Truncation-closed subfields of a Hahn field

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Abstract

Let $K((G))$ be a Hahn field, where K is algebraically closed of characteristic 0, or real closed, and G is a divisible ordered Abelian group. Let R be a truncation-closed, relatively algebraically closed subfield of $K((G))$. A *tc-basis* for R over K is a sequence $(r_\alpha)_{\alpha<\gamma}$, forming a transcendence basis for R over K, such that for each $\alpha < \gamma$, either $r_\alpha = t^g$ for some $g \in G$ or else r_α has limit length, with all proper truncations algebraic over $K \cup \{r_\beta : \beta < \alpha\}$. Building on partial results in [3] and [2], we give sharp bounds on the lengths of elements of R in terms of the length of a tc-basis, assuming that the length is a countable ordinal.

Keywords: Hahn field, generalized power series, truncation-closed fields, length

1 Introduction

Let K be a field that is either algebraically closed of characteristic 0, or real closed.

Definition 1.1 (Puiseux series). *The* Puiseux series *over K are formal sums of the form $\sum_{z\geq k} a_z t^{\frac{z}{n}}$, for some integer k and some positive integer n, where the terms correspond to integers $z \geq k$, and the coefficients a_z are in K.*

Each Puiseux series is a Laurent series in the indeterminate $t^{\frac{1}{n}}$ for some positive integer n. The Puiseux series were studied first by Newton [8], [9], and then by Puiseux [10], [11].

Theorem 1.2 (Newton-Puiseux Theorem)**.**

1. *If K is algebraically closed of characteristic 0, then the set of Puiseux series over K forms an algebraically closed field.*

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2. *If K is a real closed field, then the set of Puiseux series over K forms a real closed field.*

Hahn series were introduced by Hahn [1] and were studied more generally by Mal'tsev [5] and B. Neumann [7].

Definition 1.3 (Hahn series)**.** *Let G be an ordered Abelian group, and let K be a field. The* Hahn field *$K((G))$ consists of the formal sums $s = \sum_{g\in S} a_g t^g$, where $S \subseteq G$ is well-ordered, and $a_g \in K - \{0\}$ for each $g \in S$. The set S, called the* support *of s, may be denoted by $Supp(s)$. The* length *of s is the order type of $Supp(s)$. For $g \in G$, the* coefficient *of t^g in s is a_g if $g \in S$, and 0 otherwise. We let 0 be the element of $K((G))$ with empty support; i.e., all coefficients are 0.*

In defining the operations, we consider the coefficients of t^g for all $g \in G$, not just those in the support

Definition 1.4 (Operations on Hahn series)**.** *Let $s, s' \in K((G))$, where the coefficients of t^g in s and s' are, respectively, a_g and b_g (possibly 0). We define the sum and product, by specifying the coefficients, as follows:*

1. *In $s + s'$, the coefficient of t^g is $a_g + b_g$.*
2. *In $s \cdot s'$, the coefficient of t^g is $\sum_{(h,h')\in P} a_h \cdot b_h$, where P is the set of pairs $(h, h') \in Supp(s) \times Supp(s')$ such that $h + h' = g$.*

If G is an ordered Abelian group and K is a field, then $K((G))$ is a valued field, with the *natural valuation*

$$w(s) = \begin{cases} \mu g[g \in Supp(s)] & \text{if } s \neq 0 \\ \infty & \text{if } s = 0 \end{cases}$$

If K is an ordered field, then $K((G))$ is also an ordered field, where s is positive iff the coefficient $a_{w(s)}$ is positive in K. In what follows, G will always be a divisible ordered Abelian group, and K will always be a field that is either algebraically closed of characteristic 0 or real closed. We may also suppose that $K \subseteq K((G))$, identifying $a \in K$ with $at^0 \in K((G))$. Maclane [4] generalized the Newton-Puiseux Theorem to Hahn fields.

Theorem 1.5 (Generalized Newton-Puiseux Theorem)**.** *Let G be a divisible ordered Abelian group.*

1. *If K is an algebraically closed field of characteristic 0, then so is $K((G))$.*
2. *If K is a real closed, then so is $K((G))$.*

We will consider subfields R of a Hahn field $K((G))$ with some special features. Our goal is to bound the lengths of elements of R. We need some definitions describe the subfields R that we shall consider and to state our results.

Definition 1.6. *A subfield R of $K((G))$ is* relatively algebraically closed *if $K \subseteq R$ and R contains all elements of $K((G))$ that are roots of polynomials over R.*

Note that if K is algebraically closed, then all relatively algebraically closed subfields are algebraically closed, and if K is real closed, then all relatively algebraically closed subfields are real closed.

