Limit computable integer parts

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Abstract

Let R be a real closed field. An *integer part* I for R is a discretely ordered subring such that for every $r \in R$, there exists an $i \in I$ so that $i \leq r < i + 1$. Mourgues and Ressayre [11] showed that every real closed field has an integer part. In [6], it is shown that for a countable real closed field R, the integer part obtained by the procedure of Mourgues and Ressayre is $\Delta^0_{\omega^\omega}(R)$. We would like to know whether there is a much simpler procedure, yielding an integer part that is $\Delta^0_2(R)$—limit computable relative to R. We show that there is a maximal Z-ring $I \subseteq R$ which is $\Delta^0_2(R)$. However, this I may not be an integer part for R. By a result of Wilkie [14], any Z-ring can be extended to an integer part for *some* real closed field. Using Wilkie's ideas, we produce a real closed field R with a Z-ring $I \subseteq R$ such that I does not extend to an integer part for R. For a computable real closed field, we do not know whether there must be an integer part in the class Δ^0_2. We know that certain subclasses of Δ^0_2 are not sufficient. We show that for each $n \in \omega$, there is a computable real closed field with no n-c.e. integer part. In fact, there is a computable real closed field with no n-c.e. integer part for any n.

1 Introduction

Real closed fields are models of the theory of the ordered field of real numbers. Tarski [13] proved that the theory is decidable. To do this, he extracted a computable set of axioms sufficient to carry out an elimination of quantifiers, and to decide the quantifier-free sentences. In addition to the axioms for ordered fields, there are axioms saying that positive elements have square roots, and every polynomial of odd degree has a root.

An integer part for a real closed field R is a subring that sits in R the way the integers sit in the reals. If R is archimedean, then $\mathbb{Z}$ is the unique integer part for R. If R is non-archimedean, then there are many possible integer parts, differing in isomorphism type (see [3], [1]). Integer parts have a strong connection to fragments of arithmetic. Let $IOpen$ denote the weak fragment of PA with induction axioms only for open (quantifier-free) formulas. Shepherdson [12] showed that a discrete ordered ring I is an integer part for some real closed ordered field if and only if it is a model of $IOpen$. In [4], Starchenko and the first two authors showed that if R is a countable real closed field with an integer part satisfying PA, then R is recursively saturated (in fact, the fragment $I\Sigma_4$, with induction axioms for Σ_4 formulas, is enough). If R is recursively saturated, then there are integer parts satisfying all computably axiomatizable extensions of $IOpen$.

Mourgues and Ressayre [11] showed that every real closed field has an integer part. They produced an integer part for a real closed field R by embedding R into a field of formal power series, in which the terms correspond to elements of a well-ordered subset S of the *value group* G, and the coefficients come from the *residue field* k. Below, we say just a little about these objects. For more information, see [11] or [7].

Suppose R is a real closed field. For $x, y \in R^+$, $x \sim y$ if there exist $m, n \in \mathbb{N}$ such that $nx > y$ and $my > x$. The *value group* consists of the $\sim$-equivalence classes, apart from $\{0\}$, under the operation inherited from multiplication on R^+, and with an ordering that is the reverse of the one inherited from R. The value group is a divisible ordered abelian group. The valuation map $w : R^+ \to G$ takes each element to its $\sim$-class. The *residue field* is the quotient of the ring of finite and infinitesimal elements by the ideal of infinitesimal elements. The residue field is an archimedean real closed field. A *section for the value group* G is a subgroup of $(R^+, \cdot)$ that is isomorphic to G, and, similarly, a *section for the residue field* k is a subfield of $(R, +, \cdot)$ that is isomorphic to k. We may use the terms "value group" and "residue field" to refer to these sections.

Given a divisible ordered abelian group G and an archimedean real closed field k, we form $k\langle\langle G\rangle\rangle$, consisting of the formal sums $\sum_{i\in S} a_i g_i$, where $\{\, g_i \mid i \in S\}$ is a well-ordered subset of G and $a_i \in k$. We define addition and multiplication on $k\langle\langle G\rangle\rangle$ as for ordinary power series. The structure $k\langle\langle G\rangle\rangle$ is a real closed field, with value group G and residue field k (up to isomorphism). Moreover, Mourgues and Ressayre showed that any real closed field with value group G and residue field k can be mapped isomorphically onto a subfield F of $k\langle\langle G\rangle\rangle$ that is "truncation closed" (closed under initial segments). Now, F has an integer part, consisting of the elements $s + z$, where s has all infinite support, and $z \in \mathbb{Z}$. The procedure of Mourgues and Ressayre is complicated. We can say *exactly* how complicated it is, using a standard measure from computability.

Recall the jump $X' = \{e : \varphi_e^X(e)\downarrow\}$. Iterating the jump function, and collecting at limit levels, we obtain a sequence of sets of strictly increasing information content: $X^{(0)} = X$, $X^{(\alpha+1)} = (X^\alpha)'$, and for limit ordinals α, $X^\alpha = \{(\beta, x) : x \in X^{(\beta)}, \beta < \alpha\}$. (Although we could continue further, we shall stop with ω^ω.) For finite n, a set S is $\Delta^0_n(X)$ if it is computable relative to $X^{(n-1)}$. For infinite α, a set S is $\Delta^0_\alpha(X)$ if it is computable relative to $X^{(\alpha)}$. A set is $\Delta^0_1(X)$ if it is computable relative to X. A $\Delta^0_2(X)$ set is computable in the limit; i.e., there is a guessing function $g : \omega \times \omega \to \{0, 1\}$, computable relative to X, such that for each x, the value of $g(x, s)$ is eventually constant, with value equal to 1 iff x is in the set. For a $\Delta^0_3(X)$ set, there is a $\Delta^0_2(X)$ guessing function g, etc.

