t + z \leq s < t + z + 1$.

□

In [20, Corollary 4.2], they proved the following restatement of Theorem A.

Theorem A$'$ (Mourgues and Ressayre)**.** *Let R be a real closed field with value group section G and residue field section k. Then there is an order preserving embedding δ from R onto a truncation closed subfield F of $k((G))$ (over k and G). Thus $\delta^{-1}(Z_F)$ is an integer part for R.*

We refer to δ as a "development function" δ.

2.2 Exponential integer parts

In [23] Ressayre imposed further conditions on the value group section G and the development function δ which ensure that the truncation integer part is also closed under exponentiation. The following is a rephrasing of Theorem B and of [23, Theorem 4].

Theorem B$'$. *Let $(R, 2^x)$ be an exponential real closed field. Fix a residue field section $k \subseteq R$. Then there is a value group section $G \subseteq R^{>0}$ and an order preserving embedding from R onto a truncation closed subfield F of $k((G))$ (over k and G) such that*

$$\delta(\log(G)) = \delta(R) \cap k((G^{>1})). \tag{1}$$

Condition (1) is equivalent to the following:

- for all $g \in G$, we have $\delta(\log(g)) \in k((G^{>1})$.
- if $r \in R$ satisfies $\delta(r) \in k((G^{>1}))$, then $2^r \in G$.

If G and δ satisfy Condition (1), and F is the image of R under δ, then the exponential function 2^x on R induces an exponential function on F. We let $2^y = \delta(2^x)$, where $y = \delta(x)$. We have the natural integer part Z_F, defined as in the construction of Mourgues and Ressayre. The simple lemma below, which appears in [3, Proposition 5.2], says that this is an exponential integer part for F. It follows that $\delta^{-1}(Z_F)$ is an exponential integer part for R.

Lemma 2.4. *Let G and δ satisfy Condition (1), and let F be the image of R under δ. Then Z_F is an exponential integer part of F with respect to the induced exponential function.*

Proof. Let $z \in Z_F$ and $z > 0$, then $z = a + y$ where $y \in F \cap k((G^{>1}))$ and $a \in \mathbb{Z}$. We compute $2^z = 2^a 2^y$. If $y = 0$ then $a > 0$, so $2^z \in \mathbb{N} \subseteq Z_F$. If $y \neq 0$ then $y > 0$, and $2^y > 1$. We now show that $2^y \in G$. By (1) $y = \delta(\log(g))$ for some $g \in G$. Then $2^y = \delta(2^{\log(g)}) = \delta(g) = g$, as required. Therefore, $2^y \in G^{>1}$, and so $2^z = 2^a 2^y$ belongs to $k((G^{>1}))$, and also to $F = \delta(R)$. So, $2^z \in F \cap k((G^{>1})) \subseteq Z_F$. $\square$

We shall return to the proof of Theorem B$'$.

3 Development Triples

Mourgues and Ressayre proved Theorem A$'$ by showing how to extend a partial embedding ϕ from a subfield A of R onto a truncation closed subfield F of $k((G))$ to a larger domain while preserving truncation closure.

Definition 15 (Development triple). *Suppose R is a real closed field, with residue field section k. We say that (A, H, ϕ) is a* development triple *with respect to R and k if*

1. *A is a real closed subfield of R containing k,*
2. *$H \subseteq A^{>0}$ is a value group section for A, and*
3. *ϕ is an embedding from A onto a truncation closed subfield of $k((H))$ (over k and H).*

Notation. We write $(A', H', \phi') \supseteq (A, H, \phi)$, if $A' \supseteq A$, $H' \supseteq H$, and $\phi' \supseteq \phi$. We also write $RC(X)$ to denote the real closure of X for any $X \subseteq R$.

Given a development triple (A, H, ϕ) and an element $r \in R - A$, we want a development triple $(A', H', \phi') \supset (A, H, \phi)$ with $r \in A'$. We use the following definitions to describe $\phi'(r)$.

Definition 16. *Let α be an ordinal. The* development of $r \in R$ over (A, H, ϕ) of length α *(if it exists) is an element $t_\alpha \in k((H))$ satisfying:*

- *$t_0 = 0$ if $\alpha = 0$, and otherwise,*
- *$t_\alpha = \sum_{i<\alpha} a_i g_i$ where*

$$(\forall \beta < \alpha)(\exists \hat{r}_\beta \in A)[\phi(\hat{r}_\beta) = \sum_{i<\beta} a_i g_i \ \& \ g_\beta = v(r - \hat{r}_\beta) \in G] \qquad (2)$$

As an example, for any $r \in R$, $\hat{r}_0 = 0$, and assuming that r has valuation in H, we have $g_0 = v(r)$, a_0 is the unique element of k such that $v(r - a_0 g_0) < g_0$, and $\hat{r}_1 = a_0 b_0$ in Definition 16 above.

It is straightforward to prove the next lemma.

Lemma 3.1. *Let (A, H, ϕ) be a development triple, $r \in R$, and γ an ordinal for which t_γ exists. Then,*

1. *t_γ is unique and, for all $\beta \leq \gamma$, $t_\beta = (t_\gamma)_{<\beta}$.*
2. *There is a development t_α of r over (A, H, ϕ) of maximum length α.*

Lemma 3.1 allows us to make the following definition.

Definition 17. *The* maximum development of r over (A, H, ϕ) *is the unique development of r over (A, H, ϕ) of maximum length α.*

We now restate the key lemma of Theorem A by Mourgues and Ressayre [20] in the language of development triples, since we will use development triples with additional properties in the exponential case.

Lemma 3.2 (Mourgues-Ressayre). *Suppose (A, H, ϕ) is a development triple with respect to a real closed field R and $r \in R - A$. There is a development triple $(A', H', \phi') \supseteq (A, H, \phi)$ such that $r \in A'$. Moreover, if the maximum development of r over (A, H, ϕ) is $t_\alpha \in k((H))$, then $\phi'(r)_{<\alpha} = t_\alpha$.*

We say just a little about the proof. Let $t = t_\alpha$ be the development of r over (A, H, ϕ). There are two cases to consider.

Case 1: Suppose $t_\alpha \in \phi(A)$, that is, we have $\hat{r}_\alpha \in A$ with $\phi(\hat{r}_\alpha) = t_\alpha$, and $r - r_\alpha$ has no valuation in H. In this case, we let $g = |r - r_\alpha|$ and let $H' = \langle H \cup \{g\}\rangle_{\mathbb{Q}}$. We let $A' = RC(A(g))$. We note that r and g are inter-algebraic over A; in particular, $r = \hat{r}_\alpha + \epsilon g$ for $\epsilon = \pm 1$, so $r \in A'$. We let $\phi'(g) = g$. There is a unique extension of ϕ' onto the domain A'. In the work of [10], this is the "value transcendental" case.

Case 2: Suppose $t_\alpha \notin \phi(A)$. In this case, we let $H' = H$, and we set $\phi'(r) = t_\alpha$. We let $A' = RC(A(r))$. Again, there is a unique extension of ϕ' onto the domain A'. In the work of [10], this is the "immediate transcendental" case.

To see that the restriction of ϕ' to A and the one new element g or r is an elementary embedding, it is enough to show that for all $x \in A$, $x < g$ (or $x < r$) if and only if $\phi(x) < g$ (or $\phi(x) < t$). This can be found in [5], p. 191, Lemma 3.3. Then ϕ' is an isomorphism from A' onto $RC(\phi(A)(t))$. In [20] it is shown that $\phi'(A')$ is truncation closed.

Lemma 3.3 (Mourgues-Ressayre)**.** *Let F be a truncation closed subfield of $k(G)$, and let $t \in k((G)) - F$, where all proper truncations of t are in F. Then the real closure of $F(t)$ is also a truncation closed subfield of $k((G))$.*

Thus, in both cases, we have defined a development triple (A', H', ϕ') extending (A, H, ϕ) with $r \in A' = RC(A(r))$. This is all we shall say about the proof of Lemma 3.2.

We will use the next notion extensively in §4 on the exponential case.