Definition 1.7. *For $s, s' \in K((G))$, s' is a* truncation *of s if $Supp(s')$ is an initial segment of $Supp(s)$, and for all $g \in Supp(s')$, the coefficients of t^g in s and s' are the same.*

Definition 1.8 (Truncation-closed subfields of $K((G))$)**.** *A subfield R of $K((G))$ is* truncation-closed *if R contains all of the truncations of its elements.*

The special subfields R of $K((G))$ that we shall consider are truncation-closed and relatively algebraically closed. We will bound the lengths of elements of R in terms of the length of a sequence of elements forming a transcendence basis for R over K. The definitions below are from [3].

Definition 1.9 (tc-independent sequence, canonical sequence, tc-basis)**.** *Consider a Hahn field $K((G))$.*

1. *A* truncation-closed independent sequence*, or tc-*independent sequence*, in $K((G))$ is a sequence $(r_\alpha)_{\alpha<\gamma}$, algebraically independent over K, such that for each $\alpha < \gamma$, either $r_\alpha = t^g$, for some $g \in G$, or else r_α has limit length and all proper initial segments of r_α are algebraic over $K \cup \{r_\beta : \beta < \alpha\}$.*

2. *For a tc-independent sequence $(r_\alpha)_{\alpha<\gamma}$, for $\alpha \leq \gamma$, let R_α consist of the elements of $K((G))$ that are algebraic over $K \cup \{r_\beta : \beta < \alpha\}$. We call $(R_\alpha)_{\alpha\leq\gamma}$ the* canonical sequence *corresponding to $(r_\alpha)_{\alpha<\gamma}$.*

3. *We say that $(r_\alpha)_{\alpha<\gamma}$ is a tc-*basis *for R if $(r_\alpha)_{\alpha<\gamma}$ is tc-independent, and R is the last term of the corresponding canonical sequence $(R_\alpha)_{\alpha\leq\gamma}$, i.e., $R = R_\gamma$.*

It turns out that the subfields in a canonical sequence are all truncation-closed and relatively algebraically closed.

Proposition 1.10 (Mourgues-Ressayre [6])**.** *Let $(r_\alpha)_{\alpha<\gamma}$ in $K((G))$ be a tc-independent sequence, with corresponding canonical sequence $(R_\alpha)_{\alpha\leq\gamma}$. Then each R_α is truncation-closed and relatively algebraically closed.*

About the proof. In [6], Mourgues and Ressayre showed that every real closed field F has an "integer part", where this is a discrete ordered subring appropriate to be the range of a floor function. Their construction involves defining an embedding d from F onto a truncation-closed, relatively algebraically closed subfield R of a Hahn field $k((G))$, where k is a "residue field section" of F and G is a specially chosen value group section.

In defining the embedding d, Mourgues and Ressayre isolated a tc-basis $(r_\alpha)_{\alpha<\gamma}$ for $d(R)$, with corresponding canonical sequence $(R_\alpha)_{\alpha\leq\gamma}$ consisting of subfields of $k((G))$. To show that R, and all R_α, are truncation-closed, Mourgues and Ressayre gave lemmas saying that if P is a subfield of $K((G))$ that is truncation-closed and relatively algebraically closed, and $r \in K((G))-P$, where either $r = t^g$ for some g, or else r has limit length and all proper truncations of r are in P, then $P(r)$ is truncation-closed. Moreover, they proved that if P' is the algebraic closure in $K((G))$ of $P(r)$, then P' is truncation-closed (as well as relatively algebraically closed). □

We showed in [3] that every truncation-closed and relatively algebraically closed subfield R of $K((G))$ has a tc-independent sequence. The proof is a straightforward induction that employs an arbitrary well ordering of R.

Proposition 1.11 ([3, Proposition 1.19]). *Suppose that R is a subfield of $K((G))$ that is truncation-closed and relatively algebraically closed. Then there is a tc-basis for R over K.*

We can now state our bounds on the lengths of elements in a given truncation-closed and relatively algebraically closed subfield R of $K((G))$. For R with a tc-basis of length γ, we bound the lengths of elements of R in terms of γ. If $\gamma = 1$, then the tc-basis consists of a single group element t^g. In this case, the elements of R are all contained in the field of Puiseux series over K, and by the Newton-Puiseux Theorem, they have length at most ω. In [3], we gave bounds in the case where $\gamma \leq \omega$, and in [2], we produced examples showing that these bounds are sharp. We recall these results in the next section. In the present paper, we bound the lengths of elements of a truncation-closed subfield R of $K((G))$ with a tc-basis of countably infinite length, and we prove sharpness of these bounds. Here is the main new result.

Theorem 1.12. *Suppose R is a truncation-closed subfield of $K((G))$, and let γ be a countable limit ordinal.*

1. *If R has a tc-basis of length γ, then the elements of R have length less than ω^{ω^γ}.*
2. *If R has a tc-basis of length $\gamma+n$, where n is a positive integer and $\gamma \geq \omega$, then the elements of R have length less than $\omega^{\omega^{\gamma+n}} \cdot \omega^{\omega^\gamma}$.*

In Section 2, we recall some results from [2],[3], giving bounds on lengths of elements of R in the case where there is a tc-basis of length at most ω, and for arbitrary countable γ under special assumptions on the group G. In Section 3, we add more lemmas and prove Theorem 1.12. Finally, in Section 4, we show that the bounds given in Theorem 1.12 are sharp.