In [6], the second two authors describe a canonical version of the Mourgues and Ressayre construction, for a countable real closed field R, fixing a residue field k and an enumeration $r_1, r_2 \ldots$ of a transcendence basis for R over k. What drives the complexity of the Mourgues and Ressayre construction is the length of the power series assigned to the elements of R. For the canonical construction, the power series all have length less than ω^{ω^ω}—this bound is sharp. The canonical integer part for R is $\Delta_{\omega^\omega}(R)$.

The Mourgues and Ressayre construction produces a very special kind of integer part. In particular, $\mathbb{Z}$ is always a direct summand. It is conceivable that some other procedure would yield a much simpler integer part. In Sections 2 and 3, we consider procedures that produce maximal subrings, which are $\Delta^0_2(R)$, with at least some of the features of an integer part.

In Section 2, we observe that for a countable real closed field R, there is a maximal discrete ordered subring I that is $\Delta^0_2(R)$. This I need not be an integer part. We recall that a *Z-ring* is a discrete ordered ring in which the Division Algorithm holds for divisors in $\mathbb{N}$. Any model of *IOpen* is a Z-ring. Dave Marker and Mojtaba Moniri gave an example of a discrete ordered ring I that cannot be extended to a Z-ring. This I is a subring of a countable real closed field R. By the result of Shepherdson, I cannot be extended to an integer part for R, or for any other real closed field.

Wilkie [14] showed that any Z-ring can be extended to a model of *IOpen*. This extension is an integer part for some real closed field. In Section 3, we observe that for a countable real closed field R, there is a maximal Z-ring $I \subseteq R$ that is $\Delta^0_2(R)$. We show that this need not be an integer part for R. Using ideas of Wilkie, we produce a real closed field R with a Z-ring $I \subseteq R$ such that I cannot be extended to an integer part for R.

We leave open the following question.

Question 1. *For a countable real closed field R, must there be an integer part that is $\Delta^0_2(R)$?*

We do not know whether a computable real closed field has a Δ^0_2 integer part. We consider finite levels of Ershov's difference hierarchy for Δ^0_2 sets. The computably enumerable, or c.e., sets are 1-*c.e.* A set is $(n+1)$-*c.e.* if it has the form $S_1 - S_2$, where S_1 is c.e. and S_2 is n-c.e. In Section 4. we show that for each n, there is a computable real closed field R with no n-c.e. integer part. In fact, there is a computable real closed field R such that R has no n-c.e. integer part for any $n \in \omega$.

Most of our notation is standard. When we do not specify a real closed field, but simply speak of elements, or sets of elements, we suppose that these elements live in a monster real closed field. We may take the real closure of an element a, or a set X, in this monster real closed field. We shall consider, for various n, the subring of $\mathbb{Q}[x_1, \ldots, x_n]$ in which the constant term is an integer. We write $\mathbb{Q}[x_1, \ldots, x_n]^\bullet$ for the subring with constant term 0, and then $\mathbb{Q}[x_1, \ldots, x_n]^\bullet \oplus \mathbb{Z}$ denotes the subring with constant term in $\mathbb{Z}$.

2 Discrete ordered subrings

In this section, we recall some definitions and known results on discrete ordered rings. We prove the simple fact that for a countable real closed field R, there is a maximal discrete subring that is $\Delta^0_2(R)$. Our rings all have unity, which we denote by 1.

Definition 2.1. *A* discrete ordered ring *I is an ordered ring in which 1 is the least positive element.*

Definition 2.2 (Integer part)**.** *An* integer part *for R is a discrete ordered subring I of R such that for each $r \in R$, there exists $i \in I$ such that $i \leq r < i + 1$.*

Proposition 2.1. *For a countable real closed field R, there is a maximal discrete ordered subring that is $\Delta^0_2(R)$.*

Proof. Computably in R, we list the elements $(r_n)_{n\in\omega}$ of R. We form a chain of discrete ordered subrings $(I_n)_{n\in\omega}$. We let I_0 consist of the integers. Given I_n, we let

$$I_{n+1} = \begin{cases} I_n[r_n] & \text{if this is discrete} \\ I_n & \text{otherwise} \end{cases}$$

We let $I = \cup_n I_n$. Clearly, this is a maximal discrete subring of R. We must show that it is $\Delta_2^0(R)$. Note that each I_n is c.e. relative to R. Moreover, the sequence of indices is $\Delta_2^0(R)$, since the property of not being discrete is $\Sigma_1^0(R)$. Now, $r_n \in I$ iff $r_n \in I_{n+1}$. Using $\Delta_2^0(R)$, we can determine whether $r_n \in I$ by finding the index for I_{n+1}, and then checking whether $r_n \in I_{n+1}$.