Definition 18. *Let (A', H', ϕ') and (A, H, ϕ) be development triples.*

1. *The triple (A', H', ϕ') is a* value group preserving extension *of (A, H, ϕ) if (A', H', ϕ') extends (A, H, ϕ) and $H' = H$.*
2. *A triple (A, H, ϕ) is* maximal *if (A, H, ϕ) admits no proper value group preserving extension.*

Note that $(k, \{1\}, id)$ is a maximal development triple as is any triple of the form (R, G, δ) with respect to a real closed field R.

Lemma 3.4. *Given a real closed field R, a residue field section k, and a well ordering $\prec$ of R, there is a canonical development triple (R, G, δ) with respect to R, k, and $\prec$.*

Proof. We obtain (R, G, δ) from a chain of development triples. We let (R_0, G_0, δ_0) be $(k, \{1\}, id)$. Given (R_i, G_i, δ_i), if $R - R_i \neq \emptyset$, let r be the least element of $R - R_i$ under $\prec$. By Lemma 3.2, there is an extension $(R_{i+1}, G_{i+1}, \delta_{i+1})$ of (R_i, G_i, δ_i) with $r \in R_{i+1}$. For limit α, we obtain $(R_\alpha, G_\alpha, \delta_\alpha)$ by taking the unions of the components R_i, G_i, and δ_i, for $i < \alpha$. For some α bounded by the order type of $\prec$, we have $R_\alpha = R$. □

In the same way, we can prove the following.

Lemma 3.5. *Let (A, H, ϕ) be a development triple with respect to R and k. Given a well ordering $\prec$ of R, there is a canonical maximal development triple (A', H, ϕ') extending (A, H, ϕ).*

4 Exponential integer parts

We saw in §2.2 that it suffices to prove Theorem B′ to demonstrate the existence of exponential integer parts for real closed exponential fields. We now define a special kind of development triple, which we then use to prove Theorem B′.

Definition 19 (Weak dyadic development triple)**.** *Let R be a real closed exponential field, and let k be a residue field section. Let (A, H, ϕ) be a development triple with respect to R and k. Then (A, H, ϕ) is a* weak dyadic triple *with respect to R and k if*

$$\phi(\log H) = \phi(A) \cap k((H^{>1})). \tag{3}$$

Equivalently, (A, H, ϕ) is weak dyadic if

1. *for all $r \in H$, $\log(r) \in A$ and $\phi(\log r) \in k((H^{>1}))$, and*
2. *for all $r \in A$, if $\phi(r) \in k((H^{>1}))$, then $2^r \in H$.*

Definition 20 (Dyadic development triple)**.** *Let R be a real closed exponential field, and let k be a residue field section. The triple (A, H, ϕ) is a* dyadic development triple *for R and k if it is a weak dyadic triple for R and k that is also maximal.*

Theorem B′ is equivalent to showing that every real closed exponential field R with a residue field section k has a dyadic triple (R, G, δ) with respect to R and k.

4.1 Extending dyadic triples

Proposition 4.1 below allows us to produce the desired dyadic triple (R, G, δ) needed for Theorem B′ as a union of a chain of dyadic triples.

Proposition 4.1. *Suppose (A, H, ϕ) is a dyadic triple with respect to a real closed exponential field R and a residue field section k, and $y \in R - A$. Then there is a dyadic triple $(A', H', \phi') \supseteq (A, H, \phi)$ such that $y \in A'$.*

Proof. Since (A, H, ϕ) is maximal, we may suppose that $v(y) \notin H$. Otherwise, we could replace y by $y - y_\beta$, where $\phi(y_\beta)$ is the maximum development of y over (A, H, ϕ). We may further suppose that $y > 0$, since otherwise we could replace y by $-y$. Finally, we may suppose that $v(y) > 1$, since otherwise we could replace y by y^{-1}. We will obtain the required dyadic triple (A', H', ϕ') as the union of a chain of development triples (B_i, H_i, ϕ_i) satisfying the following properties.

1. $H_0 \supseteq H$ is a value group section for $RC(A \cup \{\log^i(y) \mid i \in \omega\})$ where $\log^0(y) = y$ and $\log^{i+1}(y) = \log(\log^i(y))$ for all $i \in \omega$.
2. If $r \in H_i$, then $\log(r) \in B_i$ and $\phi_i(\log(r)) \in k((H_i^{>1}))$.

3. If $r \in B_i$ and $\phi_i(r) \in k((H_i^{>1}))$, then $2^r \in H_{i+1}$.
4. (B_i, H_i, ϕ_i) is a maximal development triple.

We focus first on building the group H_0, starting with the dyadic triple (A, H, ϕ), and $y \in R - A$, where $y > 0$, $v(y) > 1$, and $v(y) \notin H$. The group H_0 is an extension of H so that any maximal triple (A', H_0, ϕ') extending (A, H, ϕ) has $\log^i(y) \in A'$ for all $i \in \omega$. In the case where $(A, H, \phi) = (k, \{1\}, id)$, H_0 will be $\langle \{\log^i(y)\}_{i\in\omega} \rangle_{\mathbb{Q}}$. In general, H_0 will be generated by the elements of H and a specially chosen sequence $(y_i)_{i\in\omega}$.

Definition of $(y_i)_{i\in\omega}$. We define a sequence $(y_i)_{i\in\omega}$ inductively so that:

$$y_i > 0 \ \& \ v(y_i) > 1 \ \& \ v(y_i) \notin H. \tag{4}$$

We let $y_0 = y$. Given y_i satisfying (4), we let p_{i+1} be the development of $\log(y_i)$ over (A, H, ϕ). We claim that ϕ assigns p_{i+1} to some r'_{i+1} in A. Assuming this, we let $y_{i+1} = |\log(y_i) - r'_{i+1}|$. Lemma 4.2 justifies the definition and claim.

Lemma 4.2 (Inductive Lemma)**.** *Suppose (A, H, ϕ) is a dyadic triple. Let $y_i \in R - A$, where y_i satisfies* (4)*. Let p_{i+1} be the development of* $\log(y_i)$ *over (A, H, ϕ). Then*

1. $(\exists r'_{i+1} \in A)[\phi(r'_{i+1}) = p_{i+1}]$,
2. $y_{i+1} := |\log(y_i) - r'_{i+1}|$ *satisfies* (4)*, and*
3. $p_{i+1} \in k((H^{>1}))$.

Proof. Suppose that y_i satisfies (4). Let p_{i+1} be the development of $\log(y_i)$ over (A, H, ϕ). Since (A, H, ϕ) is maximal, there exists some $r'_{i+1} \in A$ such that $\phi(r'_{i+1}) = p_{i+1}$. By the definition of (maximum) development, we have that $v(y_{i+1}) \notin H$, and, in particular, $v(y_{i+1}) \neq 1$. Suppose (expecting a contradiction) that $v(y_{i+1}) < 1$ or $y_{i+1} = 0$. We have $\log(y_i) = r'_{i+1} + \pm y_{i+1}$, and $r'_{i+1} = s + s'$, where $\phi(s) \in k((H^{>1}))$ is the truncation of $\phi(r'_{i+1})$ so that $\phi(s') \in k((H^{\leq 1}))$. So, $y_i = 2^s 2^{s'} 2^{\pm y_{i+1}}$. Since $v(s') \leq 1$, we have $2^{s'}$ equals some c with $v(c) = 1$. If $v(y_{i+1}) < 1$ or $y_{i+1} = 0$, then $2^{\pm y_{i+1}} = (1 + d)$, where d is 0 or $v(d) < 1$. Since (A, H, ϕ) is weak dyadic and $\phi(s) \in k((H^{>1}))$, we have $2^s \in H$. Then, $v(y_i) = 2^s$, contradicting our assumption that $v(y_i) \notin H$. So, $y_{i+1} \neq 0$ and $v(y_{i+1}) > 1$. Since $v(y_{i+1}) < v(g)$ for all $g \in Supp(\phi(r'_{i+1}))$, we see that $\phi(r'_{i+1}) = p_{i+1} \in k((H^{>1}))$.

