A characterization of the $\mathbf{0}$-basis homogeneous bounding degrees

Karen Lange *

Abstract

We say a countable model $\mathcal{A}$ has a $\mathbf{0}$-*basis* if the types realized in $\mathcal{A}$ are uniformly computable. We say $\mathcal{A}$ has a ($\mathbf{d}$-)decidable copy if there exists a model $\mathcal{B} \cong \mathcal{A}$ such that the elementary diagram of $\mathcal{B}$ is ($\mathbf{d}$-)computable. Goncharov, Millar, and Peretyat'kin independently showed there exists a homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis but no decidable copy. We extend this result here. Let $\mathbf{d} \leq \mathbf{0}'$ be any low$_2$ degree. We show that there exists a homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis but no $\mathbf{d}$-decidable copy. A degree $\mathbf{d}$ is $\mathbf{0}$-*basis homogeneous bounding* if any homogenous $\mathcal{A}$ with a $\mathbf{0}$-basis has a $\mathbf{d}$-decidable copy. In previous work we showed that the nonlow$_2$ Δ^0_2 degrees are $\mathbf{0}$-basis homogeneous bounding. The result of this paper shows that this is an exact characterization of the $\mathbf{0}$-basis homogeneous bounding Δ^0_2 degrees.

1 Introduction

In the 1970s, Goncharov, Harrington, Peretyat'kin, Morley, and others began studying the computable content of models and constructions from model theory. Let T be a complete decidable (CD) theory. A model is called ($\mathbf{d}$-)*decidable* if its elementary diagram $D^e(\mathcal{A})$ is ($\mathbf{d}$-)computable. Various early researchers showed that a decidable copy of a prime, saturated, or specific homogeneous model of a CD theory does not necessarily exist. Thus, the Turing degree $\mathbf{0}$ is weak in this sense with respect to these special models. On the other hand, it is easy to see that, under reasonable computability

*The material for this paper is taken from the author's thesis, which was supervised by Robert Soare and Denis Hirschfeldt. The author is grateful for their insightful conversations about the topic. She would also like to thank the referees for their helpful corrections and suggestions. While preparing this document, the author was supported in part by NSF Grant #0802961.

conditions, any of these models has a $\mathbf{0}'$-decidable copy. Given this $\mathbf{0}$ and $\mathbf{0}'$ dichotomy, recent research has focused on studying when an intermediate or other degree decides a copy of a prime, saturated, or specific homogeneous model.

Many people including Csima, Hirschfeldt, Knight, Soare, and Epstein have recently studied the prime case in [3], [10], [5], and [6]. In [5], Csima, Hirschfeldt, Knight, and Soare characterized the prime bounding degrees. A degree $\mathbf{d}$ is *prime bounding* if for *any* CD theory T with a prime model $\mathcal{P}$, $\mathbf{d}$ decides a copy of $\mathcal{P}$. They showed the following result.

Theorem 1.1 (Csima, Hirschfeldt, Knight, Soare [5])**.** *Let* $\mathbf{d} \leq \mathbf{0}'$. *The degree* $\mathbf{d}$ *is nonlow*$_2$ *if and only if* $\mathbf{d}$ *is prime bounding.*

Since every prime model is homogeneous, it is natural to see how results on homogeneous models compare. In [14], we studied what degrees decide a copy of a specific homogeneous model satisfying certain computability restrictions. We say a countable model $\mathcal{A}$ has a $\mathbf{0}$-*basis* if the types realized in $\mathcal{A}$ are uniformly computable. We showed that many of the positive results on prime models hold analogously in the homogeneous case. (For an overview of many of the recent results on prime, saturated, and homogeneous models, see [15].) In particular, we studied the following concept.

Definition 1.2. A degree $\mathbf{d}$ is $\mathbf{0}$-*basis homogeneous bounding* if for any homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis, there exists a $\mathcal{B} \cong \mathcal{A}$ such that $\mathcal{B}$ is $\mathbf{d}$-decidable.

We proved the next theorem which is an analogue to the theorem on prime bounding degrees.

Theorem 1.3 (Lange, [14])**.** *Let* $\mathbf{d} \leq \mathbf{0}'$. *If* $\mathbf{d}$ *is nonlow*$_2$*, then* $\mathbf{d}$ *is* $\mathbf{0}$-*basis homogeneous bounding.*

We now show that the $\mathbf{0}$-basis homogeneous bounding degrees exactly characterize the nonlow$_2$ degrees below $\mathbf{0}'$.

Theorem 1.4. *Let* $\mathbf{d} \leq \mathbf{0}'$. *The degree* $\mathbf{d}$ *is* $\mathbf{0}$-*basis homogeneous bounding if and only if* $\mathbf{d}$ *is nonlow*$_2$*.*

The remaining direction, which we prove in Theorem 8.4, is an extension of the result by Goncharov [7], Peretyat'kin [19], and Millar [18] that there exists a homogeneous model with a $\mathbf{0}$-basis but no decidable copy. Given a low$_2$ degree $\mathbf{d} \leq \mathbf{0}'$, we construct a nontrivial homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis but no $\mathbf{d}$-decidable copy $\mathcal{B}$. Therefore $\mathbf{d}$ is not a $\mathbf{0}$-basis homogeneous bounding degree. A model $\mathcal{A}$ is *trivial* if there is a finite set $F \subset A$ such that any permutation π of A fixing F is an automorphism of $\mathcal{A}$ (see Definition 2.4), and nontrivial otherwise. Nontrivial models are more interesting as we see in §2.2.

Convention 1.5. *We assume throughout that all theories T are complete decidable and all models $\mathcal{A}$ of T are countable.*

2 Definitions and Techniques

Let $\mathcal{L}$ be a countable language and T be a complete theory on $\mathcal{L}$. Here we fix our notation for various structures under consideration and discuss some basic model theory. See [2] or [16] for an introduction to model theory and [20] or [21] for an introduction to computability theory. For an overview of computable model theory, see [1] or [8]. Finally, for a detailed look at the following definitions and techniques in the context of prime and homogeneous models, consult [15].

Definition 2.1. Let T be a complete theory in language $\mathcal{L}$, and let $\mathcal{A}$ be a model of T.
(i) A formula $\theta(\overline{x})$ is *consistent with* T if $T \cup (\exists \overline{x})\theta(\overline{x})$ is consistent, i.e., if $(\exists \overline{x})\theta(\overline{x}) \in T$, because T is complete.

(ii) An *n-type $p(\bar{x})$ of T in the n-tuple of variables $\bar{x}$* is a maximal set of formulas in variables taken from $\bar{x}$ consistent with T.

(iii) $S_n(T)$ is the set of all n-types of T (in any n-tuple of variables), and let $S(T) = \cup_{n \geq 1} S_n(T)$.

(iv) An n-tuple $\overline{a} \in A$ *realizes* an n-type $p(\overline{x}) \in S_n(T)$ if $\mathcal{A} \models \theta(\overline{a})$ for all $\theta(\overline{x}) \in p(\overline{x})$. In this case we also say that $\mathcal{A}$ *realizes* p. The *type of* $\overline{a}$ denotes the type $p(x_0, x_1, \ldots, x_{n-1})$ that $\overline{a}$ realizes.

(v) Define the *type spectrum* of $\mathcal{A}$

$$\mathbb{T}(\mathcal{A}) = \{\, p \;:\; p \in S(T) \;\;\&\; \mathcal{A} \text{ realizes } p \,\}.$$

As we will now see, homogeneity can be described in terms of the behavior of types. Moreover, $\mathbb{T}(\mathcal{A})$ plays an important role in understanding the isomorphism class of a given homogeneous model. Let Aut $\mathcal{A}$ denote the automorphisms of $\mathcal{A}$.

Definition 2.2. (i) A countable model $\mathcal{A} \models T$ is *homogeneous* if for all n-tuples $\overline{a}$ and $\overline{b}$, if $\overline{a}$ and $\overline{b}$ realize the same n-type, then

$$(\exists \Phi \ \in \ \mathrm{Aut}\ \mathcal{A})\,[\ \Phi(\overline{a}) \ = \ \overline{b}\ ].$$

(ii) A model $\mathcal{A}$ of T is *prime* if $\mathcal{A}$ can be elementarily embedded in any other model $\mathcal{B}$ of T.

Recall that prime models are necessarily homogeneous.

2.1 Presenting Types for a Complete Decidable Theory

From now on we assume that T is a complete decidable (CD) theory in a computable language $\mathcal{L}$. We define an effective enumeration of all formulas of $\mathcal{L}$ and show how we describe types using this enumeration.

Definition 2.3. (i) Given a fixed complete theory T in a computable language $\mathcal{L}$, let $\{\theta_i\}_{i\in\omega}$ be an effective numbering of all the formulas in $\mathcal{L}$.

(ii) We associate with p a function $f \in 2^\omega$ such that $f(i) = 1$ if $\theta_i \in p$, and $f(i) = 0$ otherwise.

(iii) For any type $p \in S(T)$ define $p\restriction s \ = \ p \cap \{\theta_i\}_{i<s}$. Identify $p\restriction s$ with the function $f_p\restriction s$ where $f_p(i) = 1$ if $\theta_i \in p$ and $f_p(i) = 0$ otherwise.

2.2 Decidable and Computable Models

We first define the diagrams associated with a given model. Let $\mathcal{A}$ be a model with universe A. Let $\mathcal{L}_A$ be the language $\mathcal{L} \cup \{c_a : a \in A\}$. Let $\mathcal{A}_A = (\mathcal{A}, a)_{a\in A}$ be the expansion of model $\mathcal{A}$ for language $\mathcal{L}_A$ such that c_a is interpreted by a for every $a \in A$. The *elementary diagram* $D^e(\mathcal{A})$ (atomic diagram $D^a(\mathcal{A})$) of $\mathcal{A}$ is the set of all (atomic) sentences of $\mathcal{L}_A$ that are true in $\mathcal{A}_A$. We say $\mathcal{A}$ is *(*$\mathbf{d}$*-)computable* if $D^a(\mathcal{A})$ is ($\mathbf{d}$-)computable and $\mathcal{A}$ is *(*$\mathbf{d}$*-)decidable* if $D^e(\mathcal{A})$ is ($\mathbf{d}$-)computable.

Definition 2.4. A structure $\mathcal{A}$ is called *(automorphically) trivial* if there exists a finite set $F \subset A$ such that any permutation π of A fixing F is an automorphism of $\mathcal{A}$.

The next theorem is a useful fact about degrees of copies of structures.

Theorem 2.5 (Knight [13])**.** *Let $\mathcal{A}$ be a countable structure in a relational language.*

(i) If $\mathcal{A}$ is trivial, then degree$(D^e(\mathcal{B}))$ = degree$(D^e(\mathcal{A}))$ for all $\mathcal{B} \cong \mathcal{A}$.
(ii) If $\mathcal{A}$ is nontrivial, $\mathbf{c}$ = degree$(D^e(\mathcal{A}))$, and $\mathbf{d} > \mathbf{c}$, then there exists a model $\mathcal{B} \cong \mathcal{A}$ such that $\mathbf{d}$ = degree$(D^e(\mathcal{B}))$.
These results also hold in the atomic diagram case.

Since trivial models are structurally and degree-theoretically uninteresting, we are more interested in nontrivial models, the degrees of which are closed upwards by Theorem 2.5.

3 Decidability of Homogeneous Models

In this section, we lay out the terminology required to understand Goncharov and Peretyat'kin's characterization (discussed in §3.2) of when a homogeneous model has a decidable isomorphic copy.

3.1 0-Bases and d-Uniform Bases

Definition 3.1. We call a countable subset X of $S(T)$ a *basis*, and we say X is a *basis for a model* $\mathcal{A}$ if $\mathbb{T}(\mathcal{A}) = X$.

We encode a basis $X = \{p_i\}_{i\in\omega}$ as a function $f(i, j)$ such that for any fixed i, the first digit $f(i, 0)$ in the row $\{f(i, j)\}_{j\in\omega}$ encodes the set of free variables represented in the type p_i and the remainder of the row codes the type p_i in $S(T)$ according to the enumeration of the formulas we fixed in Definition 2.3. We further assume that each type in a basis is listed infinitely many times, i.e., if p_i is a type coded by row $\{f(i, j)\}_{j\in\omega}$, then there are infinitely many i' such that p_i equals the type coded by row $\{f(i', j)\}_{j\in\omega}$.