□

Definition 2.3. *A Z-ring I is a discrete ordered ring such that for any positive $n \in \mathbb{N}$ and any $x \in I$, there exist $q, r \in I$ such that $x = qn + r$ and $0 \leq r < n$.*

Remark 1. *If I is a discrete ordered ring, then I is a Z-ring if and only if for each positive $n \in \mathbb{N}$, $I/(n) \cong \mathbb{Z}/(n)$. If I is a Z-ring, then for each prime p, there is a unique ring homomorphism from I to $\mathbb{Z}_p$—the p-adic integers. Coversely, if I is a discrete ordered ring and for each prime p there is a ring homomorphism $\varphi_p : I \to \mathbb{Z}_p$, then I can be embedded into a Z-ring S. The ring homomorphisms φ_p define S canonically:*

$$S = \{\frac{a}{n} : n \in \mathbb{N},\ a \in I \text{ s.t. } n \text{ divides } \varphi_p(a) \text{ in } \mathbb{Z}_p \text{ for all primes } p\}.$$

Definition 2.4. *A* Euclidean ring *is a discrete ordered ring I such that for all $x, y \in I$, $y > 0$ there exist $q, r \in I$ such that $x = qy + r$ with $0 \leq r < y$.*

Clearly, every Euclidean ring is a Z-ring, but it is not true that every Z-ring is a Euclidean ring, as the following example shows.

Example 1. Let R be the real closure of a, where a is infinite, and let I equal $\mathbb{Q}[a^{\frac{5}{2}}, a]^{\bullet} \oplus \mathbb{Z}$. We cannot express $a^{\frac{5}{2}}$ in the form $aq(a) + r(a)$, where $0 \leq r(a) < a$. The problem is that we did not include $a^{\frac{3}{2}}$ in I. So, I is a Z-ring that does not satisfy the Euclidean Algorithm.

Proposition 2.2. *Any model of IOpen is a Euclidean ring. Hence, it is a Z-ring.*

Proof. Let I be a model of *IOpen* and let $x, y \in I$. Let z be the least element of I such that $x < (z+1)y$. Then $zy \leq x < zy + y$. Let $r = x - zy$. Then $0 \leq r < y$.

□

We give the example (due to Marker and Moniri) of a discrete ordered ring I such that I does not extend to a Z-ring, and I is a subring of a real closed field.

Example 2. Let I be $\mathbb{Z}[x, \frac{x^2+1}{4}]$, where x is an infinite element in our monster real closed field. It is easy to show that the order on $\mathbb{Z}[x, \frac{x^2+1}{4}]$ is discrete. $\mathbb{Z}[x, \frac{x^2+1}{4}]$ does not extend to a Z-ring. Let J be a Z-ring extending $\mathbb{Z}[x, \frac{x^2+1}{4}]$, and $\varphi_2 : J \to \mathbb{Z}_2$ the endowed homomorphism. In Z_2 the remainder of the division of $\varphi_2(x^2+1)$ by 4 is 0, which is in contradiction with the only two possible cases, 4 divides x^2 or 4 divides $x^2 + 3$.

3 Extending Z-rings

In this section, we focus on Z-rings. We show that for a countable real closed field R, there is a maximal Z-ring $I \subseteq R$ such that I is $\Delta^0_2(R)$. Next, we recall Wilkie's proof that every Z-ring extends to a model of *IOpen*. We then give our example of a real closed field R with a Z-ring $I \subseteq R$ such that I does not extend to an integer part for R.

The following simple result says when it is possible to extend a Z-ring so as to include a further element, while staying inside a given real closed field R.

Proposition 3.1. *Suppose R is a real closed field, I is a Z-ring such that $I \subseteq R$, and $a \in R - I$. The following are equivalent:*

1. *there is a Z-ring J such that $I \subseteq J \subseteq R$ and $a \in J$,*
2. *for all $p(x) \in I[x]$, $p(a)$ is either infinite or in $\mathbb{Z}$,*
3. *for all $p(x) \in I[x]$, $p(a) \notin (0,1)$.*

Proof. To prove $1 \Rightarrow 2$, suppose $p(x) \in I[x]$ and $a \in J$. Then $p(a) \in J$, so if it is finite, then it is in $\mathbb{Z}$. To prove $2 \Rightarrow 3$, suppose $p(x) \in I[x]$. Then $p(a)$ cannot be in the interval $(0,1)$ or it would be finite and not in $\mathbb{Z}$. To prove $3 \Rightarrow 1$, we let J consist of the elements of R of the form $\frac{p(a)}{n} + z$, where $p(x) \in I[x]$ has constant term equal to 0, $n \in \mathbb{N}$, and $z \in I$. First, we show that J is discrete. If $p(x)$ has constant term 0, then $p(a)$ must be 0 or infinite, and then $\frac{p(a)}{n}$ is also 0 or infinite. Then $\frac{p(a)}{n} + z$ cannot be in $(0,1)$. Next, we show that J is a Z-ring. Given an element $\frac{p(a)}{n} + z$ and $m \in \mathbb{N}$, we have $\frac{p(a)}{n} + z = \frac{p(a)}{nm} \cdot m + z'm + k$, where $z = z'm + k$.

□

The third condition of Proposition 3.1 is clearly $\Delta^0_2(R)$. Moreover, if I is c.e. relative to R, then so is J, with an index that we can compute from that of I. Using this, we show the following.

Theorem 3.2. *If R is a countable real closed field, then there is a maximal Z-ring $I \subseteq R$ such that I is $\Delta^0_2(R)$.*

Proof. Computably in R, we make a list of the elements, $(r_n)_{n\in\omega}$ in R. We form a chain of Z-rings $(I_n)_{n\in\omega}$, putting r_n into I_{n+1}, if possible. Again, each I_n will be c.e. relative to R, and the sequence of indices will be $\Delta^0_2(R)$. We let I_0 consist of the integers. Given the Z-ring I_n, we check whether I_n and r_n satisfy Condition 3 of Proposition 3.1. If this is the case, then I_{n+1} is the resulting extension that includes r_n, and otherwise, $I_{n+1} = I_n$. Then $I = \cup_n I_n$ has the desired properties. It is a Z-ring that is maximal in R, and it is $\Delta^0_2(R)$.