□

Lemma 4.3. *For all $i, n \in \omega$, $(y_{i+1})^n < y_i$. Hence, $v(y_i) \neq v(y_j)$ for $i \neq j$.*

Proof. From the definition of y_{i+1}, we see that $y_{i+1} < \log(y_i)$, so $y_{i+1}^n < \log(y_i)^n$. Since $v(y_i) > 1$, $\log(y_i)^n < 2^{\log(y_i)} = y_i$ by the growth axioms for real closed exponential fields (Axiom 5 in Definition 3).

□

Let $H_{0,n} = \langle H \cup \{y_i \mid i < n\} \rangle_{\mathbb{Q}}$. Let $H_0 = \cup_{n\in\omega} H_{0,n}$.

Lemma 4.4. *For each n, $v(y_n) \notin H_{0,n}$. Hence, $H_0 \supset H$ is a value group section for $RC(A \cup H_0)$.*

Proof. The statement is clear for $n = 0$. Assuming that it holds for n, we prove it for $n+1$. We assume (expecting a contradiction) that $v(y_{n+1}) \in H_{0,n}$, i.e., $y_{n+1} = cgy_0^{q_0} \cdot \ldots \cdot y_n^{q_n}$, where $c \in R$, $v(c) = 1$, $g \in G$, and $q_i \in \mathbb{Q}$. Taking logs, we obtain

$$\log(y_{n+1}) = \log(c) + \log(g) + q_0 \log(y_0) + \ldots + q_n \log(y_n). \tag{5}$$

Recall that, by definition, $\log(y_i) = r'_{i+1} + \epsilon_{i+1} y_{i+1}$, where $\phi(r'_{i+1})$ is the development of $\log(y_i)$ over (A, H, ϕ) and $\epsilon_{i+1} = \pm 1$. Then, by substitution and rearranging terms, we have that $\epsilon_{n+2} y_{n+2}$ equals

$$\log(c) + [\log(g) + q_0 r'_1 + \ldots + q_n r'_{n+1} - r'_{n+2}] + [q_0 \epsilon_1 y_1 + \ldots + q_n \epsilon_{n+1} y_{n+1}]$$

We have $v(\log(c)) = 1$, $v(\log(g) + q_0 r'_1 + \ldots + q_n r'_{n+1} - r'_{n+2}) \in H^{>1}$, and $v(q_0 \epsilon_1 y_1 + \ldots + q_n \epsilon_{n+1} y_{n+1}) = v(y_1)$ by Lemmas 4.2 and 4.3. Thus, $v(y_{n+2})$ is either in H or equals $v(y_1)$, contradicting either Lemma 4.2 or Lemma 4.3. □

Lemma 4.5. *If $h \in H_0$, then the development of* $\log(h)$ *over (A, H, ϕ) is in $k((H_0^{>1}))$.*

Proof. Let $h \in H_0$ with $h = g \prod_{i=0}^{n} y_{l_i}^{q^i}$. So,

$$\log(h) = \log(g) + \sum_{i=0}^{n} q_i (r'_{i+1} + \pm y_{i+1}) \tag{6}$$

where $q_i \in \mathbb{Q}$. The developments of $\log(g)$ and r'_{i+1} are in $k((H^{>1}))$ since (A, H, ϕ) is weak dyadic and by construction. Since $v(y_{i+1}) > 1$, $\log(h)$ has a development in $k((H_0^{>1}))$. □

Defining the sequence (B_j, H_j, ϕ_j). We have described the group H_0. By Lemma 3.2 and Lemma 3.5, we obtain B_0 and ϕ_0 such that (B_0, H_0, ϕ_0) is maximal and extends (A, H, ϕ). The rest of the chain (B_i, H_i, ϕ_i) is built on this foundation so that the union of the chain is dyadic. We define

$$H_1 = \langle H_0 \cup \{2^r \mid r \in B_0 \ \& \ \ \phi_0(r) \in k((H_0^{>1}))\}\rangle_{\mathbb{Q}}.$$

As done for H_0, we ensure that for all $h \in H_1$, $\phi_0(\log(h)) \in k((H_0^{>1}))$, and that H_1 is a value group section for $RC(B_0 \cup H_1)$. We then apply Lemma 3.2 and Lemma 3.5, to get B_1 and ϕ_1 such that (B_1, H_1, ϕ_1) extends (B_0, H_0, ϕ_0) and is maximal (for H_1).

In general, given $(B_i, H_i, \phi_i)_{i<\alpha}$ such that (B_i, H_i, ϕ_i) is maximal and $\phi_i(h) \in k((H_i^{>1}))$ for all $h \in H_i$, we let

$$H_{j+1} = \langle H_j \cup \{2^r \mid r \in B_j \ \& \ \ \phi_j(r) \in k((H_j^{>1}))\}\rangle_{\mathbb{Q}}.$$

We must check that for all $h \in H_{j+1}$, $\phi_j(\log(h)) \in k((H_j^{>1}))$ and that H_{j+1} is a value group section for $RC(B_i \cup H_\alpha)$. We apply Lemma 3.2 and

Lemma 3.5 to obtain B_{j+1} and ϕ_{j+1} such that $(B_{j+1}, H_{j+1}, \phi_{j+1})$ extends (B_j, H_j, ϕ_j) and is maximal (for H_{j+1}).

For α a limit ordinal, we let $H_\alpha = \cup_{i<\alpha} H_i$. If $(\cup_{i<\alpha} B_i, H_\alpha, \cup_{i<\alpha}\phi_i)$ is not maximal, then we use Lemma 3.5 to find a maximal triple $(B_\alpha, H_\alpha, \phi_\alpha)$ extending $(\cup_{i<\alpha} B_i, H_\alpha, \cup_{i<\alpha}\phi_i)$ and continue. If $(\cup_{i<\alpha} B_i, H_\alpha, \cup_{i<\alpha}\phi_i)$ is maximal, then we set $B_\alpha = \cup_{i<\alpha} B_i$ and $\phi_\alpha = \cup_{i<\alpha}\phi_i$. In this case, $(B_\alpha, H_\alpha, \phi_\alpha)$ is the desired dyadic triple (A', H', ϕ').

The analogue of the proof of Lemma 4.5 shows the following.

Lemma 4.6. *For all α, if $h \in H_\alpha$, then $\phi_\alpha(\log(h)) \in k((H_\alpha^{>1}))$.*

The next lemma shows that H_α is a value group section for $RC(\cup_{i<\alpha} B_i \cup H_\alpha)$.

Lemma 4.7. *For all α, if $h, h' \in H_\alpha$ and $v(h) = v(h')$, then $h = h'$.*

Proof. If $v(h) = v(h')$, then $h = ch'$, for some $c \in R^{>0}$ with $v(c) = 1$. By Lemma 4.6, we have $\phi_\alpha(\log(h)), \phi_\alpha(\log(h')) \in k((H_\alpha^{>1}))$. Since $\log(h) = \log(c) + \log(h')$, we must have $\phi_\alpha(\log(h)) = \phi_\alpha(\log(h'))$ and $\log(c) = 0$, so $c = 1$.

□

Since R is a set, there exists some limit ordinal λ such that $B_\lambda = \cup_{i<\lambda} B_i$ and $\phi_\lambda = \cup_{i<\lambda}\phi_i$, that is $(\cup_{i<\lambda} B_i, H_\alpha, \cup_{i<\lambda}\phi_i)$ is maximal, so the chain as constructed is completed. Then $(B_\lambda, H_\lambda, \phi_\lambda)$ is a dyadic triple extending (A, H, ϕ) and for which $y \in B_\lambda$, as required for Proposition 4.1. □

Lemma 4.8. *Given a real closed exponential field R, a residue field section k, and a well ordering $\prec$ of R, there is a canonical dyadic triple (R, G, δ) with respect to R, k, and $\prec$.*

Proof. Like in Lemma 3.4, we obtain a dyadic triple (R, G, δ) as the union of a chain of dyadic triples (R_i, G_i, δ_i), where $(R_0, G_i, \delta_0) = (k, \{1\}, id)$, and when we pass from (R_i, G_i, δ_i) to $(R_{i+1}, G_{i+1}, \delta_{i+1})$, we include in R_{i+1} the $\prec$-first element of $R - R_i$. At limit steps in our construction we use the following corollary. □

Corollary 4.9. *Suppose (A, H, ϕ) is the union of a chain of dyadic triples. Then there is a dyadic triple (A', H', ϕ') extending (A, H, ϕ).*

Proof. The triple (A, H, ϕ) may not be maximal. By Lemma 3.5, we extend (A, H, ϕ) to a maximal triple $(\hat{A}, H, \hat{\phi})$. If $\hat{A} = R$, then $(R, H, \hat{\phi})$ is a dyadic triple. If not, we can extend $(\hat{A}, H, \hat{\phi})$ to a dyadic triple (A', H', ϕ') using the second half of the proof of Proposition 4.1.