If $\mathcal{A}$ has a decidable copy, then there exists a uniformly computable encoding of $\mathbb{T}(\mathcal{A})$. We generalize this idea.

Definition 3.2. Let $\mathbf{d}$ be a degree. We say $\mathcal{A}$ has a $\mathbf{d}$-*uniform basis* $X = \{p_j\}_{j\in\omega}$ if X is a $\mathbf{d}$-uniformly computable encoding of $\mathbb{T}(\mathcal{A})$. If $\mathbf{d} = \mathbf{0}$, we use the shorter term $\mathbf{0}$-*basis* for $\mathbf{0}$-uniform basis.

Goncharov, Millar, and Peretyat'kin separately showed that a $\mathbf{0}$-basis alone does not guarantee the existence of a decidable copy of a homogeneous model by building counterexamples [7], [18], [19]. Goncharov and Peretyat'kin, however, exactly characterized when a homogeneous model has a decidable copy. We now discuss their characterization. For more detail on Goncharov's and Peretyat'kin's characterization and how it relates to the analogous characterizations for prime and saturated models, see [15].

3.2 Monotone Extension functions

Although a $\mathbf{0}$-basis for a homogeneous model $\mathcal{A}$ computably tells us what types are realized in $\mathcal{A}$, Goncharov and Peretyat'kin realized that to produce a decidable copy, we need computable information about how these types extend one another.

Definition 3.3. **[Monotone Extension Function (MEF)]**
Let $\mathcal{A}$ be a homogeneous model of a CD theory T, and let $X = \{p_i\}_{i\in\omega}$ be a $\mathbf{0}$-basis for $\mathcal{A}$.

(i) A function f is an *extension function (EF)* for X if, for every n and for every n-type $p_i(\overline{x}) \in X$ and $(n+1)$-ary $\theta_j(\overline{x}, y)$ consistent with $p_i(\overline{x})$, the $(n+1)$-type $p_{f(i,j)} \in X$ extends both $p_i(\overline{x})$ and $\theta_j(\overline{x}, y)$, i.e.,

$$p_i(\overline{x}) \cup \{\theta_j(\overline{x}, y)\} \subseteq p_{f(i,j)}(\overline{x}, y).$$

In this case, we call $p_{f(i,j)}$ an *amalgamator* for p_i and θ_j.

(ii) A function f is a *monotone extension function (MEF)* if there exists a computable function $g(i, j, s)$ such that

- $f(i, j) = \lim_s g(i, j, s)$ is an extension function and
- [Formula Monotonicity] $\quad p_{g(i,j,s)}{\restriction} s \subseteq p_{g(i,j,s+1)}{\restriction} s$.

A monotone extension function is a computable function that, given any n-type $p_i(\bar{x})$ and any consistent $(n+1)$-ary formula $\theta_j(\bar{x}, y)$, monotonically approximates the index of an amalgamating $(n+1)$-type. Specifically, the approximate amalgamator $p_{g(i,j,s)}(\overline{x}, y)$ at stage s agrees with the true amalgamator $p_{f(i,j)}(\overline{x}, y)$ on the first s formulas of $\mathcal{L}$.

Notice that if a $\mathbf{0}$-basis X has an MEF g, then there exists another MEF g' for X such that for all s and all $t < g'(i, j, s)$, we have $p_{g'(i,j,s)}{\restriction} s \neq p_t{\restriction} s$. (In other words, g' is an MEF that rests on the least possible row in X at each stage s.) We can compute g' from g and X by calculating $g(i, j, s)$ and setting $g'(i, j, s)$ to the index of the least row in X that corresponds to a type in the same variables as $p_{g(i,j,s)}$ and agrees with this type on the first s many formulas of $\mathcal{L}$. Any MEF $g'(i, j, s)$ satisfying this property is also a monotonic function in s. The next result is our main tool for obtaining new results.

Theorem 3.4 (Relativization of Goncharov [7], Peretyat'kin [19])**.** *Let T be a CD theory, and let $\mathcal{A}$ be a homogeneous model of T with a $\mathbf{d}$-uniform basis. Then the following are equivalent:*

1. *$\mathcal{A}$ has a* **d***-decidable isomorphic copy.*

2. *Every* **d***-uniform basis for $\mathcal{A}$ has a* **d***-monotone extension function.*

3. *Some* **d***-uniform basis for $\mathcal{A}$ has a* **d***-monotone extension function.*

Suppose a homogeneous model $\mathcal{A}$ has a $\mathbf{0}$-basis X. Since a $\mathbf{0}$-basis can be effectively viewed as a $\mathbf{d}$-uniform basis, to show $\mathcal{A}$ has a $\mathbf{d}$-decidable isomorphic copy, we can build a $\mathbf{d}$-monotone extension function on the original $\mathbf{0}$-basis X. Then by Theorem 3.4, $\mathcal{A}$ has a $\mathbf{d}$-decidable isomorphic copy $\mathcal{B}$.

4 The Overall Strategy

In [14], we proved Theorem 1.3 that any nonlow$_2$ $\mathbf{d} \leq \mathbf{0}'$ ($\mathbf{d}'' > \mathbf{0}''$) is $\mathbf{0}$-basis homogeneous bounding. We now show that the $\mathbf{0}$-basis homogeneous bounding degrees exactly characterize the nonlow$_2$ degrees below $\mathbf{0}'$.

Theorem 4.1. *Let $\mathbf{d} \leq \mathbf{0}'$. The degree $\mathbf{d}$ is $\mathbf{0}$-basis homogeneous bounding if and only if $\mathbf{d}$ is nonlow$_2$.*

We fix a low$_2$ degree $\mathbf{d} \leq \mathbf{0}'$ and construct a nontrivial homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis but no $\mathbf{d}$-decidable copy. By Theorem 3.4 and the discussion above, we build a homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis such that the $\mathbf{0}$-basis has no $\mathbf{d}$-monotone extension function. Then $\mathcal{A}$ has no $\mathbf{d}$-decidable copy.

To build such a counterexample, we satisfy two general requirements that will be described in more detail later. First, we ensure that we are building a *homogeneous* model $\mathcal{A}$ with a $\mathbf{0}$-basis X.

P: $\mathcal{A}$ is a homogeneous model with a $\mathbf{0}$-basis X.

Second, we require that $\mathcal{A}$ has no $\mathbf{d}$-decidable copies. In other words,

N: The $\mathbf{0}$-basis X for $\mathcal{A}$ has no $\mathbf{d}$-monotone extension function.

5 The Positive Requirements

First, we explore how to satisfy **P**. The next section describes how to build a basis that is realized by some homogeneous model.

5.1 Building Homogeneous Models

Our counterexample will model a CD theory T with unary relations $\{P_i\}_{i\in\omega}$ and binary relations $\{R_i\}_{i\in\omega}$ that we describe later. We will construct a $\mathbf{0}$-basis of T that satisfies the closure properties described in Theorem 5.1. Then, this $\mathbf{0}$-basis will equal $\mathbb{T}(\mathcal{A})$ for some homogeneous model $\mathcal{A}$ of T.

Theorem 5.1 (Goncharov [7], Peretyat'kin [19])**.** *Let T be a complete theory, and suppose X is a countable set contained in $S(T)$. Then there exists a homogeneous model realizing exactly the types in X if and only if*

1. $T \in X$
2. *X is closed under permutations of the variables of $\mathcal{L}$.*
3. *X is closed under taking subtypes.*
4. **Extension Property (EP)**
   If $p(\bar{x}) \in X$ and $\theta(\bar{x}, y)$ are consistent, there exists a type $q(\bar{x}, y) \in X$ such that $p \cup \{\theta\} \subseteq q$.
5. **Type Amalgamation Property (TAP)**
   For any pair of types $p_1(\bar{x}, y), p_2(\bar{x}, z) \in X$ such that $p_1 \restriction \bar{x} = p_2 \restriction \bar{x}$, there exists a type $q(\bar{x}, y, z) \in X$ containing p_1 and p_2.

5.2 Refining the Positive Requirements

Recall the $\mathbf{0}$-basis $X = \{p_i\}_{i\in\omega}$ is encoded as a function $f(i, j)$, where the restriction of this function to the domain $\{i\} \times \omega$ encodes the type p_i. We build X computably in stages, and we view X as a uniformly computable infinite matrix where the i^{th} row corresponds to a type p_i in $\mathbb{T}(\mathcal{A})$.

To ensure that a given homogeneity closure condition with respect to a row or pair of rows of X is satisfied, we place a marker H on an empty row of X. Then we ensure that H moves to a new row finitely often during the construction and that the row on which it settles satisfies the given homogeneity closure condition. Hence, $\mathbf{P}$ can be restated as:

$\mathbf{P}$: All homogeneity markers settle, and the rows that they settle on satisfy the required homogeneity closure condition.

More specifically, we must satisfy for all i and j the following requirements.

$\mathbf{Q}_j$: Let π_k denote the k^{th} permutation of the free variables in $\mathcal{L}$. We assign markers $Perm_{j,k}$ to $\mathbf{Q}_j$. Then $Perm_{j,k}$ settles on a row corresponding to the type generated by the permutation π_k of the free variables applied to p_j.

$\mathbf{S}_j$: Let V_k denote the k^{th} distinct subset of the free variables in p_j. For each V_k, marker $Sub_{j,k}$ will settle on a row corresponding to the subtype of p_j generated by the free variables in V_k.

$\mathbf{R}_{i,j}$: If $\theta_j(\bar{x}, y)$ is consistent with $p_i(\bar{x})$, then marker $Tr_{i,j}$ will settle on a row k such that p_k extends p_i and $\{\theta_j\}$.

$\mathbf{T}_{i,j}$: If rows i and j correspond to types $p_i(\bar{x}, y)$ and $p_j(\bar{x}, z)$ such that $p_i \restriction \bar{x} = p_j \restriction \bar{x}$, then marker $Tap_{i,j}$ settles on a row k whose type $q(\bar{x}, y, z)$ contains p_i and p_j. (We allow the possibility that $\bar{x}$ is empty).

In the construction, we place the theory T, a 0-type, on row 0 of X. Note that the closure conditions for permutations of variables, subtypes, and type amalgamation are trivially satisfied for row 0 (together with any other row in the case of type amalgamation). To satisfy $\mathbf{R}_{0,j}$, we will ensure that for every 1-ary formula θ_j consistent with T, there is some row that corresponds to a 1-type containing θ_j. Hence, we only use homogeneity markers to satisfy the above requirements for rows that correspond to n-types for $n \geq 1$. We will not use homogeneity markers to satisfy positive requirements involving row 0 or other rows that correspond to the 0-type T.

6 The Negative Requirements

Let $\mathbf{d} \leq \mathbf{0}'$ be a low$_2$ degree. Our goal is to build a $\mathbf{0}$-basis $X = \{p_i\}_{i \in \omega}$ for a homogeneous model $\mathcal{A}$ such that X has no $\mathbf{d}$-monotone extension function. For X to be a $\mathbf{0}$-basis for a homogeneous model, X must satisfy the homogeneity closure conditions described above in Theorem 5.1.

6.1 A Characterization of Low$_2$ Δ^0_2 Degrees

We use the following characterization of the low$_2$ Δ^0_2 degrees to enumerate all the $\mathbf{d}$-computable functions. We use this enumeration to ensure that no $\mathbf{d}$-computable function can be a $\mathbf{d}$-monotone extension function for the $\mathbf{0}$-basis X that we are building.

Theorem 6.1 (Derived from Jockusch [12], see [5] p. 1125)**.** *A degree* $\mathbf{d} \leq \mathbf{0}'$ *is low*$_2$ *if and only if the* $\mathbf{d}$*-computable functions are* $\mathbf{0}'$*-uniform. In other words, there exists a function* $g \leq \mathbf{0}'$ *such that if* $g_e(x) = g(e, x)$*, then* $\{g_e\}_{e \in \omega} = \{f : f \leq \mathbf{d}\}$.

Applying the limit lemma to the above, we obtain:

Corollary 6.2. *If* $\mathbf{d} \leq \mathbf{0}'$ *is low*$_2$*, then there exists a computable function* $g(e, x, s)$ *such that the function* $g_e(x) = \lim_s g(e, x, s)$ *exists for all* e *and* x *and* $\{g_e : e \in \omega\} = \{f : f \leq \mathbf{d}\}$.