□

We recall the following result of Wilkie [14].

Theorem 3.3. *Every Z-ring extends to a model of IOpen.*

The proof rests on the following.

Lemma 3.4 (Wilkie)**.** *Let R be a real closed field extending $\mathbb{R}$, and let I be a countable Z-ring of R. If $r \in R$ and the interval $[r, r+1)$ contains no element of I, then there is a some $\epsilon \in [0,1) \cap \mathbb{R}$ such that if $\beta = r + \epsilon$, then for all polynomials $p(x) \in I[x]$, the value $p(\beta)$ is infinite.*

We give the proof of this lemma, since we want to use ideas from it.

Proof. We choose $\epsilon \in \mathbb{R}$, in the interval $[0,1)$, such that $r + \epsilon$ is not infinitesimally close to any element of $acl(I)$, the algebraic closure of I. To see that such an ϵ exists, we list the elements of $acl(I)$ as $(x_i)_{i \in \omega}$. For each i, we choose an interval $O_i \subseteq [0,1)$, with rational endpoints, such that for $y \notin O_i$, the distance of y from $x_i - r$ is at least $\frac{1}{2^{i+1}}$ and O_i has length at most $\frac{1}{2^{i+2}}$. In the interval $[0,1)$, there is a real ϵ outside the intervals O_i. Let $\beta = r + \epsilon$, as planned.

Claim: For all $p(x) \in I[x]$, $p(\beta)$ is infinite.

Proof of Claim. We can write $p(x)$ in the form

$$a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n \ ,$$

where $a_i \in I$, or in the form

$$a_n (x - \theta_1) \cdots (x - \theta_n) \ ,$$

where $\theta_i \in acl(I)$. Then, $p(\beta) = a_n(\beta - \theta_1) \cdots (\beta - \theta_n)$. If $p(\beta)$ is finite, then a_n must be finite, since no $|\beta - \theta_i|$ is infinitesimal. Therefore, $a_n \in \mathbb{Z}$ since I is a Z-ring. Moreover, all the $|\beta - \theta_i|$ are finite. Let $N \in \mathbb{N}$ be a bound on $\sum_{i=1}^n |\beta - \theta_i|$.

Matching the coefficients of x^{n-1} in the two expressions for $p(x)$, and writing c for $\pm a_{n-1}$, we get $\sum_{i=1}^n \theta_i = \frac{c}{a_n}$. Then $\frac{N}{n} \geq \sum_{i=1}^n |\frac{\beta - \theta_i}{n}| \geq |\beta - \frac{c}{n a_n}| \geq 0$. There exist $b \in I$ and a natural number $k < n a_n$ such that $c = b n a_n + k$. So, $|\beta - b|$ is finite. Hence, there exists $z \in \mathbb{Z}$ such that $b + z$, an element of I, is in the interval $[r, r+1)$, a contradiction.

□

Using the claim, together with Proposition 3.1, we gets a Z-ring J extending I and containing β.

□

Remark 2. *Instead of assuming that I is countable and R extends $\mathbb{R}$, we could use either of the following assumptions.*

- *R is κ^+-saturated, where κ is the cardinality of the Z-ring $I \subseteq R$.*
- *R is recursively saturated, and I is finitely generated, in the sense that for some finite tuple $a_1, \ldots, a_n$, $I \subseteq \mathbb{Q}[a_1, \ldots, a_n]$.*

In either case, if $r \in R$ and there is no element of I in $[r, r+1)$, then there exists $\beta \in [r, r+1)$ such that we can extend I to a Z-ring $J \subseteq R$ containing β.

We now turn to the main new result of the section, showing that a maximal Z-ring in a real closed field R need not be an integer part for R. The following notation will be used here, and also in Section 4.

Notation: If a, b are elements of a real closed field and $a, b > 0$, we write $a << b$ if $a^n < b$ for all $n \in \mathbb{N}$.

Theorem 3.5. *There is a real closed field R with an element r and a Z-ring $I \subseteq R$ such that I cannot be extended to a discrete ordered subring of R with an element in the interval $[r, r+1)$.*

Proof. Let $r, i_0, \ldots, i_n, \ldots$ be a sequence of infinite elements (taken from the monster model) such that $r << i_0 << \ldots << i_n << \ldots$. In the course of the construction, we choose a sequence of reals $(\epsilon_n)_{n \in \omega}$, where $\epsilon_n \in (0,1) \cap \mathbb{R}$. Our R will be the real closure of

$$\{r\} \cup \{i_n : n \in \omega\} \cup \{\epsilon_n : n \in \omega\} \ .$$

The value group of R will be generated by $\{r\} \cup \{i_n : n \in \omega\}$; i.e., the group elements are the finite products of rational powers of these elements. The residue field of R will be the real closure of $\{\epsilon_n : n \in \omega\}$.

We build a chain $(I_n)_{n \in \omega}$ of countable Z-rings, where I_{2k} is included in the real closure of

$$\{r\} \cup \{i_n : n \leq k\} \cup \{\epsilon_k : k < n\} \ ,$$

and I_{2k+1} is included in the real closure of

$$\{r\} \cup \{i_n : n \leq k\} \cup \{\epsilon_k : k \leq n\} \ .$$

(The new elements of I_{2k} may depend on i_k, and the new elements of I_{2k+1} may depend on ϵ_k.)