□

5 Recursive saturation, Barwise-Kreisel Compactness, and Σ-saturation

In this section, we give some background on "recursive saturation" and "Barwise-Kreisel Compactness", which we use in proving our main new result, Theorem C. We also include some background on Σ-saturation, used for Theorem D.

5.0.1 Recursive saturation and models of PA

Recursive saturation is a weak version of ω-saturation, which was defined by Barwise and Schlipf [2].

Definition 21 (Recursive saturation)**.** *A structure $\mathcal{A}$ is* recursively saturated *if for all tuples $\overline{a}$ in $\mathcal{A}$ and all c.e. sets of formulas $\Gamma(\overline{y}, x)$, if every finite subset of $\Gamma(\overline{a}, x)$ is satisfied in $\mathcal{A}$, then some $b \in \mathcal{A}$ satisfies all of $\Gamma(\overline{a}, x)$.*

Countable recursively saturated structures can be expanded as follows.

Theorem 5.1 (Barwise-Schlipf)**.** *Let $\mathcal{A}$ be a countable recursively saturated L-structure. Let Γ be a c.e. set of sentences, in a language $L' \supseteq L$. If the consequences of Γ in the language L are true in $\mathcal{A}$, then $\mathcal{A}$ can be expanded to a model of Γ.*

In [17], it is shown that a countable model of Presburger arithmetic can be expanded to a model of PA iff it is either standard or recursively saturated. In [4], it is shown that a countable real closed field has an integer part satisfying PA iff it is either Archimedean or recursively saturated. Recall that a set X is *low* if $X' \leq_T \emptyset'$. Gödel's Incompleteness Theorem implies that there is no computable completion of PA. The next result implies that there is a low completion of PA.

Theorem 5.2 (Low Basis Theorem, Jockusch-Soare)**.** *Given a countable c.e. set A of elementary first order sentences that is consistent, there is a low completion of A.*

Scott [25] looked at families of sets coded in completions of PA.

Definition 22 (Scott set)**.** *A* Scott set *$\mathcal{S}$ is a subset of $P(\omega)$ satisfying the following conditions:*

1. *If $X \in \mathcal{S}$ and $Y \leq_T X$, then $Y \in \mathcal{S}$*
2. *If $X, Y \in \mathcal{S}$, then the join of X and Y, denoted $X \oplus Y$, is in $\mathcal{S}$.*
3. *If $T \in \mathcal{S}$ encodes a consistent set of first order sentences, then there is a completion $\hat{T}$ of T in $\mathcal{S}$.*

Scott showed that the families of sets coded in a natural way in completions of PA are exactly the countable Scott sets.

Definition 23 (Enumeration of a family of sets)**.** *For a countable family of sets $\mathcal{S} \subseteq P(\omega)$, an* enumeration *of $\mathcal{S}$ is a binary relation E such that $\mathcal{S} = \{E_n : n \in \omega\}$, where $E_n = \{k : (n, k) \in E\}$.*

Remark 1. *If K is a completion of PA, and $(\varphi_n(x))_{n\in\omega}$ is a computable list of the formulas with just x free, we obtain a natural enumeration E of the Scott set associated with K by taking the set of pairs (n,k) such that $\varphi_n(k) \in K$. The relation E is clearly computable in K.*

For more information about arithmetic, see [1] §19.1.

In [18] Macintyre and Marker considered the complexity of recursively saturated models. They proved the following.

Theorem 5.3 (Macintyre-Marker)**.** *Suppose E is an enumeration of a countable Scott set $\mathcal{S}$. Let T be a complete theory in $\mathcal{S}$. Then T has a recursively saturated model $\mathcal{A}$ such that $D^c(\mathcal{A}) \leq_T E$.*

The next result may be well-known. The proof will be obvious to anyone familiar with the proof of Theorem 5.1.

Proposition 5.4. *Suppose $\mathcal{A}$ is a countable recursively saturated structure, say with universe ω, and let Γ be a c.e. set of finitary sentences, in an expanded language, such that the consequences of Γ are all true in $\mathcal{A}$. Then $\mathcal{A}$ can be expanded to a model $\hat{\mathcal{A}}$ of Γ such that $D^c(\hat{\mathcal{A}})$ is computable in the jump of $D^c(\mathcal{A})$.*

Proof Sketch. We carry out a Henkin construction, as Barwise and Schlipf did, and we observe that the jump of $D^c(\mathcal{A})$ is sufficient for this construction. We make a computable list of the sentences $\varphi(\overline{a})$ in the expanded language, with names for the elements of ω. We also make a computable list of the c.e. sets $\Lambda(\overline{a},x)$. At each stage s, we have put into $D^c(\hat{\mathcal{A}})$ a c.e. set $\Sigma_s(\overline{a})$ of sentences involving a finite tuple of constants, such that the consequences in the language of $\mathcal{A}$ are true in $\mathcal{A}$ of the constants $\overline{a}$. At stage $s+1$, we consider the next sentence $\varphi(\overline{a})$. We add $\varphi(\overline{a})$ to $\Sigma_s(\overline{a})$ if our consistency condition is satisfied (which is computable in the jump of $D^c(\mathcal{A})$), and otherwise we add the negation. Then we consider the next c.e. set $\Lambda(\overline{a},x)$. To check consistency, we see if the consequences of adding this, with some new constant e for x, are true of $\overline{a}$ (again using the jump of $D^c(\mathcal{A})$). If so, we can find some $b \in \omega$ such that for $b = x$ the consequences are satisfied by $\overline{a}, b$ by the recursive saturation of $\mathcal{A}$.

$\square$

5.0.2 Compactness for infinitary logic

Recall that an *admissible set* is a transitive model of KP, a weak fragment of set theory. If A is an admissible set, and $B \subseteq A$, then B is Σ_1 *on* A if it is defined in A by a finitary formula with only existential and bounded quantifiers—the formula may have parameters. A set is A-*finite* if it is an element of A. The least admissible set is $A = L_{\omega_1^{CK}}$. In this case, a set $B \subseteq \omega$ is Σ_1 on A if it is "Π^1_1"—this means that $n \in B$ iff $(\forall f \in \omega^\omega)\,(\exists s)\,R(n, f$ ¼ $s)$, where the relation $R(n,u)$ (on numbers and finite sequences) is computable. A set $B \subseteq \omega$ is A-finite in $L_{\omega_1^{CK}}$ if it is hyperarithmetical. For a countable language L, there are uncountably many formulas of $L_{\omega_1\omega}$. For a countable admissible set A, the *admissible fragment* L_A consists of the $L_{\omega_1\omega}$ formulas that are elements of A, so it is countable.

Theorem 5.5 (Barwise Compactness)**.** *Let A be a countable admissible set, and let L be an A-finite language. Suppose Γ is a set of L_A-sentences that is Σ_1 on A. If every A-finite subset of Γ has a model, then Γ has a model.*

In the case where A is the least admissible set, the L_A-formulas are essentially the computable infinitary formulas. These are formulas of $L_{\omega_1\omega}$ in which the infinite disjunctions and conjunctions are over c.e. sets. For more on computable infinitary formulas, see [1]. As a special case of Barwise Compactness, we have the result below.

Theorem 5.6 (Barwise-Kreisel Compactness)**.** *Let L be a computable language. Suppose Γ is a Π^1_1 set of computable infinitary L-sentences. If every hyperarithmetical subset of Γ has a model, then Γ has a model.*

Ressayre's notion of Σ-saturation, defined in [21], [22] is associated with Barwise Compactness.