Let g_e denote the e^{th} $\mathbf{d}$-computable function. Let $g_{e,s}(x) = g(e, x, s)$ be the computable approximation to g_e at stage s. At each stage s, we have a computable approximation $\{g_{e,s}\}_{e\in\omega}$ to the list $\{g_e\}_{e\in\omega}$, and, hence, to a list of the $\mathbf{d}$-computable functions.

6.2 Refining the Negative Requirements

Given this listing of $\mathbf{d}$-computable functions, $\mathbf{N}$ can be restated as, for all e,

$\mathbf{N}_e$: The function g_e is not an MEF for the $\mathbf{0}$-basis X.

To show that g_e is not an MEF, it suffices to show that g_e does not behave like an MEF on a particular $p_i(\bar{x})$ and $\theta_j(\bar{x}, y)$. Fix i and j. Let $\Lambda_s(t) = g_{e,s}(\langle i, j, t\rangle)$ and $\Lambda(t) = \lim_{s\to\infty} \Lambda_s(t) = g_e(\langle i, j, t\rangle)$.

Definition 6.3. Let $X = \{p_i\}_{i\in\omega}$ be a basis. Let $p_i(\bar{x})$ be an n-type, and let $\theta_j(\bar{x}, y)$ be an $(n+1)$-ary formula consistent with p_i. Let $\Lambda(t)$ be a function from ω to ω.

1. We say Λ *rests on row* k *at level* t if $\Lambda(t) = k$ and Λ *settles* on row k if $\lim_{t\to\infty} \Lambda(t) = k$ exists.

2. We say that Λ *traces out an amalgamator through level* t for an n-type $p_i(\bar{x})$ and $(n + 1)$-ary formula $\theta_j(\bar{x}, y)$ if:

   (a) $\Lambda(t) = k$
   (Λ rests on row k at level t),

   (b) $p_{\Lambda(t')}\restriction t' = p_{\Lambda(t'+1)}\restriction t'$ for all $t' < t$
   (Λ respects the formula monotonicity property of MEFs),

   (c) $p_k(\bar{x}, y)$ is an $(n + 1)$-type containing θ_j if $j < t$ and consistent with the partial type $p_i\restriction t$
   (Through level t, the type p_k extends p_i and $\{\theta_j\}$), and

   (d) $p_{\Lambda(t')}\restriction t' \neq p_j\restriction t'$ for all $j < \Lambda(t')$ for all $t' \leq t$.
   (Λ rests on the least possible row in X that satisfies the above conditions at each level t. As discussed in Section 3.2, given an arbitrary MEF, we can compute an MEF with this property.)

3. We say Λ *traces out an amalgamator* for p_i and θ_j if Λ does so through t for all $t \in \omega$.

Let $p_i(\bar{x})$ be any n-type and $\theta_j(\bar{x}, y)$ be any $(n+1)$-ary formula consistent with p_i. If g_e is an MEF, there is an MEF $g_{e'}$ whose corresponding Λ will both trace out an amalgamator for p_i and θ_j and settle on some row (which corresponds to the amalgamator being traced out). To ensure that $g_{e'}$ (and hence g_e) is not an MEF, we fix a 1-type $p_i(x)$ and a 2-ary formula $\theta_j(x, y)$ consistent with p_i and build X so that if Λ appears to be tracing out an amalgamator for p_i and θ_j, Λ does not settle on any row.

Suppose Λ rests on row k at level t. If Λ appears to be tracing out an amalgamator, then we build X so that Λ may not settle on row k and continue to trace out an amalgamator. Then, for Λ to continue tracing out an amalgamator, Λ must move from row k to a row l for $l > k$, i.e., $\Lambda(t') = l$ for some $t' > t$ by Condition 2d above. If Λ continues to trace out an amalgamator, we will ensure that Λ must move infinitely often, preventing Λ from settling. Hence, we can describe the requirement $\mathbf{N}_e$ as, for all k,

$\mathbf{N}_{e,k}$: If Λ_e traces out an amalgamator for p_i and θ_j, then Λ_e does not settle on row k.

When it is clear from context, we drop the subscript on Λ_e and simply refer to Λ. Note that we use "stage" and "level" to describe different concepts. We use the stage of a construction to obtain our computable approximation Λ_s to Λ. "Level" refers to the length of time Λ_s has been tracing out an amalgamator (as in Definition 6.3).

6.3 Two Examples

Before venturing into the construction, we start with two examples in order to give some intuition for how the positive and negative requirements interact and how they can be satisfied. Some details are left out here but will be fully described in the proof. We fix a 1-type $p_i(x)$ and a 2-ary formula $\theta_j(x, y)$ consistent with p_i on which to satisfy $\mathbf{N}_{e,k}$ for all k.

6.3.1 Example 1

Suppose that at stage s of the construction there exists some $t \leq s$ such that Λ_s traces out an amalgamator through level t for p_i and θ_j. Furthermore suppose that Λ_s rests on row k at level t. Since p_k extends p_i and θ_j through level t, we wish to force Λ off of row k of X to satisfy $\mathbf{N}_{e,k}$. To do this, we want to extend row k so that p_k is inconsistent with p_i. Then Λ

cannot remain on row k if g_e is an MEF regardless of whether Λ_s is a good approximation of Λ through level t.

Suppose that no homogeneity markers rest on row k. Then there are no constraints from the positive requirements on how we build row k. By the flexibility of the theory that we will define and the fact that at stage s only finitely much of rows k and i have been filled, we can find some unary relation P_l that has not appeared in any formula in p_k or p_i by this stage in the construction. We extend X so $p_k(x, y)$ and $p_i(x)$ disagree on the formula $P_l(x)$ and hence p_k cannot extend p_i. Then $\mathbf{N}_{e,k}$ is forever satisfied because if Λ settles on row k, Λ cannot trace out an amalgamator. In this case, it is easy to satisfy $\mathbf{N}_{e,k}$ because no homogeneity marker rests on row k.

6.3.2 Example 2

Now suppose that row k (on which Λ_s is resting at level t) has a homogeneity marker H resting on it. In the most extreme case, suppose that H is the marker that requires that we build an amalgamator for p_i and θ_j. In this case, we say row k and H are *dependent* on row i.

We wish to satisfy $\mathbf{N}_{e,k}$ by making p_k inconsistent with $p_i \cup \{\theta_j\}$. But marker H requires that row k be built so that p_k extends $p_i \cup \{\theta_j\}$. Thus, our need to satisfy a homogeneity condition directly conflicts with $\mathbf{N}_{e,k}$. We ensure that p_i and θ_j have an amalgamator in the $\mathbf{0}$-basis X but that this amalgamator row cannot be found $\mathbf{d}$-monotonically.

Suppose Λ traces out an amalgamator (otherwise $\mathbf{N}_{e,k}$ is satisfied automatically). To resolve the tension between the positive and negative requirements, we exploit our assumption that Λ satisfies the formula monotonicity property, i.e., condition 2b. in Definition 6.3. We also allow homogeneity marker H to move finitely often to a different row. We will find two possible consistent extensions for row k that differ on some formula $\theta_{split^k_k}$. Since rows i and k and formula θ_j contain only finitely much information at stage s, they have not commented on some binary relation R_l. By the flexibility of the theory, there are two ways to extend row k so that one extension contains $R_l(x, y)$ and the other contains $\neg R_l(x, y)$ and both extensions are consistent with p_i and θ_j.

At this stage we extend row k in one direction of the splitting and we build the other direction on an empty row $k' > k$ (i.e., one row contains $R_l(x, y)$ and the other contains $\neg R_l(x, y)$). Let $\theta_{split^k_k}$ be the formula $R_l(x, y)$ in the effective enumeration of all formulas, and let $split^{k'}_k = split^k_k$. We say that row k and k' *split* at $\theta_{split^k_k}$. We call rows k and k' and the types they correspond to a *splitting* of the partial type corresponding to row k at stage

s, and we call this process *building a splitting* of row k in marker H on rows k and k'. Then we will see which extension (if any) Λ_s traces out.

If Λ_s traces out an amalgamator through level $split_k^k$, either Λ_s will move off of rows k and k' by level $split_k^k$ or Λ_s will decide whether the type it is tracing out will include $R_l(x, y)$ or $\neg R_l(x, y)$, i.e., which direction of the splitting $split_k^k$ it will follow. At that point, we will make row i inconsistent with the row that Λ_s chose, and we will move the marker H onto the other row. In either case, if Λ_s correctly approximates Λ through level t, $\mathbf{N}_{e,k}$ will be satisfied forever (since Λ respects formula monotonicity). Moreover, we continue to have a row on which to satisfy H. If Λ_s does not correctly approximate Λ through level t, we repeat this procedure at the same splitting formula. Eventually, the approximation of Λ will be correct and the strategy will succeed.

We will show later that only finitely many Λ can move a given homogeneity marker such as H and that each such Λ can only move a single marker finitely many times. Thus, each homogeneity marker moves only finitely often as desired.

The strategy of setting and monitoring splittings can be generalized to deal with any case where both row k and Λ depend on row i.

7 Enacting the Strategy

In the second example above, we saw how a row in the matrix of types X can be dependent on another row via a homogeneity marker. We can think of these dependent rows as being generated (according to the homogeneity markers) by other rows. In the next section we formalize this notion of dependency.

7.1 Row Dependencies from Homogeneity Markers

Definition 7.1. Suppose that H rests on row k at stage s and that H is a homogeneity marker made to satisfy some closure condition for row j. Then row k is *directly dependent* on row j via H at stage s. A row without a homogeneity marker at stage s is an *independent row* at stage s.

Definition 7.2. We define $R_{k,s}$, the *dependency graph for row k at stage s*, as follows. Let the rows of X denote the nodes in a directed graph G. We include an edge from row i to row j in G if row i is directly dependent on row j at stage s. Let $R_{k,s}$ be the subgraph of G consisting of all nodes and edges that are on some path in G beginning at row k. We say that row k is *generated by* or *depends on* row j if row j is a node in $R_{k,s}$ and $j \neq k$.

Our construction will ensure that if row k depends on row j, then $k > j$. The nodes in $R_{k,s}$ with no outgoing edges will be independent rows at stage s. Recall that the closure conditions rows corresponding to the theory T will be satisfied without using homogeneity markers. Thus, for $k > 0$, no $R_{k,s}$ contains such a row as a node.

Definition 7.3. We inductively define the *rank of a row k at stage s* as follows:

(i) $rank_s(k) = 0$ if k is an independent row at stage s, and

(ii) $rank_s(k) = \max_{j \in R_{k,s}, j \neq k} rank_s(j) + 1$, otherwise.

Rank is well defined by the definition of $R_{k,s}$ and by our construction assumption that if k depends on j, then $k > j$.

Dependency graphs help us determine whether we will act as in the first example above or the second. Suppose Λ_e appears to be tracing out an amalgamator for an independent row corresponding to p_{i_e} and θ_{j_e}. If Λ_e rests on row k and row $i_e \notin R_{k,s}$, we will show that there is a way to consistently extend rows i and k while respecting the homogeneity marker on row k so that p_i and p_k are inconsistent as in the first example. We will also prove that we can enact the splitting strategy described in the second example if $i_e \in R_{k,s}$.

7.2 A Simple Complete Decidable Theory

We define a flexible CD theory T that will be the theory of our counterexample $\mathcal{A}$. We utilize the flexibility of the theory to satisfy requirements **P** and **N** simultaneously.

7.2.1 The Theory T

The language of T is $L = \{P_0, P_1, P_2, P_3, \ldots, R_0, R_1, R_2, R_3, \ldots\}$ where P_i is a unary relation symbol and R_i is a binary relation symbol for all $i \in \omega$. Let $L_s = \{P_0, \ldots, P_{s-1}, R_0, \ldots, R_{s-1}\}$ for $s \geq 0$; note $L_0 = \emptyset$.

We say an L_s-formula $\delta(x_0, x_1, \ldots x_n)$ is *atomically complete* if δ is a conjunction of atomic and negated atomic L_s-formulas such that every atomic L_s-formula in the variables $x_0, \ldots, x_n$ occurs exactly once (either positively or negatively) in this conjunction. (In other words, δ describes the atomic diagram generated by $x_0, \dots, x_n$.) Let $T = \bigcup_{s\in\omega} T_s$ where T_s theory in L_s generated by the following axiom schema.

$$Ax_s \quad (\forall x_0 \dots \forall x_{n-1})[\delta'(x_0, x_1, \dots, x_{n-1}) \rightarrow (\exists x_n)\delta(x_0, x_1, \dots, x_{n-1}, x_n)],$$

where δ and δ' are consistent atomically complete formulas in variables $x_0, \dots, x_n$ and variables $x_0, \dots, x_{n-1}$ respectively and δ' is a subformula of δ.