As the construction proceeds, we enumerate the elements of R in the interval $[r, r+1)$. Our enumeration $(r_k)_{k \in \omega}$ has the feature that r_k is in the real closure of

$$\{r\} \cup \{i_n : n < k\} \cup \{\epsilon_n : n < k\} \ .$$

We make sure that r_k cannot be included in any discrete ordered ring J such that $I_{2k+1} \subseteq J$. Thus, the union $I = \cup_n I_n$ will be a Z-ring that cannot be extended to a discrete ordered ring containing any element of R in the interval $[r, r+1)$.

We let $I_0 = \mathbb{Q}[i_0]^\bullet \oplus \mathbb{Z}$. We let $b_0 = i_0 r_0 + \epsilon_0$, for a suitably chosen ϵ_0, and we let $I_1 = \mathbb{Q}[i_0, b_0]^\bullet \oplus \mathbb{Z}$. In general, I_{2k} will be $\mathbb{Q}[i_0, \ldots, i_k; b_0, \ldots, b_{k-1}]^\bullet \oplus \mathbb{Z}$ and I_{2k+1} will be $\mathbb{Q}[i_0, \ldots, i_k; b_0, \ldots, b_k]^\bullet \oplus \mathbb{Z}$, where $b_j = i_j r_j + \epsilon_j$, for suitably chosen ϵ_j.

Below, we say inductively how to choose the ϵ_j's, and we show that for each n, the ring I_n is a Z-ring.

$\mathbf{n = 0}$. Using the fact that $i_0 >> r$, together with Proposition 3.1, we can see that $\mathbb{Q}[i_0]^\bullet \oplus \mathbb{Z}$ is a Z-ring. We let this be I_0. Notice that $\{i_0^q \mid q \in \mathbb{Q}\}$ serves as a value group for the real closure of I_0, and the elements of I_0 all have valuations of the form i_0^h for $h \in \mathbb{N}$. There is no element of valuation r.

$\mathbf{n = 1}$. Let $r_0 = r$. Clearly, no element of I_0 is an integer part for $r_0 i_0$. By Lemma 3.4, we can choose $\epsilon_0 \in (0,1) \cap \mathbb{R}$ such that if $b_0 = r_0 i_0 + \epsilon_0$, then for all polynomials $p(x) \in I_0[x]$, the value $p(b_0)$ is infinite. By Proposition 3.1, $\mathbb{Q}[i_0, b_0]^\bullet \oplus \mathbb{Z}$ is a Z-ring. We let this be I_1. Clearly, $I_0 \subseteq I_1$. The elements of I_1 all have valuations of the form $i_0^h (r i_0)^j$, for some $h, j \in \mathbb{N}$. Again, there is no element of valuation r.

$\mathbf{n = 2k}$. We consider i_k. By Proposition 3.1, $\mathbb{Q}[i_0, b_0, \ldots, i_{k-1}, b_{k-1}, i_k]^\bullet \oplus \mathbb{Z}$ is a Z-ring. We let this be I_{2k}. We assume by induction that I_{2k-1} has no element of valuation r. Since $i_k >> a$ for all $a \in I_{2k-1} \cup \{r\}$, there is no element of valuation r in I_{2k}.

$\mathbf{n = 2k+1}$. Consider $r_k = r + \delta_k$, where δ_k is an element of the real closure of $r, i_0, b_0, \ldots, i_{k-1}, b_{k-1}$ in the interval $(0,1)$. We claim that $r_k i_k$ has no integer part in I_{2k}. Any element of I_{2k} is the sum of an integer and terms (with coefficients in $\mathbb{Q}$) of the form

$$i_k^{h_k} \prod_{j=0}^{k-1} i_j^{h_j} b_j^{l_j} = i_k^{h_k} \prod_{j=0}^{k-1} i_j^{h_j} (i_j r_j + \epsilon_j)^{l_j} = i_k^{h_k} \prod_{j=0}^{k-1} i_j^{h_j} [i_j (r + \delta_j) + \epsilon_j]^{l_j} . \tag{1}$$

We have

$$[i_j(r+\delta_j) + \epsilon_j]^{l_j} = [i_j(r+\delta_j)]^{l_j} + z_1 [i_j(r+\delta_j)]^{l_j - 1} \epsilon_j + \ldots + z_{l_j - 1} [i_j(r+\delta_j)] \epsilon_j^{l_j - 1} + \epsilon_j^{l_j} , \tag{2}$$

where $z_j \in \mathbb{Z}$. Expanding $[i_j(r+\delta_j)]^k \epsilon_j^l = [i_j r + i_j \delta_j]^k \epsilon_j^l$, and substituting in (1), we get a sum of terms of the form $\alpha (i_j r)^{l_1} (i_j \delta_j)^{l_2}$, where $\alpha \in \mathbb{R}$. Thus, every element of I_{2k} is a sum of terms of the form

$$\alpha i_k^{h_k} \prod_{j=0}^{k-1} i_j^{h_j} (i_j r)^{l_j} (i_j \delta_j)^{l'_j} , \tag{3}$$

for $\alpha \in \mathbb{R}$.

We cannot claim that these terms all have distinct valuations. In particular, if $\delta_j = \frac{1}{r}$, then $(i_j r)(i_j \delta_j) = (i_j)^2$. Even so, the valuation of any element of I_{2k} must match the valuation of one of the terms. Recall that $\delta_j \in (0,1)$ and that δ_j is in the real closure of r, $i_0, \ldots, i_{j-1}$, and $b_0, \ldots b_{j-1}$ for $j > 0$. Since $i_j >> i_{j-1}$, and δ_j is in a real closed field with value group generated by r and i_m for $j < k$, it follows that $i_j \delta_j >> i_{j-1} >> r$ as well. The valuation of $i_j \delta_j$ will be i_j if δ_j has valuation 1 (i.e., if it is not infinitesimal. Otherwise, the valuation of δ_j will be an infinitesimal member of the value group generated by r and i_m for $m < j$.