2 Earlier results

Suppose R is a truncation-closed subfield of $K((G))$ with a tc-basis of length γ. In [3], we gave bounds on the lengths of elements of R for $\gamma \leq \omega$. We also gave bounds for arbitrary countable ordinals γ under the assumption that the group G is "Archimedean."

Definition 2.1. *Let G be a divisible ordered Abelian group. For positive elements $a, b \in G$, we write $a \approx b$ if there exist m, n such that $ma > b$ and $nb > a$. The group G is* Archimedean *if there is just one $\approx$-class of positive elements.*

Note that $(\mathbb{Q}, +)$ is Archimedean. In fact, the Archimedean divisible ordered Abelian groups are all isomorphic to subgroups of $(\mathbb{R}, +)$. We can now state the main results from [3].

Theorem 2.2 ([3, Theorem 1.3])**.** *Let R be a truncation-closed subfield of $K((G))$.*

1. *If R has a tc-basis of length $n \geq 1$, then the elements of R have length at most $\omega^{\omega^{(n-1)}}$.*
2. *If R has a tc-basis of length ω, then the elements have length less than $\omega^{\omega^{\omega}}$.*

Theorem 2.3 ([3, Theorem 1.4])**.** *Let R be a truncation-closed subfield of $K((G))$ with tc-basis of countable length $\alpha \geq \omega$. If G is Archimedean, then the elements of R have length at most $\omega^{\omega^{\alpha}}$.*

In [3], we showed that the above bounds are sharp.

Theorem 2.4 ([2, Theorem 7.1], [3, Theorem 5.1])**.** *Let K be a field that is real closed or algebraically closed of characteristic 0.*

1. *There is a truncation-closed subfield R of $K((\mathbb{Q}))$ with a tc-basis of length ω, and corresponding canonical sequence $(R_n)_{n \leq \omega}$, such that for all $n \geq 1$, R_n has an element of length $\omega^{\omega^{(n-1)}}$.*
2. *For each countable ordinal $\alpha \geq \omega$, there is a truncation-closed subfield R of $K((\mathbb{Q}))$ with a tc-basis of length α such that if α is a successor ordinal, then R has an element of length $\omega^{\omega^{\alpha}}$, and if α is a limit ordinal, then R has elements of all lengths less than $\omega^{\omega^{\alpha}}$.*

To extend the results in Theorems 2.2 and 2.3, we need several results from [3] here, which we list below. We generally omit the proofs for those that are more straightforward and sketch the arguments for those that are less so.

2.1 The tc-independent sequence

Recall that if $(r_\alpha)_{\alpha<\gamma}$ is a tc-independent sequence in $K((G))$, with corresponding canonical sequence $(R_\alpha)_{\alpha\leq\gamma}$, then for each $\alpha < \gamma$, either r_α has the form t^g for some $g \in G$, or else r_α has limit length, with all truncations in R_α.

Lemma 2.5 ([3, Lemma 3.3])**.** *Suppose $(r_\alpha)_{\alpha<\gamma}$ is a tc-independent sequence, with corresponding canonical sequence $(R_\alpha)_{\alpha\leq\gamma}$. If $R_{\alpha+1} - R_\alpha$ has an element of form t^g, then r_α must have this form.*

Lemma 2.6 ([3, Lemma 3.4])**.** *Let $(r_\alpha)_{\alpha<\gamma}$ be a tc-independent sequence in $K((G))$, with corresponding canonical sequence $(R_\alpha)_{\alpha\leq\gamma}$. Then there is a tc-independent sequence $(r'_\alpha)_{\alpha<\gamma}$, with the same canonical sequence, such that for all α, $w(r'_\alpha) > 0$.*

Given Lemma 2.6, we adopt the following convention.

Convention. If $(r_\beta)_{\beta<\alpha}$ is a *tc*-independent sequence, then for all $\beta < \alpha$, $w(r_\beta) > 0$.

2.2 Convex subgroups

In [3], we used the bounds obtained under the assumption that the group is Archimedean in proving Theorem 2.2, where the group is not assumed to be Archimedean. In the remainder of this section, we recall the ideas behind this, so that we can use them again in Sections 3 and 4.

Definition 2.7. *A subgroup H of an ordered group G is* convex *if for all $g, h \in G^{>0}$, if $h \in H$ and $g < h$, then $g \in H$.*

Note: If H is a convex subgroup of G, then $H^{>0}$ is closed under $\approx$; i.e., if $g, h \in G^{>0}$, where $g \in H$ and $h \approx g$, then $h \in H$.