Theorem 7.4. *The set $T = \bigcup_{s\in\omega} T_s$ is a complete decidable theory, and T admits quantifier elimination.*

This follows from the next series of lemmas.

Lemma 7.5. *For $s \geq 0$, T_s is consistent.*

Proof. The theory T_0 is consistent. Assume $s > 0$. Let $\mathcal{A}_0$ be a model in L_s of T_0. We construct a model $\mathcal{A}$ of T_s. Let $(\theta_1, \bar{a}_1), (\theta_2, \bar{a}_2), \dots$ be an enumeration of all pairs consisting of an axiom in Ax_s and a tuple from $\mathcal{A}_0$ whose length equals the number of variables in the δ' of the axiom. We extend $\mathcal{A}_0$ to $\mathcal{A}_1$ in such a way that $\mathcal{A}_1$ satisfies the axioms in Ax_s when the universal quantifiers in Ax_s are restricted to the universe of $\mathcal{A}_0$. The universe of the model $\mathcal{A}_1$ consists of A_0 and infinitely many new constants $\{b_1, b_2, \dots\}$.

Let θ_1 be of the form

$$(\forall x_0 \dots \forall x_{n-1})[\delta'(x_0, x_1, \dots, x_{n-1}) \rightarrow (\exists x_n)\delta(x_0, x_1, \dots, x_{n-1}, x_n)].$$

If $\mathcal{A}_0 \models \delta'(\bar{a}_1)$, extend the definitions of the predicates in L_s to the set $\{a_0, a_1, \dots, a_{n-1}, b_1\}$ so that

$$\mathcal{A}_1 \models \delta(a_0, a_1, \dots, a_{n-1}, b_1).$$

Repeat this process for all θ_i, $\bar{a}_i$, and b_i in order to obtain $\mathcal{A}_1$. Then obtain an extension $\mathcal{A}_2$ of $\mathcal{A}_1$ by taking an enumeration of all pairs of axioms and tuples from $\mathcal{A}_1$ of the appropriate length, adding infinitely many new constants and proceeding as above. Continuing similarly, we construct a chain of models

$$\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$$

If $\mathcal{A} = \bigcup_{s\geq 0} \mathcal{A}_s$, then $\mathcal{A} \models T_s$. $\square$

Lemma 7.6. *For $s \geq 0$, T_s is $\aleph_0$-categorical.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be countable models of T_s. We show that they are isomorphic. Assume f is a finite (partial) isomorphism from $\mathcal{A}$ to $\mathcal{B}$ and $dom(f) = \{a_0, a_1, \ldots, a_{n-1}\}$. Let $\delta'(x_0, x_1, \ldots, x_{n-1})$ be the finite diagram of $\mathcal{A}$ determined by $dom(f)$, let a be an element of $A \setminus dom(f)$, and let $\delta(x_0, x_1, \ldots, x_{n-1}, x_n)$ be the finite diagram of $\mathcal{A}$ given by $dom(f) \cup \{a\}$. Now $\mathcal{B} \vDash \delta'(f(a_0), f(a_1), \ldots, f(a_{n-1}))$. Thus, there is a $b \in B$ such that

$$\mathcal{B} \vDash \delta(f(a_0), f(a_1), \ldots, f(a_{n-1}), b).$$

Then $f_1 = f \cup \{(a, b)\}$ is a finite isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Symmetrically, we can extend f to include a given $b \in B$ in the range of an extension of f. □

Lemma 7.7. *For $s \geq 0$, T_s is complete and decidable.*

Proof. T_s has no finite models since $(\exists x_0, \ldots, x_n)[\bigwedge_{i \neq j, i, j \leq n} x_i \neq x_j]$ is provable from T_s for all n. Since T_s is $\aleph_0$-categorical, by the Łoś-Vaught Test, it is complete. Since T_s is complete and computably axiomatizable, it is decidable. □

Lemma 7.8. *For $s \geq 0$, T_s admits elimination of quantifiers.*

Proof. We show that T_s is submodel complete and hence admits quantifier elimination. Let $\mathcal{A}, \mathcal{B} \vDash T_s$ and $\mathcal{D} \subseteq \mathcal{A}, \mathcal{B}$. We show $\mathcal{A}$ and $\mathcal{B}$ satisfy the same existential sentences in L_s with parameters from $\mathcal{D}$. Let $\mathcal{A} \vDash \theta(d_0, \ldots, d_{n-1}, a_0, \ldots, a_{m-1})$, where $d_0, \ldots, d_{n-1} \in D$ and $a_0, \ldots, a_{m-1} \in A \backslash D$. Extend the identity function on $\{d_0, \ldots, d_{n-1}\}$ to a finite isomorphism f from $\mathcal{A}$ to $\mathcal{B}$ such that $a_0, \ldots, a_{m-1} \in dom(f)$ as in the categoricity lemma above. Then

$$\mathcal{B} \vDash (\exists x_0 \ldots \exists x_{m-1})\, \theta(d_0, \ldots, d_{n-1}, x_0, \ldots, x_{m-1}).$$

□

7.3 Satisfying Positive Requirements

Let X_s be the stage s approximation of the $\mathbf{0}$-basis X we are building for the homogeneous model $\mathcal{A}$, and let $QF(\theta)$ be the least-indexed quantifier-free formula equivalent to θ. We say a formula θ is *decided* in row k at stage s if the partial type p corresponding to row k in X_s includes θ or $\neg\theta$. We consider each homogeneity closure requirement, and we assume that at each stage s of the construction we have the following conditions.

7.3.1 Conditions on X

- Each row in X_s is consistent with T.
- For all $k > 0$, only finitely many formulas in row k of X_s have been decided.
- For all $k > 0$, if $\theta \in p_k$ at stage s, then all the atomic formulas in $QF(\theta)$ are decided in row k at stage s.

  Suppose row k is directly dependent on row i via homogeneity condition H at stage s. Let ρ denote the conjunction of all of the literals included in row k at stage s.

- If $H = Perm_{i,m}$ or $H = Sub_{i,m}$, then row k is the correct permutation of variables or subtype of the type in row i.
- If $H = Tr_{i,j}$ and $p_i(\bar{x})$ and $\theta_j(\bar{x}, y)$ are consistent, then

$$T \cup p_i(\bar{x}) \vdash (\exists y)[\rho(\bar{x}, y) \wedge QF(\theta_j(\bar{x}, y))]. \tag{1}$$

- Suppose $H = Tap_{i,j}$, and $p_i(\bar{x}, y)$ and $p_j(\bar{x}, z)$ share the free variables $\bar{x}$. If there exists some $q(\bar{x})$ extending the subtypes in $\bar{x}$ of the types corresponding to rows i and j at stage s (i.e., p_i and p_j agree on $\bar{x}$), then $T \cup p_i(\bar{x}, y) \vdash (\exists z)\rho(\bar{x}, y, z)$ and $T \cup p_j(\bar{x}, z) \vdash (\exists y)\rho(\bar{x}, y, z)$.

We show that we can computably build X by finite extensions while ensuring that these conditions are maintained.

Lemma 7.9. *Suppose X_s satisfies the conditions above, and let row k of X_s correspond to partial type $p_k(\bar{x})$. Let F be a finite set of formulas in $\bar{x}$. Then there exists a finite extension X_{s+1} of X_s so that all the formulas in F are decided in row k and the above conditions are maintained. Moreover, only the rows in $R_{k,s}$ are extended in creating X_{s+1}, and X_{s+1} is uniformly computable from X_s.*

Proof. We prove the lemma by induction on the rank of row k at stage s. Let ρ be the conjunction of all the literals included in row k at stage s.

Suppose $rank_s(k) = 0$. Then row k has no homogeneity marker resting on it. We show how to extend k to decide some formula θ. First, extend row k to include all the (positive) atomic subformulas of $QF(\theta)$ that are not decided in row k and exclude their negations. This is a consistent extension by definition of T and the consistency of row k of X_s. Check whether

$T \cup \{\rho'\} \vdash QF(\theta)$ or $T \cup \{\rho'\} \vdash \neg QF(\theta)$ where ρ' is the conjunction of ρ and the included atomic formulas. Since all literals in $QF(\theta)$ are decided, one of the statements holds. Include or exclude θ in row k respectively. Note that all conditions are maintained and that this is a computable extension. To decide a finite set of formulas, repeat this process.

Suppose $rank_s(k) = n + 1$ and that the lemma holds for all rows of rank less than or equal to n. Suppose marker $Tr_{i,j}$ rests on row k, row i corresponds to an l-type $p_i(\bar{x})$, $rank_s(i) = n$, and $\theta_j(\bar{x}, y)$ is an $(l + 1)$-formula.

First, suppose row k is empty. By induction, extend row i to decide all the atomic subformulas of $QF(\,(\exists y)\,[\theta_j(\bar{x}, y)]\,)$ if they are not already decided. If this extension implies $(\exists y)\,[\theta_j(\bar{x}, y)]$, then p_i and θ_j are consistent. We extend row k to include θ_j and consistently decide all the atomic formulas in $QF(\theta_j)$ and extend row i so that this extension ρ of row k proves $QF(\theta_j)$ and row i proves $(\exists y)[\rho(\bar{x}, y) \wedge QF(\theta_j)]$. This extension is computable since T is decidable. Otherwise p_i and θ_j are not consistent, and we can treat row k as in the rank zero case.

Assuming $p_i(\bar{x})$ and $\theta_j(\bar{x}, y)$ are consistent and row k is nonempty, we show how to extend row k to decide some formula θ. Let S be the set of all the (positive) atomic subformulas of $QF(\theta)$ that are not subformulas of ρ. By the conditions, $T \cup p_i(\bar{x}) \vdash (\exists y)[\rho(\bar{x}, y) \wedge QF(\theta_j)]$. Let $S_{\bar{x}}$ be all the formulas in $\bar{x}$ in S. Let $S_y = S \setminus S_{\bar{x}}$, i.e., atomic formulas in variables including y in S.

By the inductive hypothesis, we can finitely (and computably) extend row i and rows dependent on row i to decide the atomic formulas in $S_{\bar{x}}$ and continue satisfying the conditions. Extend row k to decide the atomic formulas in $S_{\bar{x}}$ the same way row i did. Let $\nu_{\bar{x}}$ be the conjunction of the literals included in row i with subformulas in $S_{\bar{x}}$. Let $\rho' = \rho \wedge \nu_{\bar{x}}$. By the axioms of T, we have $T \cup p_i \vdash (\exists y)[\rho'(\bar{x}, y) \wedge QF(\theta_j)]$.

Extend row k to include all the positive atomic formulas in S_y, and include or exclude θ accordingly. Since the relations in S_y have not been mentioned in row k, this extension of row k is consistent by the axioms of T. Let ρ'' denote the conjunction of all literals included in row k at this point. Once again, by the axioms of T, $T \cup p_i \vdash (\exists y)[\rho''(\bar{x}, y) \wedge QF(\theta_j)]$ so the conditions are maintained.

The case where marker $Tap_{i,j}$ rests on row k and rows i and j have rank at most n is similar, and the cases where markers $Perm_{l,m}$ or $H = Sub_{l,m}$ rest on row k are straightforward. $\square$

7.4 Defeating Λ_e on row k if $i_e \notin R_{k,s}$ and $rank(i_e) = 0$

Here we show that if Λ_e rests on row k at stage s and $i_e \notin R_{k,s}$ for i_e an independent row, then we can extend rows k and i_e so that p_k and p_{i_e} are inconsistent and X_{s+1} satisfies the conditions in §7.3. Thus, if we are in this case, we can immediately extend the **0**-basis X so that Λ_e cannot settle on row k and trace out an amalgamator for p_{i_e} and θ_{j_e}. We show this lemma for independent rows i since all rows i_e in the construction will be independent.