The element $i_k r_k = i_k(r + \delta_k)$ has valuation $i_k r$, since $\delta_k \in (0,1)$. We can show that no element of I_{2k} has this valuation. Any term in an element of I_{2k} has the form in Equation (3). For a term to have a valuation in part (nontrivially) generated by r, either $(i_j r)^{l_j}$ for $l_j > 0$ or $(i_j \delta_j)^{l'_j}$ for $l'_j > 0$ must appear in Equation (3) for some $j < k$. In either case, the entire term will then have a valuation nontrivially generated by i_j for $j < k$. Hence, we can see that none of these terms has valuation $i_k r$. Thus, $i_k r_k$ has no integer part in I_{2k}. By Lemma 3.4, there exists some $b_k = i_k r_k + \epsilon_k$ with $\epsilon_k \in (0,1) \cap \mathbb{R}$ such that $\mathbb{Q}[i_0, b_0, \ldots, i_{k-1}, b_{k-1}, i_k, b_k]^{\bullet} \oplus \mathbb{Z}$ is a Z-ring extending I_{2k}. Let I_{2k+1} be this Z-ring. By a similar analysis of valuations of terms in I_{2k+1}, we see that no element of I_{2k+1} has valuation r.

Let R be the real closure of r and all i_k and ϵ_k for $k \in \omega$. The Z-ring $I = \cup_{n \in \omega} I_n$ and the real closed field R satisfy the statement of the theorem by the argument above.

□

Remark 3. *The Z-ring I that we constructed in the proof of Theorem 3.5 is not a Euclidean ring. In particular, b_0 and i_0 are elements of I, and there do not exist q and ρ in I such that $b_0 = i_0 q + \rho$ and $0 \leq \rho < i_0$, for then $|\frac{b_0}{i_0} - q| < 1$ and $|\frac{b_0}{i_0} - r| < 1$, so r would have an integer part in I.*

4 Computable real closed fields with no n-c.e. integer part

We do not know whether Δ^0_2 is sufficient to calculate an integer part for a computable real closed field. Ershov [5] defined a hierarchy within the class Δ^0_2, based on differences of c.e. sets. All Δ^0_2 sets appear in this hierarchy. We consider only the finite levels, which are not exhaustive. We say what it means to be n-c.e. For more on Ershov's hierarchy, see [2].

Definition 4.1.

- *A set is* 2-c.e., *or* d-c.e., *if it has the form $S_1 - S_2$, where S_1 and S_2 are c.e.*
- *A set is $(n+1)$-c.e. if it has the form $S_1 - S_2$, where S_1 is c.e. and S_2 is n-c.e.*

For a c.e. set S, there is an effective guessing strategy that allows at most one change. We start off guessing that $x \notin S$. If x enters S at some stage, then we know $x \in S$. For a 2-c.e. set $S = S_1 - S_2$, there is an effective guessing strategy that allows at most two changes. We start off guessing that

$x \notin S$. If x enters S_1 at some stage s_1, without also entering S_2, then we change and guess that $x \in S$. If at some stage $s_2 > s_1$, the element x enters S_2, then we change once more, and then we know $x \notin S$. In general, if S is n-c.e., we have an effective guessing strategy that, for all x, starts with the guess that $x \notin S$, and, after at most n changes, stops on the correct answer.

We obtain natural indices for n-c.e. sets from the standard codes for n-tuples. If $e = \langle e_1, e_2 \rangle$, then e is an index for the 2-c.e. set $W_{e_1} - W_{e_2}$. If $e = \langle e_1, \ldots, e_n \rangle$, then e is an index for the n-c.e. set $W_{e_1} - S$, where S is the $(n-1)$-c.e. set with index $\langle e_2, \ldots, e_n \rangle$.

Some computable real closed fields have c.e. integer parts. Here is an example.

Example 3. *If R is the real closure of a single infinite element a and R is computable, then R has a c.e. integer part.*

Let G be the divisible ordered abelian (multiplicative) group with elements a^q, for $q \in \mathbb{Q}$, and let k be the set of real algebraic numbers. The group G is a value group for R, and k is a residue field. Shepherdson [12] showed that $k[\{a^q : q \in \mathbb{Q}^+\}]^\bullet \oplus \mathbb{Z}$ is an integer part for R. Given a and 1, we can computably enumerate the elements.

The main result of this section is the following.

Theorem 4.1. *There exists a computable real closed field with no n-c.e. integer part for any $n \in \omega$.*

We begin with some special cases of Theorem 4.1 to give the ideas of the strategy. Here is the first special case.

Proposition 4.2. *There is a computable real closed field with no c.e. integer part.*

Proof. We shall produce a computable real closed field R with universe ω, which we think of as a set of constants. We satisfy the following requirements.

$\mathbf{R_e}$: W_e is not an integer part.

We start with a computable version of the monster real closed field, with infinite elements (given effectively)

$$(a_0 << b_0) << (a_1 << b_1) << (a_2 << b_2) << \ldots \ .$$

Our R will be isomorphic to a subfield of the computable monster. At each stage s, we have a finite partial $1-1$ function p_s from ω into the computable monster, and we have enumerated into the atomic diagram of R a finite set d_s of atomic sentences and negations of atomic sentences such that the constants mentioned in d_s are all in $dom(p_s)$, and p_s, by interpreting the constants, makes the sentences true in R.