Definition 2.8. *Let H be a convex subgroup of G. For $x \in K((G))$ with support in $G^{\geq 0}$, we let $o_H(x)$ be the truncation of x consisting of the terms with support in H. If $Supp(x) \cap H = \emptyset$, then $o_H(x) = 0$.*

Proposition 2.9 ([3, Proposition 3.7])**.** *Let H be a convex subgroup of G, and let $x, y \in K((G))$, both with support in $G^{\geq 0}$. Then*

1. $o_H(x + y) = o_H(x) + o_H(y)$,
2. $o_H(x \cdot y) = o_H(x) \cdot o_H(y)$.

Notation. For a polynomial $p(x) = A_0 + A_1x + \ldots + A_sx^s$ over $K((G))$, with $Supp(p) \subseteq G^{\geq 0}$, we write $o_H(p)(x)$ for the polynomial

$$o_H(A_0) + o_H(A_1)x + \ldots + o_H(A_s)x^s \ .$$

Proposition 2.9, yields the following.

Corollary 2.10 ([3, Proposition 3.9])**.** *Let H be a convex subgroup of G, let $p(x)$ be a polynomial over $K((G))$ with $Supp(p) \subseteq G^{\geq 0}$, and let r be a root of p with $w(r) \geq 0$. Then $o_H(r)$ is a root of $o_H(p)(x)$.*

Definition 2.11 (respecting a convex subgroup [3, Definition 3.10])**.** *Let H be a convex subgroup of G, and let $(r_\alpha)_{\alpha<\gamma}$ be a tc-independent sequence in $K((G))$, with corresponding canonical sequence $(R_\alpha)_{\alpha\leq\gamma}$. We say that $(r_\alpha)_{\alpha<\gamma}$* respects *$H$ if the following conditions hold:*

1. *if $R_{\alpha+1} - R_\alpha$ has an element of form t^h for $h \in H$, then r_α has this form,*

2. *if r_α has limit length, then $Supp(r_\alpha)$ is entirely in H or entirely outside H.*

Lemma 2.5 says that every tc-independent sequence in $K((G))$ respects G. Moreover, we can adjust any tc-independent sequence to make it respect a given convex subgroup.

Lemma 2.12 ([3, Proposition 3.11])**.** *Let H be a convex subgroup of G, and let $(r_\alpha)_{\alpha<\gamma}$ be a tc-independent sequence in $K((G))$. Then there is a tc-independent sequence $(r'_\alpha)_{\alpha<\gamma}$, with the same canonical sequence, such that $(r'_\beta)_{\beta<\alpha}$ respects H.*

Proof. Let $(R_\alpha)_{\alpha\leq\gamma}$ be the canonical sequence corresponding to $(r_\alpha)_{\alpha<\gamma}$. We proceed by induction on α. First, suppose that r_α has the form t^g for $g \in G$. If $R_{\alpha+1} - R_\alpha$ has an element of the form t^h, for $h \in H$, then we let r'_β have this form, and otherwise, we let $r'_\alpha = r_\alpha$. Next, suppose that r_α has limit length. By Lemma 2.5, this means that $R_{\alpha+1} - R_\alpha$ has no element of form t^g. Say that $r_\alpha = s + s'$, where s has support in H and s' has support in $G - H$. If $s' \neq 0$, then set $r'_\alpha = s'$ (note that r_α is interalgebraic with s'), while if $s' = 0$, then take $r'_\alpha = r_\alpha = s$. □

Note: For G, H, $(r_\alpha)_{\alpha<\gamma}$ and $(r'_\alpha)_{\alpha<\gamma}$ as in Lemma 2.12, if G is generated (as a rational vector space) by those g such that some r_β has the form t^g, then $G = H \oplus \tilde{G}$, where H is generated by those $g \in H$ such that some $r_{\beta'}$ has form t^g and $\tilde{G}$ is generated by the remaining $g \in G$ such that some $r_{\beta'}$ has form t^g.

The next lemma (not in [3]) says that we can adjust a tc-independent sequence to make it respect more than one convex subgroup.

Lemma 2.13. *Suppose $H_1, \ldots, H_k$ are convex subgroups of G, with $H_i \subseteq H_{i+1}$. Let $(r_\alpha)_{\alpha<\gamma}$ be a tc-independent sequence in $K((G))$. Then there is a tc-independent sequence $(r'_\alpha)_{\alpha<\gamma}$, with the same canonical sequence, such that $(r'_\alpha)_{\alpha<\gamma}$ respects all the H_i.*

Proof. We apply Lemma 2.12 first to get a sequence that respect H_1. We apply the lemma again to get a sequence that respects H_2. In fact, the new sequence respects both H_1 and H_2. We continue until we have adjusted for all the H_i. □

We will re-order a tc-independent sequence, putting first the elements with support in a convex subgroup. The next lemma is used to show in Proposition 2.15 below that the re-ordered sequence is still tc-independent.