Definition 24. *Suppose A is an admissible set and let L be an A-finite language. An L-structure $\mathcal{A}$ is* Σ_A-saturated *if*

1. *for any tuple $\overline{a}$ in $\mathcal{A}$ and any set Γ of L_A-formulas, with parameters $\overline{a}$ and free variable x, if Γ is Σ_1 on A and every A-finite subset is satisfied, then the whole set is satisfied.*
2. *let I be an A-finite set, and let Γ be a set, Σ_1 on A, consisting of pairs (i, φ), where $i \in I$ and φ is an L_A-sentence. For each i, let $\Gamma_i = \{\varphi : (i, \varphi) \in \Gamma\}$. Similarly, if $\Gamma' \subseteq \Gamma$, let $\Gamma'_i = \{\varphi : (i, \varphi) \in \Gamma'\}$. If for each A-finite $\Gamma' \subseteq \Gamma$, there is some i such that all sentences in Γ'_i are true in $\mathcal{A}$, then there is some i such that all sentences in Γ_i are true in $\mathcal{A}$.*

Proposition 5.7. *A countable structure $\mathcal{A}$ is Σ_A-saturated iff it lives in a fattening of A.*

Countable Σ-saturated models have the property of expandability.

Theorem 5.8 (Ressayre)**.** *Suppose $\mathcal{A}$ is a countable Σ_A-saturated L-structure. Let $L' \supseteq L$, and let Γ be a set of L'_A-sentences, Σ_1 on A, s.t. the consequences of Γ, in the language L, are all true in $\mathcal{A}$. Then $\mathcal{A}$ has an expansion satisfying Γ. Moreover, we may take the expansion to be Σ_A-saturated.*

Some people omit the axiom of infinity from KP, so that L_ω qualifies as an admissible set. For $A = L_\omega$, Σ_A-saturation is the same as recursive saturation. Ressayre worked independently of Barwise and Schlipf, and the first version of his definition, in [21], was actually earlier than [2]. For more on infinitary logic, see [11], [12].

6 Complexity of exponential integer parts

We have seen (in Lemma 4.8) that given a countable real closed exponential field R, a residue field section k, and a well ordering $\prec$ of R, Ressayre's construction of an exponential integer part is canonical. Each step is sufficiently effective that the whole construction is constructible. However, there may be many steps. We shall determine a Δ^0_3 well ordering $\prec$ of R, of order type $\omega+\omega$ such that when we run Ressayre's procedure, it is not finished in $L_{\omega_1^{CK}}$; in fact, we do not even reach the first non-trivial dyadic triple. Recall that the procedure starts with the trivial dyadic triple $(R_0, G_0, \delta_0) = (k, \{1\}, id)$. Let y be the $\prec$-first element of $R - k$, adjusted so that y is positive and infinite. We extend (R_0, G_0, δ_0) to a first non-trivial dyadic triple (R_1, G_1, δ_1) with $y \in R_1$. Specifically, we show the following.

Theorem C. There is a low real closed exponential field R, with a Δ^0_3 residue field section k and a Δ^0_3 ordering $\prec$ of type $\omega+\omega$, such that Ressayre's construction, even of the first non-trivial dyadic triple (R_1, G_1, δ_1), is not completed in $L_{\omega_1^{CK}}$.

Lemma 6.1. *There is a recursively saturated real closed exponential field R such that $D^c(R)$ is low.*

Proof. By Theorem 5.2, there is a low completion K of PA. Let $\mathcal{S}$ be the Scott set naturally associated with K. By Remark 1, there is an enumeration E of $\mathcal{S}$ such that $E \leq_T K$. Since $\mathcal{S}$ is a Scott set, it contains a completion T of the set of axioms for real closed exponential fields. By Theorem 5.3, of Macintyre and Marker [18], there is a recursively saturated model R of T such that $D^c(R) \leq_T E$. □

Next, we choose the residue field section k for R. In [14], the second and fourth authors proved the following result.

Proposition 6.2. *For a countable real closed field R, there is a residue field section k that is $\Pi^0_2(R)$.*

Corollary 6.3. *There is a Δ^0_3 residue field section k for R.*

Proof. By Proposition 6.2, there is a residue field section k that is $\Pi^0_2(R)$. Since R is low and k is co-c.e. relative to R', k is co-c.e. relative to $\emptyset'$. Then k is Π^0_2, so it is Δ^0_3.

□

To obtain (R_1, G_1, δ_1) in Proposition 4.1, we form a chain of development triples $(B_j, H_j, \phi_j)_{j<\zeta}$, satisfying the conditions below.

1. $H_0 = \langle \{y_i = \log^i(y) \mid i \in \omega\} \rangle_{\mathbb{Q}}$,
2. $H_{j+1} = \langle H_j \cup \{2^r \mid r \in B_j \ \& \ \ \phi_j(r) \in k((H_j^{>1}))\} \rangle_{\mathbb{Q}}$,
3. for limit j, $H_j = \cup_{j'<j} H_{j'}$,
4. for all j, (B_j, H_j, ϕ_j) is maximal.

The length of the chain is the first limit ordinal ζ such that the triple $(\cup_{j<\zeta} B_j, \cup_{j<\zeta} H_j, \cup_{j<\zeta} \phi_j)$ is maximal.

We see two possible sources of complexity.

1. Some object in the chain $(B_j, H_j, \phi_j)_{j<\zeta}$ leading to the first non-trivial dyadic triple may not be hyperarithmetical; i.e., it may lie outside $L_{\omega_1^{CK}}$.

2. The length ζ of the chain $(B_j, H_j, \phi_j)_{j<\zeta}$ leading to the first non-trivial dyadic triple may not be a computable ordinal.

We produce a Δ^0_3 well ordering $\prec$ of R such that either the chain $(B_j, H_j, \phi_j)_{j<\zeta}$ includes a non-hyperarithmetical object, or else $\zeta \geq \omega_1^{CK}$. We will apply Theorem 5.6, Barwise-Kreisel Compactness, to the set Γ consisting of the following computable infinitary sentences.

1. A sentence ψ characterizing the "ω-models" of KP.

   An *ω-model* has the feature that each element of the definable element ω has only finitely many elements, a fact that we can express using a computable infinitary sentence. An ω-model of KP contains the hyperarithmetical sets. In particular, we have the real closed exponential field R and the residue field section k, with the indices we have chosen for them.

2. A sentence $\varphi_\prec$ saying of a new symbol $\prec$ that it is a Δ^0_3 ordering of R of order type $\omega + \omega$.

   The sentence says that $\prec$ is an ordering of type $\omega + \omega$ and for some $e \in \omega$ (this is a disjunction) the e^{th} partial Δ^0_3 function f on pairs is total with $m \prec n$ for $f(m,n) = 1$ and $m \not\prec n$ for $f(m,n) = 0$.

3. A sentence φ_α, for each computable ordinal α, saying that for all limit $\beta < \alpha$, if the sequence $(B_j, H_j, \phi_j)_{i<\beta}$ is in L_α, then $B_\beta \neq \cup_{j<\beta} B_j$.

   We identify an element of $k((H_i))$ of ordinal length $\beta < \alpha$ with a function from β to $k \times H_i$ such that the second components (the elements of H_i) are decreasing.

We must show that every hyperarithmetical subset of Γ has a model. For a computable ordinal α, let Γ_α consist of ψ, $\varphi_\prec$ and $\varphi_{\alpha'}$ for $\alpha' \leq \alpha$. Each hyperarithmetical subset of Γ is included in one of the sets Γ_α. It is enough to show that for each computable ordinal α, Γ_α has a model. We must produce a Δ^0_3 ordering $\prec_\alpha$ on R, of order type $\omega + \omega$, such that if we run Ressayre's construction according to the well ordering $\prec_\alpha$, then for each limit $\beta < \alpha$, if the sequence $(B_j, H_j, \phi_j)_{j<\beta}$ is in L_α, then $B_\beta \neq \cup_{j<\beta} B_j$.

Throughout the remainder of this section, when we refer to the "development" of an element $r \in R_1$ (where $R_1 = \cup_{i<\zeta} B_i$), we mean the image $\delta_1(r) \in k((G_1))$.