For a single requirement $\mathbf{R_e}$, our strategy is as follows. We map constants to a_e and b_e, and we continue adding to the current d_s facts made true by the current p_s. When W_e provides integer parts a'_e for the constant mapped to a_e and b'_e for the constant mapped to b_e, then we ensure that a'_e and b'_e cannot both be in an integer part for R by adding a sentence of the form $(a'_e)^n = b'_e + \frac{1}{2}$, choosing n large enough that this statement is consistent with the current d_s. We adjust the partial isomorphism, letting p_{s+1} map b'_e to the appropriate element closer to a_e, and similarly for the pre-image of b_e, so that b_e leaves the range of the function. This action permanently spoils W_e as an integer part.

At stage s in the construction, we consider the first s requirements. We include the first s elements of ω in $dom(p_s)$. In the range, we include the elements a_e, for $e < s$, plus any b_e, for $e < s$, that have not been removed from the range. We also include the first s elements of the real closure of these elements in the range of p_{s+1}. For $e < s$, if W_e provides integer parts for the elements mapped under p_{s+1} to a_e and b_e, we act according to the strategy described above.

The requirement $\mathbf{R_e}$ is satisfied if W_e never provides integer parts for the elements mapped to a_e and b_e. If W_e *does* provide such elements, then the action taken against W_e ruins it forever as an integer part for R. Notice that no two requirements interfere with one another.

□

Here is the second special case, introducing a little more of the strategy.

Proposition 4.3. *There is a computable real closed field R with no 2-c.e. integer part.*

Proof. We start with a computable version of the monster real closed field with infinite elements

$$(a_0 << b_0^0 << c_0^0 << b_0^1 << c_0^1 << \ldots) << (a_1 << b_1^0 << c_1^0 << \ldots) << \ldots.$$

As above, our field R will be isomorphic to a subfield of the computable monster. We have the following requirements.

$\mathbf{R_e}$: The 2-c.e. set with index e is not an integer part for R.

To satisfy $\mathbf{R_e}$, where $e = \langle e_1, e_2 \rangle$, we use a_e and the pairs b_e^i and c_e^i. We arrange that the constant a mapped to a_e has no integer part. To remove the first potential integer part, a', enumerated into $W_{e_1} - W_{e_2}$, we map constants to the first pair b_e^0 and c_e^0. We wait until these elements have potential integer parts, b' and c', enumerated into $W_{e_1} - W_{e_2}$. We let $(b')^n = c' + \frac{1}{2}$ for some large enough n so that this statement is consistent with the atomic diagram so far, and we change our partial isomorphism appropriately. One of b' and c' must leave $W_{e_1} - W_{e_2}$ (by entering W_{e_2}), or $\mathbf{R_e}$ is satisfied. Say c' enters W_{e_2}. We then add the statement $(a')^n = c'$ to the atomic diagram for some n large. Since c' cannot change again, a' must leave $W_{e_1} - W_{e_2}$ (i.e., enter W_{e_2}).

We consider further integer parts for a suggested by $W_{e_1} - W_{e_2}$. Say a'' is the second one. We map constants to the next pair b_e^1 and c_e^1, and we watch for potential integer parts, b'' and c'', to appear in $W_{e_1} - W_{e_2}$. As before, we introduce a polynomial relationship between b'' and c'' as above, ruining discreteness, so that one of the two has to leave the integer part. Suppose c'' leaves $W_{e_1} - W_{e_2}$. We then add a polynomial relationship between a'' and c'', ensuring that if a'' is in an integer part, then c'' must be in as well. Hence, a'' must leave $W_{e_1} - W_{e_2}$. We continue in this way. Having removed the first k potential integer parts for a, we remove the next one, using the integer parts suggested for a pair of constants mapped to b_e^k and c_e^k.

If $W_{e_1} - W_{e_2}$ does not respond as described to our strategy, it is not an integer part for the resulting real closed field R. On the other hand, if $W_{e_1} - W_{e_2}$ does respond as above, then it does not provide an integer part for the constant a mapped to a_e. Either way, $\mathbf{R_e}$ is satisfied. Again, no two requirements interfere with one another.

□

Here is the third special case, introducing still more of the strategy.

Proposition 4.4. *There is a computable real closed field R with no 3-c.e. integer part.*

Proof. The construction is similar to the ones above. Again, we build the desired real closed field R by defining an increasing chain of finite partial isomorphisms, but for notational simplicity (and since it is similar to the cases above), we omit these details here and also in the next proof. We have the following requirements.

$\mathbf{R_e}$: The 3-c.e. set with index e is not an integer part for R.

We consider a single requirement $\mathbf{R_e}$. Let $S = W_{e_1} - (W_{e_2} - W_{e_3})$. For this requirement, our computable monster field includes a special element a and infinitely many b_i, each with infinitely many pairs $c_{i,j}, d_{i,j}$, such that

$$a << (b_0 << c_{0,0} < d_{0,0} << c_{0,1} << d_{0,1} << \ldots) << (b_1 < c_{1,0} < d_{1,0} < \ldots) << \ldots .$$

Our strategy will guarantee that either the constant mapped to some b_i (or one of its corresponding $c_{i,j}, d_{i,j}$) does not have an integer part in S, or else the constant mapped to a does not. We focus first on b_0. Suppose b_0' is the first integer part suggested by S for the constant mapped to b_0. When the constants mapped to $c_{0,0}$ and $d_{0,0}$ are given integer parts c', d' in S, we create a polynomial relationship between c', d', as before, so as to force one of the two to leave S—this means that it enters W_{e_2} without also entering W_{e_3}. Say c' leaves S. We relate c' and b_0' by a polynomial so that if b_0' is in an integer part for R then c' must be as well. Either b_0' must leave S, or c' must re-enter S—this means that it enters W_{e_3}. In the latter case, c' will be permanently in S.