Lemma 7.10. *If row $i \notin R_{k,s}$, $rank_s(i) = 0$, and X_s satisfies the conditions in §7.3, then there exists a finite extension X_{s+1} of X_s satisfying the conditions in which $p_i(\bar{x})$ and $p_k(\bar{x}, y)$ are inconsistent. Moreover, X_{s+1} extends only rows i and rows in $R_{k,s}$ and is uniformly computable from X_s.*

Proof. Choose the least unary relation P_l such that P_l is not mentioned in any of the rows in $R_{k,s}$ or in row i. Use Lemma 7.9 to finitely extend row k to decide $P_l(x)$ where x is the first variable of p_k. This requires only finitely extending the rows in $R_{k,s}$. Since $i \notin R_{k,s}$ and $rank_s(i) = 0$, we can finitely extend row i to decide P_l in the opposite manner to row k on x. This is a consistent extension by definition of T. Note that all conditions still hold, and this is a computable extension.

□

7.5 The Splitting Strategy for Defeating Λ_e on k if $i_e \in R_{k,s}$

We now formally develop the splitting strategy described in the second example in §6.3. If Λ_e rests on row k at stage s and $i_e \in R_{k,s}$, i.e., row k is dependent at stage s on row i_e, then we may not be able to make p_{i_e} inconsistent with p_k *and* respect the homogeneity marker on row k.

We show that we can compute two incompatible extensions of row k so that both extensions respect the homogeneity marker H on row k. We extend row k to one of the extensions and build the other on an empty row $k' > k$. We wait for Λ_e to decide the splitting or leave both rows. If Λ_e decides the splitting, we then make the extension Λ_e is following inconsistent with p_{i_e} and satisfy H on the other extension.

If H is $Tr_{i,j}$ or $Tap_{i,j}$, the theory has enough flexibility to set a splitting directly in row k. However, if H is $Perm_{j,l}$ or $Sub_{j,l}$, row k must be built according to row j. In this case, we cannot directly build a splitting of row k. Instead we must create a splitting of the rows on which it depends. Then this splitting will "percolate up" to provide a splitting of row k. The following definitions and lemma describe the pairs of variables in which a splitting can be made.

Definition 7.11. Let p be the partial type corresponding to row k in X_s. Suppose H is a homogeneity marker resting on row k at stage s.

1. Marker H is a *marker in variables $\bar{x}$ at stage s* if $\bar{x}$ is the set of free variables in p.

2. Let x and y be variables in H. We say H *allows a splitting in variables x and y* if there exists some formula $\theta(x, y)$ so that q_1 and q_2 are partial types such that $p \subseteq q_i$, we have $\theta(x, y) \in q_1$ and $\neg\theta(x, y) \in q_2$, both q_1 and q_2 respect the homogeneity condition H, and q_1 and q_2 are uniformly computable from X_s and $R_{k,s}$. We say that q_1 and q_2 *split* p at θ. We will choose a particular θ satisfying the above and denote its index in our enumeration of formulas as $split_k^k = split_k^{k'}$.

Lemma 7.12. *Let H rest on row k at stage s.*

1. *If $H = Tr_{i,j}$ is the homogeneity marker for type $p_i(\bar{x})$ and formula $\theta_j(\bar{x}, y)$, then H allows a splitting at stage s between any variable in $\bar{x}$ and y.*

2. *If $H = Tap_{i,j}$ is the homogeneity marker for types $p_i(\bar{x}, y)$ and $p_j(\bar{x}, z)$ and these types agree on $\bar{x}$, then H allows a splitting at stage s between variables y and z.*

Note if H is $Perm_{j,m}$ or $Sub_{j,m}$, H does not allow any (direct) splittings between any variables at any stage.

Proof. Suppose $H = Tr_{i,j}$, and let p_k be the partial type generated by row k at stage s. Suppose $p_i(\bar{x})$ and $\theta_j(\bar{x}, y)$ have not proved themselves inconsistent by stage s. Let x be a variable in $\bar{x}$. Let R_l be the least two-ary relation such that R_l is not mentioned in θ_j or any partial n-type corresponding to a row in X_s for $n \geq 1$. Then $R_l(x, y)$ and $\neg R_l(x, y)$ are both consistent with p_i, p_k, and θ_j by definition of T. Let q_1 be $p_k \cup \{R_l(x, y)\}$ and q_2 be $p_k \cup \{\neg R_l(x, y)\}$. Let $split_k^k$ and $split_k^{k'}$ equal the least index j such that $\theta_j(x, y) = R_l(x, y)$. Then q_1 and q_2 split p_k at $\theta_{split_k^k}$. The other case is similar.

□

Note that the finite extensions that generate the splitting in the proof above can be found effectively and will satisfy the conditions in §7.3 by the same argument as in Lemma 7.9. Lemma 7.12 shows that splittings can be made in some homogeneity markers H and between certain variables.

We develop a method to build splittings in other situations that relies on Lemma 7.12.

The next lemma states that when row k is dependent on row i, we will be able to implement either the direct diagonalization strategy (like in Example 1) or the splitting strategy (like in Example 2). We need some definitions first.

Definition 7.13. Let $l \in R_{k,s}$. We say that *row k has a forced subtype $p(\bar{x})$ of row l at stage s* if $p(\bar{x})$ is a subtype of row k and the homogeneity markers relating the rows in $R_{k,s}$ require that $p(\bar{x})$ is a subtype of row l under some permutation of variables.

Let $p(x,y)$ be a subtype of row k. Suppose that $p(x,y)$ is a forced subtype of a row $l \in R_{k,s}$ at stage s and that $p(x,y)$ corresponds to the subtype $\hat{p}(\hat{x},\hat{y})$ of row l under some permutation of variables. Suppose a splitting in variables $\hat{x}$ and $\hat{y}$ is built on rows l and l'. By construction, there is a row k' related (via homogeneity markers) to row l' in the same way row k is related to row l. Row k and row k' form a splitting of $p(x,y)$ in variables x and y. We say the splitting on rows l and l' *generates the splitting* of $p(x,y)$ on rows k and k', and we refer to the splitting on rows k and k' as a *generated* (rather than a *direct*) splitting. If we require a splitting of row k as above to satisfy $\mathbf{N}_{e,k}$, we denote the index of the formula θ that splits rows l and l' by $split^{l}_{e,k} = split^{l'}_{e,k}$ and the index of the corresponding formula that splits rows k and k' by $split^{k}_{e,k} = split^{k'}_{e,k}$. As before, we supress the index e referring to $\mathbf{N}_{e,k}$ in this notation if the particular requirement is not specified.

In the construction, row i_e of the basis we are building we be an independent 1-type in variable x_0 for all e, and θ_{j_e} is a formula in x_0 and x_1 for all e. Hence, if Λ_e is an MEF, Λ_e may only rest on rows that correspond to 2-types in variables x_0 and x_1. By definition of $p_{i_e}(x)$ and $\theta_{j_e}(x,y)$, we also ensure that the 1-subtypes $r_1(x)$ and $r_2(y)$ of any 2-type amalgamating p_{i_e} and θ_{j_e} are distinct.

Lemma 7.14. *Suppose H rests on a row k corresponding to a 2-type $p(x,y)$ with distinct 1-types at stage s, and suppose an independent row i_e corresponding to type $p_{i_e}(x)$ is in $R_{k,s}$. Given X_s, there exists an extension X_{s+1} of X_s that satisfies the conditions in § 7.3 such that one of the following cases holds. Moreover, X_{s+1} and determining the case that holds are uniformly computable in X_s.*

- *(Immediate Diagonalization) The partial type $p(x,y)$ in X_{s+1} is inconsistent with $p_{i_e}(x)$ but respects marker H.*

- *(Direct Splitting) Row k and some row k' in X_{s+1} split the partial type p at a formula indexed by $split^k_k = split^k_{k'}$. If H is a homogeneity marker resting on row k at stage s, then H rests on k at stage $s+1$, and H', a copy of H, rests on row k'. For any n, the basis X_{s+1} can be chosen so that the index $split^k_k = split^k_{k'}$ is greater than n.*

- *(Generated Splitting) There exists a row $l \in R_{k,s}$ such that row l and some row l' in X_{s+1} split the partial type q corresponding to row l of X_s at the formula indexed by $split^l_k = split^{l'}_k$. This splitting on rows l and l' generates a splitting of the partial type p on row k. A homogeneity marker L rests on row l at stages s and $s+1$, and L', a copy of L, is placed on l' at stage $s+1$. For any row m on the shortest path from row k to row l in $R_{k,s}$, let row m' denote the row related to row l' in the same way row m is related to row l, and denote the index of the formula at which rows m and m' split by $split^m_k = split^{m'}_k$. (Formula $\theta_{split^m_k}$ corresponds to $\theta_{split^l_k}$ via the relationship described between rows m and l given in $R_{k,s}$.) Finally, for any n, the basis X_{s+1} can be chosen so that for every row m as above, the index $split^m_k = split^{m'}_k$ is greater than n.*

Proof. Suppose H is $Tr_{i,j}$ or $Tap_{i,j}$. Then Lemma 7.12 shows that we can effectively find incompatible extensions $q_1(x,y)$ and $q_2(x,y)$ of $p(x,y)$ that respect H. We finitely extend row k to correspond to q_1, and we build q_2 on the first unmarked empty row k' of X_s not attended to by stage s. Then we place a marker H', a copy of H, on row k'. Let the resulting matrix be X_{s+1}. By the proof of Lemma 7.12, for any n, we can make a splitting on rows k and k' such that $split^k_k = split^k_{k'} > n$. Note that these extensions satisfy the conditions of §7.3.

Now suppose $H = Perm_{j,m}$ or $H = Sub_{j,m}$. If row k is a forced 2-type of an independent row l in $R_{k,s}$ at stage s, then $l \neq i_e$ because row l corresponds to an n-type for $n > 1$ (the subtypes of $p(x,y)$ are distinct). Since row l is independent, we can make a consistent extension of row l that causes row i_e to be inconsistent with the extension of row k generated by the extension of row l (as in Lemma 7.10).

If row k is not a forced 2-type of some independent row l, we can create a splitting in x and y in some row of $R_{k,s}$ with marker L equal to $Tr_{i,j}$ or $Tap_{i,j}$ using Lemma 7.12. We say the *depth* of some row $n \in R_{k,s}$ is the length of the shortest path from row k to row n. Let row l be the row of least depth in the dependency tree $R_{k,s}$ for which such a splitting can be made. Row l exists because k is not a forced 2-type of an independent row.

Specifically, if we trace the ancestry of the 1-subtypes of row k through $R_{k,s}$, row l is the least depth node where these types were joined into a higher arity (at least a 2) type. Let q be the partial type generated by row l in X_s, and let the free variables $\hat{x}$ and $\hat{y}$ in q correspond to x and y in p (via the relationships prescribed by $R_{k,s}$).

By Lemma 7.12, there exist incompatible extensions q_1 and q_2 of q that split in variables $\hat{x}$ and $\hat{y}$ and respect L. Finitely extend row l to correspond to q_1 and build q_2 on the first fresh row l' of X_s. Place a marker L', a copy of marker L, on row l'. Let row k' be related to row l' in the same way as row k is related to row l in $R_{k,s}$. The splitting on rows l and l' between variables $\hat{x}$ and $\hat{y}$ generates a splitting on rows k and k' in variables x and y. Since we will computably keep track of where the homogeneity makers rest, we will be able to compute row k' from l'.

By the proof of Lemma 7.12, for any n, we can make a splitting on row l and row l' such that $split_k^l = split_k^{l'} > n$. There are only finitely many rows on the shortest path in $R_{k,s}$ from row k to row l. For any such row m, the 2-subtype of row m corresponding to the 2-subtype $q_1(\hat{x}, \hat{y})$ on row l is a forced 2-type of row l. Hence, for any n, by knowing $R_{k,s}$, we can create a splitting of row l at a large enough index such that for every row m on the shortest path from row k to row l, $split_k^m = split_k^{m'} > n$. □

Before continuing with the construction, we point out an important difference between direct and generated splittings. Suppose we build a splitting on rows l and l' in marker L in order to generate a splitting on rows k and k' as in the third case of Lemma 7.14. Let row m be some row on the shortest path in $R_{k,s}$ from row k to row l (other than l) with marker M. If we later build a splitting *split* directly in marker M on row m, the splittings *split* and $split_k^m$ do not interfere with one another since the subtype of row m that splits at $split_k^m$ is a forced 2-type of row l, whereas the subtype of row m that splits at *split* is not. Regardless of how *split* is decided, the row marked by M after the decision contains the forced 2-type of row l, so the half of the splitting $split_k^m$ on row m can be transferred if deciding *split* moves M to row m'. On the other hand, if deciding $split_k^l$ moves L from row l to l', the splitting *split* is unaffected since M depends on row l, which is now an independent row. Thus, on a given row the only splittings that can interfere with one another are direct splittings.