6.1 Special elements

Let α be a computable ordinal. To show that Γ_α has a model, we use some special elements, which we name by constants. We identify the ordinals $\beta < \alpha$ with natural number codes. (We fix a notation $a \in \mathcal{O}$ for α, and we identify each ordinal $\beta < \alpha$ with its unique notation $b <_{\mathcal{O}} a$.) Our special elements are named by constants y, y_i, $i \in \omega$, c_β, for $\beta < \alpha$ either 0 or a limit ordinal, and $c_{\beta,i}$ for all $\beta < \alpha$ and all $i \in \omega$.

We first state some properties that we would like for the constants. We define all of the constants in terms of y, c_0, and c_β for limit $\beta < \alpha$. Assuming that the sequence $(B_j, H_j, \phi_j)_{j<\beta}$ is in L_α, we want $c_\beta \in B_\beta - \cup_{j<\beta} B_j$.

To assure this, we specify a development that we want for c_β, in terms of constants $c_{j+1,i}$ for $j < \beta$, which we want in $H_{j+1} - H_j$. Next, we give a c.e. set of finitary axioms, partially describing the constants. Since R is recursively saturated, we can apply Theorem 5.1 of Barwise and Schlipf to get an expansion R_α satisfying these finitary axioms. Finally, we choose a Δ^0_3 well ordering $\prec_\alpha$ such that when we run Ressayre's construction using R, k, and $\prec_\alpha$, the constants will have all of the desired properties.

In this way, we have a model of Γ_α. We take an ω-model of KP that includes the full chain $(B_j, H_j, \phi_j)_{j<\zeta}$ of development triples leading to the first non-trivial dyadic triple. Inside this model, we can define L_α. Using the constants from R_α, we see that for each computable limit ordinal $\beta < \alpha$, if $(B_j, H_j, \phi_j)_{j<\beta}$ is in L_α, then $c_\beta \in B_\beta - \cup_{j<\beta} B_j$.

6.1.1 Desired properties of the constants

First, we list the properties that we want for our constants.

- y should be positive and infinite, and $y_i = \log^i(y) \in H_0$,
- $c_0 = c_{0,1} \in B_0$, with development $\sum_{1\le i<\omega} y_i$,
- $c_{n+1,i} \in H_{n+1} - H_n$,
- $c_\omega \in B_\omega - \cup_{n\in\omega} B_n$, with development $\sum_{1\le i<\omega} c_{i,i}$,
- In general $c_{\beta+1,i} \in H_{\beta+1} - H_\beta$, and
- for limit β, $c_\beta \in B_\beta - \cup_{\gamma<\beta} B_\gamma$.

6.1.2 Finitary descriptions of the constants

Next, we give a c.e. set of finitary axioms, saying what we can about the constants, in an effort to make them eligible for the developments we want for them, or for membership in the appropriate value group section.

- $y > n$ for all positive integers n, and $y_i = \log^i(y)$

  This guarantees that y is positive and infinite, and all the y_i are also positive and infinite, with $y_0 > y_1 > y_2 > y_3 \ldots$. We can make sure that $y_i \in H_0$ by putting y first in the well ordering $\prec_\alpha$. Then $\sum_{1\le i<\omega} y_i$ is in $k((H_0^{>1}))$.
- $y_1 < c_0 < 2y_1$, $y_2 < c_0 - y_1 < 2y_2$, $y_3 < c_0 - y_1 - y_2 < 2y_3$, etc.

  This guarantees that if $H_0 = \langle \{y_i\}_{i\in\omega}\rangle_{\mathbb{Q}}$, then c_0 is a possible candidate for the development $\sum_{1\le i<\omega} y_i$. It will get this development provided that no other element is assigned it first.
- $c_{0,i} = c_0 - \sum_{j=1}^{i-1} y_j$.

  This guarantees that if c_0 is assigned the development $\sum_{1\le i<\omega} y_i$, then $c_{0,i}$ will have the development $\sum_{i\le j<\omega} y_j$.
- Let $\gamma < \alpha$ be a successor ordinal with $\gamma = \beta + 1$. We define $c_{\gamma,j}$ to be $c_{\beta+1,j} = 2^{c_{\beta,j+1}}$ for $0 < j < \omega$.

  Assuming that $c_\beta \in k((H_\beta^{>1})) \cap B_\beta$, this guarantees that $c_{\beta+1,j} \in H_{\beta+1}$.

- Finally, let $\gamma < \alpha$ be a limit ordinal, where the notation for γ gives a sequence of successor ordinals $(\gamma_i)_{i\in\omega}$ converging to γ. Assuming that $c_{\gamma_i,i} \in H_{\gamma_i}^{>1}$, we can show that $\sum_{1\le i<\omega} c_{\gamma_i,i} \in k((H_\gamma^{>1}))$.

  Our description of $c_\gamma = c_{\gamma,1}$ says

  $c_{\gamma_1,1} < c_\gamma < 2c_{\gamma_1,1}$, $c_{\gamma_2,2} < c_\gamma - c_{\gamma_1,1} < 2c_{\gamma_2,2}$, etc.

  This description guarantees that c_γ is eligible for the development $\sum_{1\le i<\omega} c_{\gamma_i,i}$. It will get this development provided that no other element is assigned it first.

- We define $c_{\gamma,i}$ to be $c_\gamma - \sum_{j=1}^{i-1} c_{\gamma_j,j}$.

This completes our descriptions of the constants y, y_i, and $c_{\gamma,i}$ for $\gamma < \alpha$ and $0 < i \in \omega$. In order for $\sum_{i\le j<\omega} c_{\gamma_j,j} \in k((H_\gamma^{>1}))$ for some limit γ, we need the fact that each element $c_{\gamma_j,j}$ is a member of the appropriate value group section $H_{\gamma_j}^{>1}$, together with the fact that $c_{\gamma_j,j} > c_{\gamma_{j+1},j+1}$ for all nonzero $j \in \omega$. The next lemma shows that the latter condition holds.

Lemma 6.4. *The descriptions of the constants $c_{\beta,i}$ for $\beta < \alpha$ and $i \in \omega$ imply that for all $\beta < \alpha$*

$$y_0 > c_{\beta,1} > y_1 > c_{\beta,2} > y_2 > c_{\beta,3} > y_3 > c_{\beta,4} > \dots \tag{7}$$

Proof. From the description of $c_{0,i}$, we can see that (7) holds for $\beta = 0$. Let $\gamma < \alpha$ be a successor ordinal with $\gamma = \beta + 1$. We inductively assume that the descriptions for the elements $c_{\beta,k}$ imply the ordering in (7). By applying a power of 2 to the inequalities in (7) and the definition of $c_{\gamma,i}$, we obtain the ordering

$$y_0 > c_{\gamma,1} > y_1 > c_{\gamma,2} > y_2 > c_{\gamma,3} > y_3 > c_{\gamma,4} > \dots \tag{8}$$

Let $\gamma < \alpha$ be a limit ordinal, where our notation for γ gives a sequence of successor ordinals $(\gamma_i)_{i\in\omega}$ converging to γ. Moreover, by induction, we have that the descriptions of the $(c_{\gamma_i,i})_{i\in\omega}$ imply that

$$y_0 > c_{\gamma_1,1} > y_1 > c_{\gamma_2,2} > y_2 > c_{\gamma_3,3} > y_3 > c_{\gamma_4,4} > \dots \tag{9}$$

By the description of $c_{\gamma,i}$, we have that

$$y_0 > c_{\gamma,1} > c_{\gamma_1,1} > y_1 > c_{\gamma,2} > c_{\gamma_2,2} > y_2 > c_{\gamma,3} > c_{\gamma_3,3} > y_3 > \dots, \tag{10}$$

completing the induction. □

For the given computable ordinal α, we may take the set of finitary sentences describing the constants to be computably enumerable. It is clearly consistent. Since R is recursively saturated, we can apply Theorem 5.1 to get an expansion R_α of R with special elements y, y_i, and $c_{\beta,i}$, for $\beta < \alpha$ and $i \in \omega$, satisfying the finitary sentences. By Proposition 5.4, we may take $D^c(R_\alpha)$ to be Δ_2^0 since $D^c(R)$ is low. This means that we can find the element of R playing the role of y or c_β using the oracle $\emptyset'$.