Lemma 2.14 ([3, Lemma 3.12])**.** *Let H be a convex subgroup of G. Let $(r_\alpha)_{\alpha<\gamma}$ be a tc-independent sequence in $K((G))$ that respects H, and let $(R_\alpha)_{\beta\leq\alpha}$ be the corresonding canonical sequence. Then for all $\alpha \leq \gamma$, all elements of R_α with support in H are algebraic over $K \cup \{o_H(r_\beta) : \beta < \alpha\}$.*

The proof of Lemma 2.14 uses Proposition 2.9 and Corollary 2.10.

Proposition 2.15 ([3, Proposition 3.13]). *Let H be a convex subgroup of G. Let $(r_\alpha)_{\alpha<\gamma}$ be a tc-independent sequence in $K((G))$, respecting H. Let $(\hat{r}_\beta)_{\beta<\gamma^*}$ be the sequence obtained by putting first those r_α with support in H, and then those with support outside H, leaving the ordering otherwise unchanged. Then $(\hat{r}_\beta)_{\beta<\gamma^*}$ is also tc-independent.*

Proof sketch. Let $(R_\alpha)_{\alpha\leq\gamma}$ be the canonical sequence corresponding to $(r_\alpha)_{\alpha<\gamma}$, and let $(\hat{R}_\beta)_{\beta\leq\gamma^*}$ be the canonical sequence corresponding to the re-ordering $(\hat{r}_\beta)_{\beta<\gamma^*}$. For each $\beta<\gamma^*$, let $\hat{r}_\beta = r_{\alpha(\beta)}$. Clearly, the sequence $(\hat{r}_\beta)_{\beta<\gamma^*}$ is algebraically independent over K, and for each $\beta<\gamma^*$, either $\hat{r}_\beta = r_{\alpha(\beta)}$ has the form t^g for some $g\in G$, or else it has limit length, and then all proper truncations are in $R_{\alpha(\beta)}$. It is enough to prove that for $\beta<\gamma^*$, if $\hat{r}_\beta$ has limit length, the proper truncations of $\hat{r}_\beta$ are all in $\hat{R}_\beta$.

Let β^* be least such that $\hat{r}_\beta$ has support outside H. If $\beta\geq\beta^*$, then we have $R_{\alpha(\beta)}\subseteq\hat{R}_\beta$, so the statement is clear. Suppose $\beta<\beta^*$. The predecessors of $\hat{r}_\beta$ in the new sequence are just the predecessors of $r_{\alpha(\beta)}$ in the old sequence that have support in H. Since the sequence $(r_\alpha)_{\alpha<\gamma}$ respects H, each r_α has support entirely in H or entirely outside of H, so $o_H(r_\alpha)$ is equal to either r_α or 0. Dropping from the sequence those $o_H(r_\alpha)$ that are equal to 0, what remains is exactly the sequence $(\hat{r}_\beta)_{\beta<\beta^*}$. Now $\hat{r}_\beta$ and its truncations have support in H. Moreover, the proper truncations of $\hat{r}_\beta$ are in $R_{\alpha(\beta)}$, so they are algebraic over $K\cup\{o_H(r_{\alpha'}):\alpha'<\alpha(\beta)\}$ by Lemma 2.14. Thus, the proper truncations of $\hat{r}_\beta$ are algebraic over $K\cup\{\hat{r}_{\beta'}:\beta'<\beta\}$, so these truncations are in $\hat{R}_\beta$. □

The next result (not in [3]) extends Proposition 2.15 to a finite chain of convex subgroups.

Proposition 2.16. *Let $H_1,\ldots,H_k$ be convex subgroups of G, where $H_i\subseteq H_{i+1}$. Let $(r_\alpha)_{\alpha<\gamma}$ be a tc-independent sequence in $K((G))$ that respects all the H_i. Let $(\hat{r}_\beta)_{\beta<\beta^*}$ be the new sequence obtained by putting those r_α's with support in H_i before those in H_{i+1}, and leaving the ordering otherwise unchanged. Then $(\hat{r}_\beta)_{\beta<\beta^*}$ is tc-independent.*

Proof. We obtain the new sequence in k steps. In the first step, we put those r_α with support in H_1 in front of those with support not in H_1, and we leave the ordering otherwise unchanged. By Proposition 2.15, applied to H_1, the resulting sequence is tc-independent. Next, we put those r_α's with support in H_2 in front of those with support not in H_2, leaving the ordering the same otherwise. The elements with support in H_1 are still at the front. By Proposition 2.15, applied to H_2, this second sequence is tc-independent. For each $i\leq k$, after i steps of this kind, we have a tc-independent sequence with the elements having support in $H_1,\ldots,H_i$ in the proper places. After k steps, we have the desired sequence. □

Convention. Given a truncation-closed, relatively algebraically closed subfield R of $K((G))$, with tc-basis $(r_\alpha)_{\alpha<\gamma}$, we may suppose that for all $g\in G$, $t^g\in R$. If this is not so initially, we replace G by the appropriate subgroup.