8 Construction

We put together the above strategic modules and lemmas to construct the desired counterexample.

8.1 Main Construction

Let $\mathbf{d} \in \Delta^0_2$ be a low$_2$ degree. We will build a $\mathbf{0}$-basis (and hence a $\mathbf{d}$-uniform basis) X for a homogeneous model $\mathcal{A}$ that has no $\mathbf{d}$-MEF.

Construction.

Let the function $g(e, x, s)$ be a computable approximation of a listing of all $\mathbf{d}$-computable functions (as defined in §6.1). Let $i_e = 4e+1$ for all e. Let j_e be the first index such that θ_{j_e} is the conjuction of $\neg P_e(x_1)$ and $x_0 = x_0$ for all e. Let

$$\Lambda_{e,s}(t) = g_{e,s}(\langle i_e, j_e, t\rangle) = g(e, \langle i_e, j_e, t\rangle, s) \tag{2}$$

be the stage s approximation to $\Lambda_e(t) = \lim_s g_{e,s}(\langle i_e, j_e, t\rangle)$.

Stage 0: On row 0, we code the CD theory T described in §7.2 and indicate that it is a 0-type. Place on row i_e for all $e \in \omega$ the finite data that corresponds to the partial 1-type containing $\neg P_i(x_0)$ for all $i < e$ and $P_e(x_0)$. We call these formulas the *coding formulas for* e. Thus, any 2-type $p(x_0, x_1)$ amalgamating $p_{i_e}(x_0)$ and $\theta_{j_e}(x_0, x_1)$ has distinct 1-subtypes. Let e^* denote the maximum of the indices of the formula θ_{j_e} and the formulas included in row i_e at this stage.

Create the countably many homogeneity markers, and effectively place them on rows $\{4j+2\}_{j\in\omega}$ so that $R_{k,0}$ is computable for any row k and $k > l$ if row k depends on l. (Recall that we do not use markers to satsify closure conditions involving row 0 or any other row corresponding to the 0-type T.)

Let $\{\theta_{w(j)}(x_0)\}_{j\in\omega}$ be an effective enumeration of all 1-ary formulas consistent with T. On row $4e+3$ place the finite data that indicates this row corresponds to the partial 1-type in x_0 containing $\theta_{w(e)}(x_0)$ and all the literals in $QF(\theta_{w(e)}(x_0))$ needed to imply $\theta_{w(e)}$. (By including these rows, we satsify the homogeneity closure conditions involving the 0-type T.) For all e, row $4e+4$ remains empty and has no homogeneity marker. Call this matrix X_0.

Stage $s+1$: We are given X_s that satisfies the conditions in §7.3 and in which θ_j has been decided in all rows i for $i, j \leq s$. The matrix X_s also has infinitely many empty unmarked rows with index greater than any row attended to at a previous stage.

<u>$\mathbf{N}_{e,\tilde{k}}$ Requires Attention</u>

We say $\mathbf{N}_{e,\tilde{k}}$ *requires attention* at stage $s+1$ if either the following primary or redecision conditions for attention hold. Requirement $\mathbf{N}_{e,\tilde{k}}$ satisfies the *primary conditions for attention* if the following conditions are satisfied.

1. $\Lambda_{e,s+1}(t') = \tilde{k}$ for some t' such that $e^* < t' < s+1$, and row $\tilde{k}$ corresponds to a 2-type in x_0 and x_1.

   ($\Lambda_{e,s+1}$ is resting on row $\tilde{k}$, and row $\tilde{k}$ contains the coding formulas for e, ensuring that only one Λ_e can require attention when resting on row $\tilde{k}$.)

2. $\Lambda_{e,s+1}$ traces out an amalgamator for p_{i_e} and θ_{j_e} through level t'.

   ($\Lambda_{e,s+1}$ appears to be an MEF.)

3. Row $\tilde{k}$ is consistent with row i_e and θ_{j_e}.

   ($\mathbf{N}_{e,\tilde{k}}$ is not already satisfied forever.)

4. We are not monitoring Λ_e on row $\tilde{k}$ on behalf of $\mathbf{N}_{e,\tilde{k}}$.

Requirement $\mathbf{N}_{e,\tilde{k}}$ satisfies the *redecision conditions for attention* if there is some row k that satisfies the conditions below.

1′ We are monitoring a decision on behalf of $\mathbf{N}_{e,\tilde{k}}$ for redecision at index $\mathit{split} = \mathit{split}^{k}_{e,\tilde{k}}$ on row k.

2′. The most recent splitting that was decided on behalf of $\mathbf{N}_{e,\tilde{k}}$ was built on rows $\hat{k}$ and $\hat{k}'$ at $\mathit{split}^{\hat{k}}_{e,\tilde{k}} = \mathit{split}$. The splitting on rows $\hat{k}$ and $\hat{k}'$ was last decided at stage s' where $s' < s+1$. At stage s', row $\Lambda_{e,s'}(\mathit{split})$ agreed with row $\hat{k}'$ through (and at) index *split*.

3′. At stage $s+1$, $\Lambda_{e,s+1}$ traces out an amalgamator for p_{i_e} and θ_{j_e} through level *split*, and row $\Lambda_{e,s+1}(\mathit{split})$ agrees with row $\hat{k}$ and row k through (and at) index *split*.

   Moreover,

   (a) We have not yet attended to this redecision on row k on behalf of $\mathbf{N}_{e,\tilde{k}}$, and

(b) Requirement $\mathbf{N}_{e,\tilde{k}}$ has not been reset since stage s'.

Attending to $\mathbf{N}_{e,\tilde{k}}$

Suppose $\langle e, \tilde{k}\rangle$ is the least number such that $\mathbf{N}_{e,\tilde{k}}$ requires attention. We attend to $\mathbf{N}_{e,\tilde{k}}$. First, suppose $\mathbf{N}_{e,\tilde{k}}$ satisfies the primary conditions for attention.

Suppose $i_e \notin R_{\tilde{k},s}$, i.e., row $\tilde{k}$ is not dependent on row i_e. Then by Lemma 7.10, we can finitely extend X_s so that row $\tilde{k}$ will be inconsistent with row i_e and this extension satisfies the conditions in §7.3. Then $\mathbf{N}_{e,\tilde{k}}$ is satisfied forever and will never act again. If $i_e \in R_{\tilde{k},s}$ and by Lemma 7.14, we can make an extension to X_s so that row $\tilde{k}$ is inconsistent with row i_e, do so. Then $\mathbf{N}_{e,\tilde{k}}$ is satisfied forever and will never act again.

Otherwise, by Lemma 7.14, we can set a splitting against Λ_e on row $\tilde{k}$ and some row k'. This splitting is either built directly on rows $\tilde{k}$ and k' or is generated by a splitting on rows l and l'. Take row l to be the row of least depth in $R_{\tilde{k},s}$ where a splitting in the appropriate variables can be made, and take row l' to be an unmarked empty row with l' greater than the index of any row attended to so far. Without loss of generality, we assume the splitting is built on rows l and l'. By Lemma 7.14, if marker L rests on row l at stage s, we place a marker L', a copy of marker L, on row l'. For each row $m \in R_{\tilde{k},s}$ on the path of least length from row $\tilde{k}$ to row l and for each row $m' \in R_{k',s}$, the row related to row l' in the same way that row m is related to l, the splitting indexed at $split^{l}_{e,\tilde{k}} = split^{l'}_{e,\tilde{k}}$ on rows l and l' generates the splitting at $split^{m}_{e,\tilde{k}} = split^{m'}_{e,\tilde{k}}$ on rows m and m'.

By Lemma 7.14, we may make the splitting on rows l and l' such that the indices of the splitting formulas satisfy the following splitting priorities.

- $split^{l}_{e,\tilde{k}} > split^{l}_{d,\tilde{n}}$ if $split^{l}_{d,\tilde{n}}$ is the index for a direct splitting on row l and $\langle e, \tilde{k}\rangle > \langle d, \tilde{n}\rangle$
- $split^{\tilde{k}}_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}} > e^*$

If there exists an active direct splitting index $split^{l}_{d,\tilde{n}}$ on row l where $\langle e, \tilde{k}\rangle < \langle d, \tilde{n}\rangle$, reset $\mathbf{N}_{d,\tilde{n}}$, and delete all of its associated splittings. Notice that no splittings have been built on any row m' at this stage by choice of row l'. Monitor Λ_e on $split^{\tilde{k}}_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$.

Now suppose $\mathbf{N}_{e,\tilde{k}}$ satisfies the redecision conditions for attention above for some row k. Then row k is half of a splitting that splits at formula

$\theta_{split^k_{e,\tilde{k}}}$. Let row l be the row of least depth in $R_{k,s}$ where the splitting at $split^k_{e,\tilde{k}}$ can be made, and suppose the formula indexed by $split^l_{e,\tilde{k}}$ in row l corresponds to the formula indexed by $split^k_{e,\tilde{k}}$ in row k. Suppose row l is marked by marker L. Let row l' be an empty unmarked row greater than all rows attended to so far, and let row k' be the row related to l' in the same way row k is related to l. As in the last case, construct a splitting on row l and row l' at $split^l_{e,\tilde{k}} = split^{l'}_{e,\tilde{k}}$ corresponding to the index $split^k_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$, where row k' is the row related to row l' in the same way row k is related to row l. Mark the row l' with a homogeneity marker L' as above. We now monitor Λ_e on $split^k_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$ on rows k and k'.

Deciding Splittings

Suppose we are monitoring Λ_e on $split^k_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$ on rows k and k' on behalf of $\mathbf{N}_{e,\tilde{k}}$. (Note k may equal $\tilde{k}$.) The splitting $split^k_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$ on rows k and k' *is decided* if:

- $\Lambda_{e,s+1}$ traces out an amalgamator for p_{i_e} and θ_{j_e} through level $split^k_{e,\tilde{k}}$.
- Row $\Lambda_{e,s+1}(split^k_{e,\tilde{k}})$ agrees with either row k or k' through (and at) $split^k_{e,\tilde{k}}$.

  ($\Lambda_{e,s+1}$ acts like an MEF on p_{i_e} and formula θ_{j_e} through level $split^k_{e,\tilde{k}}$.)

Let $\langle e, \tilde{k}\rangle$ be the least number such that the splitting $split^k_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$ is decided. Suppose row $\Lambda_{e,s}(split^k_{e,\tilde{k}})$ agrees with row k through $split^k_{e,\tilde{k}}$. As above, we suppose the splitting on rows k and k' is generated by a splitting on rows l and l'. Then we move the marker L on row l to row l' (overwriting the marker L' on row l'). We say that Λ_e *kicks* L on behalf of $\mathbf{N}_{e,\tilde{k}}$. As in Lemma 7.10, we make row k inconsistent with row i_e via an appropriate extension of row l, and delete the splitting indices on behalf of $\mathbf{N}_{e,\tilde{k}}$ previously generated by row l. This action respects the homogeneity conditions since row l is no longer dependent on any other row. Thus, we can make rows k and i_e inconsistent on some P_i not yet mentioned in any row other than row 0 as in the above-mentioned lemmas. We now monitor this decision on behalf of $\mathbf{N}_{e,\tilde{k}}$ for redecision at $split^{k'}_{e,\tilde{k}}$ on row k'.

Reset any requirement $\mathbf{N}_{d,\tilde{n}}$ such that there exists an index for an active direct splitting $split^l_{d,\tilde{n}}$ on row l at this stage and $\langle d, \tilde{n}\rangle > \langle e, \tilde{k}\rangle$. (Although row l' copied row l through $split^l_{e,\tilde{k}}$, we have no control over how row l'

decides the formula at $split^l_{d,\tilde{n}} > split^l_{e,\tilde{k}}$.) Suppose there exists an index for a splitting $split^l_{d,\tilde{n}}$ on row l at this stage that is not reset. This splitting is either a direct splitting (in the marker L) and $\langle d, \tilde{n}\rangle < \langle e, \tilde{k}\rangle$ or this splitting is a generated splitting.