6.2 The ordering

We describe a Δ_3^0 well ordering $\prec_\alpha$ of R, of order type $\omega+\omega$, such that when we run Ressayre's construction on R, k, and $\prec_\alpha$, for any limit ordinal $\beta<\alpha$, if the sequence $(B_j,H_j,\gamma_j)_{j<\beta}$ is in L_α, then $B_\beta \neq \cup_{j<\beta}B_j$. To show this, it is enough to give the constants c_0 and c_β, for $\beta<\alpha$, the intended developments. So, we locate these constants (in $\prec_\alpha$) before the other elements that might compete for the intended developments.

The elements of R are natural numbers, so R inherits from ω the usual ordering $<$ of type ω. We let y be the $\prec_\alpha$-least element of R. The special elements c_β for $\beta<\alpha$, ordered by $<$, make up the remainder of the initial segment of type ω, and the other elements, ordered by $<$, make up the remaining segment of type ω. Since $D^c(R_\alpha)$ is Δ_2^0, we can use Δ_3^0 to determine, for a given $r\in R_\alpha$, whether there exists some $\beta<\alpha$ such that $r=c_\beta$, i.e., whether r should be placed in the intial ω segment or the latter. Hence, $\prec_\alpha$ is Δ_3^0.

We have determined R, k, and $\prec_\alpha$. We run Ressayre's construction to obtain $(B_i,H_i,\phi_i)_{i<\zeta}$, the resulting chain of development triples leading to the first nontrivial dyadic triple (R_1,G_1,δ_1).

6.2.1 Lemmas about the constants

We must check that if $(B_j,H_j,\phi_j)_{j<\beta}$ is in L_α, then the constants c_γ and $c_{\gamma,i}$, for $\gamma\le\beta$, have the properties we want for them. The following lemmas are useful.

Lemma 6.5. *For all $\beta<\alpha$, for all $h\in H_\beta^{>1}$, there is some i such that $h>y_i$.*

Proof. We proceed by induction on β. Since y is the $\prec_\alpha$-least element of R, H_0 equals $\langle\{y_i\}_{i<\omega}\rangle_{\mathbb{Q}}$. So, $h\in H_0$ is a finite product of rational powers of the y_i. Let i be least such that there is a factor $y_i^{q_i}$. Since $h\in H_0^{>1}$, q_i must be positive. Then $h>y_{i+1}$. Suppose the statement holds for β, and $h\in H_{\beta+1}^{>1}$. By construction, we may assume that $h=2^r$, where $\phi_\beta(r)\in k((H_\beta^{>1}))$ has a positive initial coefficient.

Say $v(r)=h'>y_i$. Then $h>2^{y_i}>y_i$. Finally, suppose the statement holds for $\gamma<\beta$, where β is a limit ordinal. Since $H_\beta=\cup_{\gamma<\beta}H_\gamma$, the statement holds for H_β.

□

Lemma 6.6. *For all $\beta<\alpha$, if $h\in H_\beta$ and $h\neq 1$, then there is some γ with $\gamma=0$ or $\gamma<\beta$ such that* $\log(h)\in B_\gamma$ *and* $\phi_\gamma(\log(h))\in k((H_\gamma^{>1}))$.

Proof. We prove the lemma by induction on $\beta<\alpha$. If $h\in H_0$, then $h=\prod_{i=0}^n y_{l_i}^{q_i}$ with all $q_i\in\mathbb{Q}$ nonzero and $l_i<l_{i+1}$ for $0\le i<n$. Then $\log(h)=\sum_{i=0}^n q_i\log(y_{l_i})$. Since $\log(y_{l_i})=y_{l_i+1}$, $\log(h)\in B_0$ and $\phi_0(\log(h))\in k((H_\gamma^{>1}))$.

Suppose the statement holds for all $\lambda<\beta$. If β is a limit ordinal, $h\in H_\beta$ implies $h\in H_\lambda$ for some successor ordinal $\lambda<\beta$, so the statement holds by induction. Suppose $\beta=\lambda+1$ and $h\in H_\beta-H_\lambda$. By construction, $h=h'\prod_{i=0}^n 2^{t_i}$ where $h'\in H_\lambda$ and $t_i\in B_\lambda$ and $\phi_\lambda(t_i)\in k((H_\lambda^{>1}))$ for all

$1 \leq i \leq n$. Then, $\log(h) = \log(h') + \sum_{i=0}^{n} t_i$ has the desired features by induction. $\square$

To show that the elements $c_{\beta,i}$ get the developments we want for them, we must show that other elements cannot compete for these developments.

Lemma 6.7. *Suppose Ressayre's construction is run on R, k, and a well ordering $\prec$ on R such that y is the $\prec$-first element and the elements c_β for $\beta < \alpha$ form the initial ω segment. For all $\beta, \gamma < \alpha$, the following statements hold.*

1. *If γ is a limit ordinal greater than β, then $c_{\gamma,i} \notin B_\beta$ (i.e., $c_{\gamma,i}$ is not assigned a development by ϕ_β).*
2. *If γ is a successor ordinal greater than β, then $c_{\gamma,i}$ has no valuation in H_β.*
3. *If β is 0 or a limit ordinal, then $c_{\beta,i} \in B_\beta$ and $\phi_\beta(c_{\beta,i}) = \sum_{i \leq j < \omega} c_{\beta_j,j}$ in $k((H_\beta^{>1}))$ where the sequence $(\beta_i)_{i\in\omega}$ is defined as follows. If $\beta = 0$, then $\beta_i = i$ for all $i \in \omega$. If β is a limit ordinal, $(\beta_i)_{i\in\omega}$ is the sequence of successor ordinals converging to β given by our notation for β.*
4. *If β is a successor ordinal, then $c_{\beta,i}$ is in $H_\beta^{>1}$.*

Proof. We begin with the case where $\beta = 0$. Clearly, c_0 will be assigned the development $\sum_{1 \leq i < \omega} y_i$ if it is the first element after y in $\prec$. However, c_0 may not be the first such element; there may be finitely many other c_β before c_0 under $\prec$. Statements 1 and 2 imply that these finitely many c_β would not interfere with ϕ_0 assigning the development $\sum_{1 \leq i < \omega} y_i$ to c_0. Hence, Statements 1 and 2 give Statement 3.

We begin by showing for all $\gamma > 0$ and all $i \in \omega$ that $c_{\gamma,i}$ has no valuation in H_0. Suppose otherwise, and let γ be the first ordinal witnessing the failure. If γ is a limit ordinal, then the valuation of $c_{\gamma,i}$ is the same as that of $c_{\gamma_i,i}$, where γ_i is a smaller successor ordinal. So, we may suppose that $\gamma = \lambda + 1$ for some λ. The element $c_{\lambda+1,i}$ was defined to be $2^{c_{\lambda,i+1}}$. Since $c_{\lambda+1,i}$ has a valuation in H_0, we have that $c_{\lambda+1,i}$ equals $cy_0^{q_0} y_1^{q_1} \cdots y_n^{q_n}$, where c is finite and the $q_i \in \mathbb{Q}$. Then, taking logs, we have $c_{\lambda,i+1} = \log(c) + q_0 y_1 + q_1 y_2 + \cdots + q_n y_{n+1}$. We see that $c_{\lambda,i+1}$ has valuation equal to some y_i, which is in H_0, since both $c_{\lambda+1,i}$ and $c_{\lambda,i+1}$ are infinite. We must have $\lambda = 0$ and $\gamma = 1$, since otherwise we have reached a contradiction. If we use a different ordering, putting c_0 first after y, then c_0 would be assigned the desired development. Then, $c_{0,i+1}$ would be in B_0 and $c_{1,i} = 2^{c_{0,i+1}}$ would be in $H_1 - H_0$ by Ressayre's construction. Therefore, $c_{1,i}$ has no valuation in H_0. So, Statements 1, 2, 3, and 4 hold when $\beta = 0$.

Suppose $\beta < \alpha$ is arbitrary and that the statements in Lemma 6.7 hold for all $\lambda < \beta$ and all $\gamma < \alpha$.