Lemma 2.17. *Let $(r_\alpha)_{\alpha<\gamma}$ be a tc-basis for a truncation-closed, relatively algebraically closed subfield R of $K((G))$. Let H be a non-trivial convex subgroup of G. There is a tc-basis $(\hat{r}_\beta)_{\beta<\beta^*}$ for R that respects H and has the elements that have support in H first.*

Let $(\hat{R}_\beta)_{\beta\leq\beta^*}$ be the canonical sequence corresponding to this tc-basis, and let γ^* be the first ordinal such that $\hat{r}_{\gamma^*}$ has support outside H. We have:

- $R_{\beta^*} = R$.
- *G is the direct sum $H + G^*$, where G^* is the subgroup of G generated (in the sense of rational linear combinations) by those g such that $t^g = \hat{r}_\beta$ for $\beta \geq \gamma^*$.*
- *$K^* = \hat{R}_{\gamma^*}$ is a truncation-closed, relatively algebraically closed subfield of $K((H))$, with tc-basis $(\hat{r}_\beta)_{\beta<\gamma^*}$.*

Moreover, all of the objects defined above are determined from K, G, H, and the initial tc-basis $(r_\alpha)_{\alpha<\gamma}$.

Proof. Suppose R is a truncation-closed, relatively algebraically closed subfield of $K((G))$, with tc-basis $(r_\alpha)_{\alpha<\gamma}$. We may suppose that for all $g \in G$, $t^g \in R$ as in the Convention. Lemma 2.12 lets us replace the original tc-basis by one that respects H. Proposition 2.15 lets us replace the second tc-basis by a third one, with the elements that have support in H first. Let this third tc-basis be $(\hat{r}_\beta)_{\beta<\beta^*}$, and let $(\hat{R}_\beta)_{\beta\leq\beta^*}$ be the corresponding canonical sequence. It is easy to check that the described objects have the stated properties. □

Our R is a truncation-closed, relatively algebraically closed subfield of $K((G))$, but we may think of it also as a truncation-closed, relatively algebraically closed subfield of $K^*((G^*))$, with tc-basis $(\hat{r}_\beta)_{\gamma^*\leq\beta<\beta^*}$. The intuition is clear. The following lemma makes precise the identification (see [3], proof of Theorem 1.3).

Lemma 2.18. *Let R, $(r_\alpha)_{\alpha<\gamma}$, H, G^*, $(\hat{r}_\beta)_{\beta<\beta^*}$, and K^* be as in Lemma 2.17. There is a natural embedding f of R in $K^*((G^*))$ such that*

1. *f is the identity on elements at^0, for $a \in K^*$ and on elements t^g for $g \in G^*$;*
2. *the sequence $(f(\hat{r}_\beta))_{\gamma^*\leq\beta<\beta^*}$ is a tc-basis for $f(R)$ as a subfield of $K^*((G^*))$. Thus $f(R)$ is a truncation-closed, relatively algebraically closed subfield of $K^*((G^*))$.*

Proof sketch. We define the embedding f as follows. Suppose $s \in R$, where $s = \sum_{g\in S} a_g t^g$. Each $g \in S$ is uniquely expressed as a sum $g^* + h$, for $g^* \in G^*$ and $h \in H$. For simplicity, we assume that $S \subseteq G^{\geq 0}$. The set S^*, consisting of the G^*-components of elements of S, is well ordered. Also, for each $g^* \in S^*$, the set of H-components of elements of S with G^*-component g^* is well ordered. We let $f(s) = \sum_{g^*\in G^*} b_{g^*} t^{g^*}$, where $b_{g^*} = \sum_{g^*+h=g} a_g t^h$. Statement (1) is not difficult.

For (2), we note that the elements $\hat{r}_\beta$, for $\gamma^* \leq \beta < \beta^*$ are algebraically independent over K^*. For each β, if $\hat{R}_{\beta+1} - \hat{R}_\beta$ has an element of form t^g, then $\hat{r}_\beta$ has this form, for $g \in G^*$. If $\hat{r}_\beta$ has limit length, then all proper truncations are in $\hat{R}_\beta$. We identify $\hat{R}_\beta$ with the truncation-closed subfield of $K^*((G^*))$ consisting of elements algebraic over $K^* \cup \{f(\hat{r}_\gamma) : \gamma^* \leq \gamma < \beta\}$. □

The final lemma of this section says that we can re-order a *tc*-basis, moving a group element to an earlier place.

Lemma 2.19 ([3, Lemma 3.14])**.** *Suppose R is a truncation-closed subfield of $K((G))$ with tc-basis $(r_\alpha)_{\alpha\leq\gamma}$. If the last term, r_γ, has the form t^g, for some $g \in G$, and we form a new sequence, putting r_γ in an earlier position, leaving the rest of the ordering unchanged, then the resulting sequence is also a tc-basis for R.*

3 Bounding Theorem

In this section, we give some further lemmas and then prove Theorem 1.12. We consider different cases, depending on whether the value group of R has a greatest Archimedean class. In the lemma below, there is a greatest Archimedean class.