First, suppose $split^l_{d,\tilde{n}}$ corresponds to a direct splitting (in the marker L) and $\langle d, \tilde{n}\rangle < \langle e, \tilde{k}\rangle$. We have $split^l_{d,\tilde{n}} < split^l_{e,\tilde{k}}$ by the splitting priorities. Since row l' copied row l through $split^l_{e,\tilde{k}}$ (i.e., row l' agrees with row l through and at index $split^l_{d,\tilde{n}}$), we can shift the half of the splitting $split^l_{d,\tilde{n}}$ on row l on behalf of $\mathbf{N}_{d,\tilde{n}}$ to row l'. Second, suppose $split^l_{d,\tilde{n}}$ corresponds to a generated splitting. Let the formula indexed by $split^l_{d,\tilde{n}}$ be in free variables $\hat{x}$ and $\hat{y}$. Before L was kicked, the subtype $p(\hat{x}, \hat{y})$ of p_l generated by $\hat{x}$ and $\hat{y}$ was a forced 2-type of a row that generated the splitting indexed at $split^l_{d,\tilde{n}}$. Since L' was a copy of marker L and L now has been kicked to row l', $p(\hat{x}, \hat{y})$ is a forced subtype of $p_{l'}$. Regardless of how $split^l_{d,\tilde{n}}$ compares with $split^l_{e,\tilde{k}}$, row l' will agree with row l on the formula indexed at $split^l_{d,\tilde{n}}$ since both rows l and l' have the same forced 2-types (and agree on every formula in these types). Hence, we can again shift the half of the splitting $split^l_{d,\tilde{n}}$ on row l on behalf of $\mathbf{N}_{d,\tilde{n}}$ to row l'. We now describe this shift in detail.

Suppose we are monitoring splitting $split^n_{d,\tilde{n}} = split^{n'}_{d,\tilde{n}}$ on rows n and n' on behalf of $\mathbf{N}_{d,\tilde{n}}$ where this splitting is generated by $split^u_{d,\tilde{n}}$ and $split^{u'}_{d,\tilde{n}}$ and row n agrees with row $\tilde{n}$ through and at $split^n_{d,\tilde{n}}$. (We act similarly if we are monitoring a decision on behalf of $\mathbf{N}_{d,\tilde{n}}$ for redecision at $split^n_{d,\tilde{n}}$ on row n.) Let $\mathcal{P}$ denote the shortest path in $R_{n,s}$ between row n and row u, and let $\mathcal{P}'$ be the corresponding path in $R_{n',s}$ between row n' and row u'. We consider two cases.

First, suppose row $l \in \mathcal{P}$. Then, by how we construct splittings, $p(x_0)$, the subtype of row n that corresponds to variable x_0, is a forced 1-type of row l. Since row l has no marker, it is an independent row. Thus, we can extend row l so that this extension causes $p(x_0)$ to be inconsistent with p_{i_d}. Then $\mathbf{N}_{d,\tilde{n}}$ is satisfied forever. (If $n = \tilde{n}$, row n is now inconsistent with p_{i_d}. If $n \neq \tilde{n}$, row n was already made inconsistent with p_{i_d} at a previous stage.)

To ensure that the homogeneity markers settle, we continue to monitor any splittings associated with $\mathbf{N}_{d,\tilde{n}}$. For each row m between row n and row l in $\mathcal{P}$, let row m' be the row that is related to row l' in the same way row m is related to row l in $R_{k,s}$. Define $split^{m'}_{d,\tilde{n}} = split^m_{d,\tilde{n}}$ for each such m'. Let row n'' be the row related to row l' in the same way row n is related to row l, and monitor the splitting $split^{n''}_{d,\tilde{n}} = split^{n'}_{d,\tilde{n}}$ on rows n'' and n'. We call this *shifting the splitting* on behalf of $\mathbf{N}_{d,\tilde{n}}$ on rows n and n' to rows n'' and

n'. Note that row l' was not marked with any homogeneity markers when $split^{l'}_{e,\tilde{k}}$ was built on it. Since $\mathbf{N}_{e,\tilde{k}}$ was never reset since the splitting was built, no direct splittings previously existed on row l' below $split^{l'}_{e,\tilde{k}}$.

Second, suppose row $l \in \mathcal{P}'$. We similarly shift the the splitting on behalf of $\mathbf{N}_{d,\tilde{n}}$ on rows n and n' to rows n'' and n' as above, but we are unable to satisfy $\mathbf{N}_{d,\tilde{n}}$ forever.

If $\Lambda_{e,s+1}$ follows the direction of the splitting that row k' follows, our action is symmetric to that above.

Satisfying Homogeneity Conditions

Extend X_{s+1} so that θ_j is decided in all rows i for $i, j \leq s+1$ and satisfies the conditions in §7.3 using Lemma 7.9.

End Construction.

8.2 Verification.

We show that each homogeneity closure requirement is satisfied and that there is no $\mathbf{d}$-MEF for the $\mathbf{0}$-basis X we have built. We first show that $\mathbf{N}_{e,\tilde{k}}$ is satisfied for all $e, \tilde{k} \in \omega$. We then show that all homogeneity markers eventually settle, i.e., are kicked from the row they are resting on at a given stage only finitely often. Each homogeneity requirement will be satisfied if it has a homogeneity marker that eventually settles.

Lemma 8.1. *For all $e, \tilde{k} \in \omega$, $\mathbf{N}_{e,\tilde{k}}$ requires attention or has a splitting decided on its behalf only finitely often.*

Proof. Assume the statement is true for all $\mathbf{N}_{d,\tilde{n}}$ for $\langle d, \tilde{n}\rangle < \langle e, \tilde{k}\rangle$. Choose the least stage $\hat{s}$ after which there is no $\langle d, \tilde{n}\rangle < \langle e, \tilde{k}\rangle$ such that $\mathbf{N}_{d,\tilde{n}}$ requires attention or has any splittings decided on its behalf. Suppose $\mathbf{N}_{e,\tilde{k}}$ requires attention at some stage $s' \geq \hat{s}$. Suppose $\mathbf{N}_{e,\tilde{k}}$ satisfies the primary conditions for attention at stage s'. Then $\mathbf{N}_{e,\tilde{k}}$ receives attention at stage s'. The argument is the same if $\mathbf{N}_{e,\tilde{k}}$ satisfies the redecision conditions for attention at stage s' or if $\mathbf{N}_{e,\tilde{k}}$ never requires attention after stage s' but decides a splitting after stage s'.

If, in receiving attention, $\mathbf{N}_{e,\tilde{k}}$ is satisfied forever without setting a splitting, then $\mathbf{N}_{e,\tilde{k}}$ will never require attention or have a splitting decided on its behalf again. Otherwise, we set a splitting at index $split = split^{\tilde{k}}_{e,\tilde{k}}$ in rows $\tilde{k}$ and k' on behalf of $\mathbf{N}_{e,\tilde{k}}$ and monitor this splitting. Since $\mathbf{N}_{e,\tilde{k}}$ is never reset after stage s', by definition of the redecision conditions for attention,

$\mathbf{N}_{e,\tilde{k}}$ can only require attention at some stage greater than s' if $\mathbf{N}_{e,\tilde{k}}$ must redecide the splitting at index *split*. In this case, when $\mathbf{N}_{e,\tilde{k}}$ requires attention after stage s', the splitting monitored on behalf of $\mathbf{N}_{e,\tilde{k}}$ at this later stage is again indexed by *split*. Thus, after stage s', if we are monitoring a splitting on behalf of $\mathbf{N}_{e,\tilde{k}}$ on rows k and k', this splitting occurs at index *split*.

We claim that we decide a splitting on behalf of $\mathbf{N}_{e,\tilde{k}}$ only finitely many times after stage s'. By the splitting priorities, all direct splittings made on behalf of $\mathbf{N}_{d,\tilde{n}}$ for $\langle d, \tilde{n}\rangle > \langle e, \tilde{k}\rangle$ must respect the direct splittings on behalf of $\mathbf{N}_{e,\tilde{k}}$. (Recall that generated splittings do not injure direct splittings.) Moreover, these requirements cannot reset $\mathbf{N}_{e,\tilde{k}}$. Let s'' be a stage greater than s' such that for all $s \geq s''$, the approximation $\Lambda_{e,s}(x)$ equals $\Lambda_e(x)$ for all $x \leq split$. Suppose $\mathbf{N}_{e,\tilde{k}}$ decides the splitting at index *split* at some stage $s > s''$. Since $\Lambda_{e,s}$ equals Λ_e through index *split*, by the redecision conditions for attention, $\mathbf{N}_{e,\tilde{k}}$ cannot require attention after stage s. Then no more splittings can be decided on behalf of $\mathbf{N}_{e,\tilde{k}}$ after stage s.

$\square$

Lemma 8.2. *Requirement* $\mathbf{N}_{e,\tilde{k}}$ *is satisfied for all* $e, \tilde{k} \in \omega$.

Proof. Assume that $\mathbf{N}_{d,\tilde{n}}$ is satisfied for all $\langle d, \tilde{n}\rangle < \langle e, \tilde{k}\rangle$. Moreover, suppose that all such requirements $\mathbf{N}_{d,\tilde{n}}$ do not require attention or have any splittings decided on their behalf after stage s'. Suppose Λ_e traces out an amalgamator for p_{i_e} and θ_{j_e}, and $\lim_{t\to\infty} \Lambda_e(t) = \tilde{k}$. Let $t' > e^*$ be the least stage such that $\Lambda_e(t') = \tilde{k}$. Then $\Lambda_e(t) = \tilde{k}$ for all $t \geq t'$. Take $s'' \geq max\{s', t'\}$ such that $\Lambda_{e,s}(x) = \Lambda_e(x)$ for all $x \leq t'$ and $s \geq s''$.

At stage s'', requirement $\mathbf{N}_{e,\tilde{k}}$ will require and receive attention via the primary conditions for attention, or we will already be monitoring some splitting on behalf of $\mathbf{N}_{e,\tilde{k}}$ on rows k and k'. Suppose $\mathbf{N}_{e,\tilde{k}}$ receives attention at this stage. We will discuss the other case below. If possible, we extend $X_{s''}$ so that row $\tilde{k}$ is inconsistent with row i_e. This is a contradiction since we assumed Λ_e traces out an amalgamator for p_{i_e} and θ_{j_e}, and $\lim_{t\to\infty} \Lambda_e(t) = \tilde{k}$. Thus, we set a splitting at index $split^{\tilde{k}}_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$ on row $\tilde{k}$ and some empty unmarked row k'. Let $s''' \geq s''$ be a stage by which $\Lambda_e(x) = \Lambda_{e,s}(x)$ for all $x \leq split^{\tilde{k}}_{e,\tilde{k}}$ and all $s \geq s'''$. Then $\Lambda_{e,s}(split^{\tilde{k}}_{e,\tilde{k}}) = \tilde{k}$ for all $s \geq s'''$. By stage s''', our construction will have decided the splitting indexed by $split^{\tilde{k}}_{e,\tilde{k}}$ correctly. Since $\Lambda_{e,s'''}(split^{\tilde{k}}_{e,\tilde{k}}) = \tilde{k}$, we made row $\tilde{k}$ inconsistent with row i_e in $X_{s'''}$ when deciding this splitting. Hence, Λ_e does not trace out an amalgamator for p_{i_e} and θ_{j_e}, a contradiction.

Suppose we are already monitoring a splitting on behalf of $\mathbf{N}_{e,\tilde{k}}$ on rows k and k' at stage s''. If k or k' equals $\tilde{k}$, the argument above holds. Otherwise, the splitting on rows k and k' was originally set in row $\tilde{k}$ and some row $\tilde{k}'$. Since k and k' are unequal to $\tilde{k}$, the splitting on row $\tilde{k}$ was shifted at some point to another row when some splitting was decided. When this shift occurred, row $\tilde{k}$ was made inconsistent with row i_e by the way splittings are decided, a contradiction. $\square$

We now show that all the homogeneity closure conditions are satisfied.

Lemma 8.3. *Let H be any homogeneity marker placed on X at stage* 0*. Let h_s denote the row H rests on at stage s. Then $\lim_{s\to\infty} h_s$ exists.*

Proof. Let H be any homogeneity marker placed on X at stage 0. If H is a marker acting on behalf of a subtype or permutation closure requirement, then the marker H is never moved by construction since we do not set splittings in such markers.

Suppose that H is any other kind of homogeneity marker. We have $h_s \neq h_{s+1}$ if and only if marker H is kicked off of row h_s onto row h_{s+1} when a splitting was decided on behalf of some $\mathbf{N}_{e,\tilde{k}}$ at stage $s+1$. We show H is kicked on behalf of only finitely many $\mathbf{N}_{e,\tilde{k}}$. Since only finitely many splittings are decided on behalf of each $\mathbf{N}_{e,\tilde{k}}$ by Lemma 8.1, H is kicked only finitely many times and the theorem holds. For a contradiction, suppose that H is kicked on behalf of infinitely many $\mathbf{N}_{e,\tilde{k}}$.