We begin by proving Statement 4. If $\beta = \lambda + 1$ is a successor ordinal, we have that $c_{\beta,i} = 2^{c_{\lambda,i+1}}$. If λ is itself a successor ordinal, we have that $c_{\lambda,i+1} \in H_\lambda^{>1}$ by Statement 4 for λ of the induction hypothesis. Since $c_{\lambda,i+1} \in H_\lambda^{>1}$, $c_{\beta,i} = 2^{c_{\lambda,i+1}} \in H_\beta^{>1}$ by construction. If λ is a limit ordinal, we have that $c_{\lambda,i+1} \in B_\lambda$ is assigned a development in $k((H_\lambda^{>1}))$

by Statement 3 for λ of the induction hypothesis. Again, by construction, $c_{\beta,i} = 2^{c_{\lambda,i+1}} \in H_\beta^{>1}$ as desired.

We now show that Statements 1 and 2 hold for β and all $\gamma < \alpha$ by induction on γ. Given some ordinal $\gamma' > \beta$, additionally suppose that Statements 1 and 2 hold for all $\gamma < \gamma'$ with respect to β. First, suppose $\gamma' = \gamma + 1$ is a successor ordinal, and suppose for a contradiction that $c_{\gamma',i} = 2^{c_{\gamma,i+1}}$ has a valuation in H_β. By construction of H_β, we have that

$$c_{\gamma',i} = 2^{c_{\gamma,i+1}} = ch2^{b_1} \cdots 2^{b_n} \tag{11}$$

where c is finite, $h \in H_\lambda$, and $b_j \in B_\lambda$ so that $\phi_\lambda(b_j) \in k((H_\lambda^{>1}))$ for some ordinal $\lambda < \beta$. Taking logs of both sides, we have that

$$c_{\gamma,i+1} = \log(c) + \log(h) + b_1 + \ldots + b_n. \tag{12}$$

By Lemma 6.6, $\log(h) \in B_\lambda$ and $\phi_\lambda(\log(h)) \in k((H_\lambda^{>1}))$. Thus, $v(c_{\gamma,i+1}) \in H_\lambda$. If γ is a successor ordinal, this would contradict Statement 2 of the inductive hypothesis with respect to λ. So, suppose γ is a limit ordinal. Consider the sequence $(\gamma_j)_{j\in\omega}$ of successor ordinals given by the notation for γ such that $\lim_{j\to\infty} \gamma_j = \gamma$. Let l be the least natural number such that $\gamma_l > \lambda$. We have $c_{\gamma_j,j} \in H_\lambda^{>1}$ for $j < l$ by Statement 4 of the inductive hypothesis. Thus,

$$c_{\gamma,i+1} - \sum_{i+1\le j<l} c_{\gamma_j,j} = \log(c) + \log(h) + b_1 + \ldots + b_n - \sum_{i+1\le j<l} c_{\gamma_j,j} \tag{13}$$

The left hand side of the equation has the same valuation as $c_{\gamma_l,l}$ by definition of $c_{\gamma,i+1}$. The right hand side of the equation consists of elements whose developments are in $k((H_\lambda^{>1}))$ and the finite element $\log(c)$. Thus, $v(c_{\gamma_l,l}) \in H_\lambda^{>1}$, contradicting Statement 2 of the inductive hypothesis applied to λ. This completes the case where γ' is a succesor ordinal.

Second, suppose that γ' is a limit ordinal, and suppose for a contradiction that $c_{\gamma',i} \in B_\beta$, i.e., ϕ_β assigns $c_{\gamma',i}$ a development in $k((H_\beta))$. Consider the sequence $(\gamma'_j)_{j\in\omega}$ of successor ordinals given by the notation for γ' such that $\lim_{j\to\infty} \gamma'_j = \gamma'$. Let l be the least natural number such that $\gamma'_l > \beta$. By Statement 4 of the induction hypothesis, we have that $c_{\gamma'_j,j} \in H_\beta^{>1}$ for all $\gamma'_j < \beta$. If $\beta = \gamma'_j$, we also have $c_{\gamma'_j,j} \in H_\beta^{>1}$ by the proof above of Statement 4 for β. Since ϕ_β assigns $c_{\gamma',i}$ a development in $k((H_\beta))$, the embedding ϕ_β also assigns the difference $c_{\gamma',i} - \sum_{i\le j<l} c_{\gamma'_j,j}$ a development in $k((H_\beta))$. Thus, $c_{\gamma',i} - \sum_{i\le j<l} c_{\gamma'_j,j}$ has a valuation in H_β. Since $c_{\gamma',i} - \sum_{i\le j<l} c_{\gamma'_j,j}$ has the same valuation as $c_{\gamma'_l,l}$ by definition, $v(c_{\gamma'_l,l}) \in H_\beta$. Since $\gamma'_l > \beta$ is a successor ordinal less than γ', this contradicts Statement 2 of the induction hypothesis with respect to β. This completes our induction on γ'. We have proved Statements 1 and 2 for β and all $\gamma < \alpha$.

We finally prove Statement 3 for β. Suppose β is a limit ordinal. By Statements 3 and 4 of the induction hypothesis, we have that all $c_{\lambda,i}$ for $\lambda < \beta$ are in B_λ and receive their desired developments under ϕ_λ. In particular, $c_{\beta_j,j} \in H_\beta^{>1}$ for all $j \in \omega$. We have that no element $c_{\gamma,k}$ for $\gamma > \beta$ is assigned a development by ϕ_β by Statements 1 and 2 for β. Since the elements c_γ are the only elements that could come before c_β in the initial ω segment of the well ordering $\prec$, the element $c_{\beta,i}$ will enter B_β and $\phi_\beta(c_{\beta,i}) = \sum_{i\leq j<\omega} c_{\beta_j,j} \in k((H_\beta^{>1}))$. Thus, Statement 3 holds for β. This completes the proof of Lemma 6.7.

□

We now show that Γ_α is consistent. The formulas ψ and $\varphi_\prec$ are satisfied by R, k, and $\prec_\alpha$ by construction. We now show $\varphi_{\alpha'}$ holds for each ordinal $\alpha' \leq \alpha$. Let β be a limit ordinal less than α'. If $(B_j, H_j, \gamma_j)_{j<\beta}$ is in $L_{\alpha'}$, then it is in L_α. By Lemma 6.7 Statements 1 and 2, there is an element of $B_\beta - \cup_{\gamma<j} B_\gamma$, namely c_β, so $\varphi_{\alpha'}$ is satisfied. Thus, Γ_α is consistent.

We are in a position to apply Barwise Compactness. By Theorem 5.5, we obtain an ω-model of KP with R and k as elements, and a Δ_3^0 ordering $\prec$ of type $\omega+\omega$ such that if Ressayre's construction is run on R, k, and $\prec$, producing a chain of development triples $(B_i, H_i, \phi_i)_{i<\zeta}$ leading to the first non-trivial dyadic triple (R_1, G_1, δ_1), then either some triple (B_j, H_j, γ_j) for $j < \zeta$ is not in $L_{\omega_1^{CK}}$, or else the length of the chain ζ is noncomputable. This completes the proof of Theorem C. Ressayre's construction on R, k, and $\prec$ cannot be completed in $L_{\omega_1^{CK}}$.

Although Ressayre's construction may not be carried out in $L_{\omega_1^{CK}}$, we can use Σ-saturation to obtain an exponential integer part in a fattening of $L_{\omega_1^{CK}}$.

Proposition 6.8. *Let R be a hyperarithmetical real closed exponential field. Then R has is an exponential integer part Z such that (R, Z) lives in a fattening of $L_{\omega_1^{CK}}$.*

Proof. Since R is an element of A, it is trivially Σ_A-saturated. Let Γ be the natural set of sentences saying that Z is an exponential integer part. By Theorem B, R has an exponential integer part, so the consequences of Γ are true in R. Therefore, by Theorem 5.8, there is an exponential integer part Z such that (R, Z) is Σ_A-saturated. This means that (R, Z) lives in a fattening of $L_{\omega_1^{CK}}$ by Proposition 5.7.

□

The same reasoning shows that if R is a countable real closed exponential field, with universe a subset of ω, and A is the least admissible set that contains R, then R has an exponential integer part Z in a fattening of A. This establishes Theorem D of the Introduction.

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