Lemma 3.1. *Let γ be a countable limit ordinal, and let $n \in \omega$. Suppose R is a truncation-closed subfield of $K((G))$, with tc-basis $(r_\alpha)_{\alpha<\gamma+n}$. Suppose G has a last $\approx$-class, and all $r_{\gamma+k}$ have limit length, with support in this last $\approx$-class. Let H be the convex subgroup of G with positive elements in $\approx$-classes to the left of the last one. Form K^* and G^* as in Lemma 2.17. Then the elements of R, thought of as elements of $K^*((G^*))$, have length at most $\omega^{\omega^{\gamma+n}}$.*

Proof. The *tc*-basis, $(\hat{r}_\beta)$, obtained by the re-ordering process in Lemma 2.17, consists of a first part of length at most γ, with support in H, and a second part of length at most $\gamma+n$, with support in G^*. For simplicity, we suppose that these *are* the lengths, so the new basis has form $(\hat{r}_\beta)_{\beta<\gamma\cdot2+n}$, where $\hat{r}_{\gamma\cdot2+k} = r_{\gamma+k}$, and $K^* = \hat{R}_\gamma$. It is clear that G^* is Archimedean, and $(\hat{r}_\beta)_{\gamma\leq\beta<\gamma\cdot2+n}$ is a *tc*-basis for R, as a subfield of $K^*((G^*))$. By Theorem 2.3, the elements of R, thought of as elements of $K^*((G^*))$, have length at most $\omega^{\omega^{\gamma+n}}$. □

In the next two lemmas, we suppose that G has no last $\approx$-class, and R has a *tc*-basis $(r_\alpha)_{\alpha<\gamma+n}$ such that for all $k < n$, *Supp*$(r_{\gamma+k})$ is co-final in G.

Lemma 3.2. *Let γ be a countable limit ordinal, and let n be a positive integer. Let $(r_\alpha)_{\alpha<\gamma+n}$ be a tc-independent sequence in $K((G))$, with corresponding canonical sequence $(R_\alpha)_{\alpha\leq\gamma+n}$. Suppose that G has no last $\approx$-class, and for all $k < n$, Supp$(r_{\gamma+k})$ is co-final in G. Then for $k < n$, the proper truncations of $r_{\gamma+k}$ are all in R_γ.*

Proof. We proceed by induction on k. For $k = 0$, the proper truncations of r_γ are in R_γ, by the definition of tc-independent sequence. Supposing that for all $k' < k$, the proper truncations of $r_{\gamma+k'}$ are in R_γ, we show that the proper truncations of $r_{\gamma+k}$ are also in R_γ. The support of $r_{\gamma+k}$ has no last $\approx$-class. Let r be a proper truncation of $r_{\gamma+k}$. We must show that $r \in R_\gamma$. Let H be a proper convex subgroup of G containing the $\approx$-classes of all elements of $Supp(r)$. To show that $r \in R_\gamma$, it is enough to show that $o_H(r_{\gamma+k}) \in R_\gamma$, since R_γ is truncation-closed. By the definition of tc-independent sequence, all truncations of $r_{\gamma+k}$ are in $R_{\gamma+k}$, including $o_H(r_{\alpha+k})$. By Lemma 2.14, $o_H(r_{\gamma+k})$ is algebraic over $K \cup \{o_H(r_\alpha) : \alpha < \gamma+k\}$. For $\alpha < \gamma$, $r_\alpha \in R_\gamma$. By the Induction Hypothesis, for all $k' < k$, $o_H(r_{\gamma+k'}) \in R_\gamma$. Now, $o_H(r_{\gamma+k})$ is an element of $K((G))$ that is algebraic over elements of R_γ, so it must be an element of R_γ. □

Lemma 3.3. *Let γ be a countable limit ordinal, and let n be a positive integer. Suppose R is a truncation-closed subfield of $K((G))$, with tc-basis $(r_\alpha)_{\alpha<\gamma+n}$, and corresponding canonical sequence $(R_\alpha)_{\alpha\leq\gamma+n}$. Suppose that G has no last $\approx$-class, and for all $k < n$, $Supp(r_{\gamma+k})$ is co-final in G. If the elements of R_γ have length at most ω^{ω^γ}, then the elements of R also have length at most ω^{ω^γ}.*

Proof. It is enough to show that all elements of R with positive valuation have length at most ω^{ω^γ}, for if $w(r) \leq 0$, we may replace r by $t^g r$, where g is chosen so that $w(r) + g > 0$, without changing the length. Suppose $r \in R$, with $w(r) > 0$. Then r is a root of a polynomial $p(x) = A_0 + A_1 x + \ldots + A_s x^s$, where the coefficients A_i are polynomials over K and finitely many r_α's for $\alpha < \gamma + n$. By our convention, $w(r_\alpha) > 0$ for all α, so $w(A_j) \geq 0$ for all j.