If b_0' leaves S, then S must suggest another integer part for the constant mapped to b_0, say b_0''. We map constants to $c_{0,1}$, $d_{0,1}$, and we watch for S to suggest integer parts for these. We act as above, so that either b_0'' must leave S, or we have an element c'' that must remain in S permanently. Note that if b_0' leaves S, then it cannot re-enter unless c' re-enters as well, and if c' re-enters S, then it is in permanently. This puts us in the next case.

Suppose c' is in S permanently. Let a' be the first potential integer part for a suggested by S. We relate a' and c' by a polynomial (as before) so that a' and c' cannot both be in an integer part for R. Since c' is permanently in S, the element a' is forever removed from S, ensuring that the constant mapped to a is not covered by a'. Let a'' be the second potential integer part suggested by S for the constant mapped to a. We then turn to the constant mapped to b_1 and use the related pair $c_{1,i}$ and $d_{1,i}$ to either remove all suggested integer parts for the constant b_1, or find an element that is permanently removed from S.

At the end of the construction, either a', the first integer part for the constant mapped to a suggested by S, is removed permanently from S, or else all of the suggested integer parts for the constant mapped to b_0 are removed (if S provides integer parts for all $c_{0,j}, d_{0,j}$). Assuming that a' has been removed, we arrange that either all the potential integer parts for the constant mapped to b_1 are removed, or else a'', the second possible integer part for the constant mapped to a, is removed. In general, either all suggested integer parts for the constant mapped to a are removed or there is some first k such that all suggested integer parts for the constant mapped to b_k are removed if S is providing integer parts for all constants. Hence, either the constant mapped to a or the constant mapped to some b_k (or one of its corresponding $c_{k,j}, d_{k,j}$) fails to have an integer part in S.

To satisfy all of the requirements, we map constants to infinitely many a_n, each with a retinue of further elements, as above, such that all elements associated with a_n are much larger (in the sense of $<<$, introduced in Section 3) than any of the elements associated with a_m for $m < n$. At step 0, we start work on Requirement $\mathbf{R_0}$. At step 1, we carry out the second step of the strategy for Requirement $\mathbf{R_0}$, and we start work on Requirement $\mathbf{R_1}$. At step s, we carry out the next step on Requirement $\mathbf{R_k}$, for each $k < s$, and we start work on Requirement $\mathbf{R_s}$.

□

Proposition 4.5. *For each n, there is a computable real closed field R with no $(n+1)$-c.e. integer part.*

Proof. We have the following requirements.

$\mathbf{R_e}$: The $(n+1)$-c.e. set with index e is not an integer part for R.

We describe the construction inductively in terms of the n-c.e. construction. We focus on a single requirement $\mathbf{R_e}$. Let S be the $(n+1)$-c.e. set with index e. To satisfy $\mathbf{R_e}$, we have (in the computable monster) a and infinitely many b_i such that

$$a << (b_0 << \ldots) << (b_1 << \ldots) << (b_2 << \ldots) << (b_3 << \ldots) << \ldots .$$

Moreover, each b_i is accompanied by a constellation of further elements like those for the n-c.e. construction.

As before, we map constants to a and b_0 and continue building the real closed field R until S provides potential integer parts a' and b'_0 for these constants. We then act as in the n-c.e. construction, continuing to build R until S suggests integer parts for certain constants mapped to further elements in the constellation of b_0. First, suppose no proposed integer part for a constant mapped to an element in the constellation of b_0 ever enters and leaves S more than n-times. It follows, by induction, that the method of the n-c.e. construction guarantees that S does not provide an integer part for some element mapped to b_0 or an element of its constellation. Hence, either $\mathbf{R_e}$ is satisfied in this way, or else some constant c' enters and leaves S a total of $(n+1)$ times. In the latter case, we relate c' with the integer part a' for the constant mapped to a, so as to ruin a' as a potential integer part for the constant mapped to a.

Now, S must provide a new integer part a'' for the constant mapped to a. We then consider b_1 and its constellation of constants. By repeating the method of the n-c.e. construction, we ensure that either S does not provide an integer part for the constant mapped to b_1 or some element of its constellation (satisfying $\mathbf{R_e}$), or else some constant c'' enters and leaves S a total of $n+1$ times. In the latter case, we relate c'' with a'' so as to force S to remove a''. We continue in this way. In the end, either S will not provide an integer part for the constant mapped to some b_i or some element of its constellation, or S will not provide an integer part for the constant mapped to a. To satisfy all of the requirements at once, we have an element a_e, with associated $b_{e,i}$'s and other elements, for each $\mathbf{R_e}$, where the elements a_e, $b_{i,e}$, etc., associated with $\mathbf{R_e}$ are much smaller (in the sense of $<<$) than those associated with $\mathbf{R_{e+1}}$. As before, there is no interference, and at stage s, we can do one step on Requirement $\mathbf{R_k}$ for each $k \leq s$.

□

Theorem 4.1 can be proved by building a single computable real closed field R on which we diagonalize against each n-c.e. set for some $n \in \omega$ as in the constructions above.

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