We define a sequence of $\langle e_i, \tilde{k}_i\rangle$ and s_i inductively. Let $\langle e_1, \tilde{k}_1\rangle$ be the least $\langle e, \tilde{k}\rangle$ such that H is kicked on behalf of $\mathbf{N}_{e,\tilde{k}}$, and let s_1 be the greatest stage at which this occurs. Suppose we are given $\langle e_i, \tilde{k}_i\rangle$ and s_i for $i < n$ such that H is kicked on behalf of $\mathbf{N}_{e_i,\tilde{k}_i}$ at stage s_i, and H is not kicked on behalf of $\mathbf{N}_{e,\tilde{k}}$ for $\langle e, \tilde{k}\rangle \leq \langle e_i, \tilde{k}_i\rangle$ after stage s_i for all $i < n$. Let $\langle e_n, \tilde{k}_n\rangle$ be the least $\langle e, \tilde{k}\rangle$ such that H is kicked on behalf of $\mathbf{N}_{e,\tilde{k}}$ after stage s_{n-1}, and let s_n be the greatest stage at which H is kicked on behalf of this requirement. For all n, when H was kicked at stage s_n, all direct splittings associated with $\mathbf{N}_{e,\tilde{k}}$ with $\langle e, \tilde{k}\rangle > \langle e_n, \tilde{k}_n\rangle$ on that row are reset. Then, from this stage forward, all direct splittings created for $\mathbf{N}_{e,\tilde{k}}$ with $\langle e, \tilde{k}\rangle > \langle e_n, \tilde{k}_n\rangle$ satisfy $\mathit{split}^l_{e,\tilde{k}} > \mathit{split}^l_{e_n,\tilde{k}_n}$ where rows l and l' generate the splitting on behalf of $\mathbf{N}_{e_n,\tilde{k}_n}$ at this stage. Thus, $p_{h_{s_i}} \restriction \mathit{split}^{h_{s_i}}_{e_i,\tilde{k}_i} \subset p_{h_s}$ for all $s \geq s_i$. Let $p_\infty = \bigcup_{i\in\omega} p_{h_{s_i}} \restriction \mathit{split}^{h_{s_i}}_{e_i,\tilde{k}_i}$. Since $\mathit{split}^{h_{s_i}}_{e_i,\tilde{k}_i} < \mathit{split}^{h_{s_i}}_{e_{i+1},\tilde{k}_{i+1}}$

for all i, $\lim_{s\to\infty} split^{h_{s_i}}_{e_i,\tilde{k}_i} = \infty$. Hence, p_∞ is a complete n-type in variables determined by the homogeneity closure requirement associated with H.

Since H was kicked on behalf of $\mathbf{N}_{e_i,\tilde{k}_i}$ at stage s_i in deciding a splitting built on some rows k and k', row k was being built as a forced 2-type of row h_{s_i-1}. For H to be kicked, row k contains the coding formulas for e_i. Hence, row h_{s_i-1} contains these coding formulas in some of its variables. Since any splitting index is taken greater than the indices of these coding formulas, p_∞ contains these coding formulas in some of its variables.

There exist only finitely many distinct 2-types in x_0 and x_1 that correspond to a permutation of a subtype of p_∞. Thus, there are only finitely many e such that Λ_e can kick H.

Suppose $q(x_0, x_1)$ is one of the finitely many 2-types that is a permutation of a subtype $\hat{q}(\hat{x}_0, \hat{x}_1)$ of p_∞ where $\hat{x}_0$ and $\hat{x}_1$ denote the variables in p_∞ that correspond to the variables x_0 and x_1 in q. There are only finitely many pairs $(\hat{x}_0, \hat{x}_1)$ of such variables in p_∞.

We say that we *set a splitting in H in variables $\hat{x}$ and $\hat{y}$ on behalf of $\mathbf{N}_{e,\tilde{k}}$ at stage s* if $\mathbf{N}_{e,\tilde{k}}$ receives attention at stage s, and at this stage we build a splitting on rows k and k' in variables x and y that is generated by a splitting in variables $\hat{x}$ and $\hat{y}$ on rows l and l' where row l is marked by H and row l' is marked by H', a copy of H.

Let $\langle e'_1, \tilde{k}'_1\rangle$ be the least value such that we set a splitting in H in some pair of variables $\hat{x}_0$ and $\hat{x}_1$ on behalf of $\mathbf{N}_{e'_1,\tilde{k}'_1}$. Let s' be the least stage at which we set this splitting in H. Let $split = split^l_{e'_1,\tilde{k}'_1}$. By choice of $\langle e'_1, \tilde{k}'_1\rangle$, whenever H is kicked after stage s', the splitting at $split$ is preserved and shifted. Let $s'' > s'$ be a stage such that for all $s \geq s''$, we have $\Lambda_{e,s}(x) = \Lambda_e(x)$ for all $x \leq split$ and no $\mathbf{N}_{d,\tilde{n}}$ with $\langle d, \tilde{n}\rangle < \langle e'_1, \tilde{k}'_1\rangle$ acts at stage s. Suppose H rests on row $\hat{l}$ at stage s'' and that row $\hat{k}$ is related to row $\hat{l}$ in the same way row k was related to row l at stage s'. Suppose row $\Lambda_e(split)$ disagrees with row $\hat{k}$ through (and including) $split$. By the note above, this disagreement is preserved whenever H is kicked, i.e., p_∞ extends this disagreement. If row $\Lambda_e(split)$ agrees with row $\hat{k}$ through (and including) $split$, by the redecision conditions for attention and the choice of s'', we set a splitting in H in $\hat{x}$ and $\hat{y}$ on rows $\hat{l}$ and $\hat{l}'$. Let row $\hat{k}'$ be the row related to row $\hat{l}'$ in the same way $\hat{k}$ is related to $\hat{l}$. Then by choice of s'', we kick H off of row $\hat{l}$ onto row $\hat{l}'$ so that row $\Lambda_e(split)$ disagrees with row $\hat{k}$ at index $split$. As in the first case, this disagreement is preserved whenever H is kicked after this stage. Therefore, after stage s'', no $\mathbf{N}_{e,\tilde{k}}$ receives attention via a splitting in H in variables $\hat{x}_0$ and $\hat{x}_1$ unless it already required such attention by stage s''. Only finitely many $\mathbf{N}_{e,\tilde{k}}$ require such attention by stage s''. Since each $\mathbf{N}_{e,\tilde{k}}$ requires attention or decides splittings finitely often, there exists a stage s''' after which Λ_e does not kick H while deciding a splitting in H in variables $\hat{x}_0$ and $\hat{x}_1$.

We continue inductively, defining $\langle e_2', \tilde{k}_2' \rangle$ to be the least value such that there exists or we set a splitting in H in some pair of variables *other than* $\hat{x}_0$ and $\hat{x}_1$ on behalf of $\mathbf{N}_{e_2',\tilde{k}_2'}$ after stage s'''. As above, we can show that there exists a stage after which Λ_e does not kick H while deciding a splitting in H in either of these pairs of variables. Since there are only finitely many pairs of variables in p_∞, by repeating this argument, we see that there exists a stage after which Λ_e does not kick H. (Remember Λ_e can only kick H while deciding some splitting in H in some pair of variables of the row marked by H.) Since there are only finitely many e such that Λ_e kicks H, marker H is kicked only finitely often. Equivalently $\lim_{s\to\infty} h_s$ exists.

□

Theorem 8.4. *Let* $\mathbf{d} \leq \mathbf{0}'$ *be low*$_2$*. There exists a nontrivial homogeneous model* $\mathcal{A}$ *with a* $\mathbf{0}$*-basis but no* $\mathbf{d}$*-decidable copy.*

Proof. By Lemma 8.3, each original homogeneity marker comes to rest on a given row. By construction, then, all the homogeneity conditions are satisfied. Since the construction is effective, we have built a homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis. By Lemma 8.2, no $\mathbf{d}$-computable function can be an MEF for this basis. Thus, by Theorem 3.4, $\mathcal{A}$ has no $\mathbf{d}$-decidable copy. The model $\mathcal{A}$ is nontrivial since $\mathcal{A}$ realizes infinitely many distinct 1-types.

□

9 Further Directions

As mentioned before, the characterization of the Δ_2^0 $\mathbf{0}$-basis homogeneous bounding degrees in Theorem 4.1 is exactly the same as the characterization of the Δ_2^0 prime bounding degrees in Theorem 1.1. Moreover, all of the other known degree theoretic results (found in Csima [3], Hirschfeldt [10], and Lange [14]) are exactly the same for the prime and homogeneous model cases. Recently we have found some reverse mathematical connections between the prime and homogeneous cases that explain some of this degree theoretic evidence. The reverse mathematics of the prime case is studied by Hirschfeldt, Shore, and Slaman in [11]. We, with Hirschfeldt and Shore, are currently exploring these connections more deeply, and we will present them in [9].

References

[1] C.J. Ash and J.F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy*, 1st edn., Stud. Logic Found. Math., vol. 144, Amsterdam, 2000.

[2] C.C. Chang and H.J. Keisler, *Model Theory*, 3rd edn., Stud. Logic Found. Math., vol. 73, North-Holland, Amsterdam, 1990 [1st edn. 1973, 2nd edn. 1977].

[3] B. F. Csima, Degree spectra of prime models, J. Symbolic Logic, vol. 69 (2004), 430-442.

[4] B. F. Csima, V. S. Harizanov, D. R. Hirschfeldt, and R. I. Soare, Bounding homogeneous models, J. Symbolic Logic, vol. 72 (2007), 305-323.

[5] B. F. Csima, D. R. Hirschfeldt, J. F. Knight, and R. I. Soare, Bounding prime models, J. Symbolic Logic, vol. 69 (2004), 1117-1142.

[6] R. Epstein, Prime models of computably enumerable degree, J. Symbolic Logic, vol. 73 (2008), 1373-1388.

[7] S.S. Goncharov, Strong constructivizability of homogeneous models (Russian), Algebra i Logika, **17** (1978) 363-388, 490; [translated in: Algebra and Logic, **17** (1978) 247-263].

[8] V. S. Harizanov, Pure computable model theory, in *Handbook of Recursive Mathematics* (Yu. L. Ershov, S. S. Goncharov, A. Nerode, J. B. Remmel, eds.), Stud. Logic Found. Math., vol. 138–139, Elsevier Science, Amsterdam, 1998, 3–114.

[9] D. R. Hirschfeldt, K. M. Lange, and R. A. Shore, The homogeneous model theorem, in preparation.

[10] D. R. Hirshfeldt, Computable trees, prime models, and relative decidability, Proceedings of Amer. Math. Soc., **134** (2006), 1495-1498.

[11] D. R. Hirschfeldt, R. A. Shore, and T. A. Slaman, The atomic model theorem and type omitting, Trans. of Amer. Math. Soc., **361** (2009), 5805-5837.

[12] C. G. Jockusch, Jr., Degrees in which the recursive sets are uniformly recursive, Canadian J. Math., vol. 24 (1972), 1092-1099.

[13] J. F. Knight, Degrees coded in jumps of orderings, J. Symbolic Logic, vol. 51 (1986), 1034-1042.

[14] K. M. Lange, The degree spectra of homogeneous models, J. Symbolic Logic, vol. 73 (2008), 1009–1028.

[15] K. M. Lange and R. I. Soare, Computability of homogeneous models, Notre Dame J. of Formal Logic, **48** no. 1 (2007), 143-170.

[16] D. Marker, *Model Theory: An Introduction*, Grad. Texts in Math., 277, New York, 2002.

[17] T. S. Millar, Foundations of recursive model theory, Ann. Math. Logic, vol. 13 (1978), 45–72.

[18] T. S. Millar, Homogeneous models and decidability, Pacific J. Math., **91** (1980), 407-418.

[19] M. G. Peretyat'kin, A criterion for strong constructivizability of a homogeneous model (Russian), Algebra i Logika, **17** (1978) 436-454, 491; [translated in: Algebra and Logic, **19** (1980) 202-229].

[20] R. I. Soare, *Computability Theory and Applications* Springer-Verlag, Heidelberg, in preparation.

[21] R. I. Soare, *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets*, Springer-Verlag, Heidelberg, 1987.