Note that $Supp(r)$ is well ordered. We define an increasing sequence of convex subgroups of G corresponding to initial segments of $Supp(r)$. Let H_0 consist of the positive elements in the $\approx$-classes up to and including the smallest one represented in $Supp(r)$. Given H_ξ, if there is some element of $Supp(r) - H_\xi$, let $H_{\xi+1}$ consist of the positive elements in the $\approx$-classes up to and including the smallest that is represented in $Supp(r) - H_\xi$. For limit ξ, let $H_\xi = \bigcup_{\zeta<\xi} H_\zeta$. To show that r has length at most ω^{ω^γ}, it is enough to show that for all ξ, the truncation $o_{H_\xi}(r)$ has length less than ω^{ω^γ}. By Proposition 2.9, $o_{H_\xi}(r)$ is a root of $o_{H_\xi}(p)$. By Lemma 3.2, $o_{H_\xi}(r_{\gamma+k}) \in R_\gamma$, for all $k < n$. Then $o(p)$ is a polynomial over R_γ. Therefore, the root $o_{H_\xi}(r)$ is in R_γ. Hence, any ordinal that bounds the lengths of elements of R_γ also bounds the length of r. □

We are ready to prove our main result, Theorem 1.12. Recall the statement.

Theorem 1.12. Let R be a truncation-closed subfield of $K((G))$, and let γ be a countable limit ordinal.

1. If R has a tc-basis of length γ, then the elements of R have length at most ω^{ω^γ}.

2. If R has a tc-basis of length $\gamma + n$, where $n \geq 1$, then the elements of R have length less than $\omega^{\omega^{\gamma+n}} \cdot \omega^{\omega^\gamma}$.

Remarks. In the case where G is Archimedean and γ is a limit ordinal, the bound above matches that in Theorem 2.3. By Theorem 2.4, this is sharp. For $\gamma+n$, where γ is a countable limit ordinal and n is a positive integer, the bound above has an added factor. We will see that this is needed in case the elements $r_{\gamma+k}$, for $k<n$, are all in a single $\approx$-class, and there is another $\approx$-class to the right of this one.

Proof of Theorem 1.12. Our proof is inductive. For $\gamma=\omega$, Statement (1) holds by part (2) of Theorem 2.2. To complete the proof, we show the following.

(a) If γ is a countable limit ordinal for which Statement (1) holds, then Statement (2) holds for γ.

(b) If γ is a countable limit ordinal for which Statements (1) and (2) hold, then Statement (1) holds for $\gamma+\omega$.

(c) If γ is a limit of limit ordinals for which Statement (1) holds, then Statement (1) holds for γ.

We consider (c) first. Take R with a tc-basis $(r_\alpha)_{\alpha<\gamma}$, where γ is the limit of an increasing sequence of limit ordinals γ_n. Let $(R_\alpha)_{\alpha\leq\gamma}$ be the canonical sequence corresponding to the tc-basis. Each element of R is in some R_{γ_n}. For each n, the elements of R_{γ_n} all have length less than $\omega^{\omega^{\gamma_n}}$. Therefore, the elements of R have length less than $\omega^{\omega^{\gamma}}$. This proves (c).

Next, we consider (b). Take R with a tc-basis $(r_\alpha)_{\alpha<\gamma+\omega}$, and let $(R_\alpha)_{\alpha\leq\gamma+\omega}$ be the corresponding canonical sequence. Each element of R is in $R_{\gamma+n}$ for some n. Now, the elements of $R_{\gamma+n}$ have length at most $\omega^{\omega^{\gamma+n}}\cdot\omega^{\omega^{\gamma}}$. This is less than $\omega^{\omega^{\gamma+\omega}}$. Thus, all elements of R have length less than $\omega^{\omega^{\gamma+\omega}}$. This proves (b).

It remains to prove (a). Take R with a tc-basis $(r_\alpha)_{\alpha<\gamma+n}$. By Lemma 2.19, we may suppose that none of the elements $r_{\gamma+k}$ has the form t^g for $g\in G$—all have limit length. We define a chain of convex subgroups of G. For each k, we include the following.

1. If $Supp(r_{\gamma+k})$ has a greatest $\approx$-class, and there are $\approx$-classes less than this one represented in the support of the r_α's, then we include the convex subgroup with positive elements in the $\approx$-classes strictly to the left of the last one with elements in $Supp(r_{\gamma+k})$.
2. If $Supp(r_{\gamma+k})$ has a greatest $\approx$-class, then we include the convex subgroup with positive elements in the $\approx$-classes up to and including this one.
3. If $Supp(r_{\gamma+k})$ has no greatest $\approx$-class, then we include the convex subgroup with positive elements less than some element in $Supp(r_{\gamma+k})$.
4. We include the trivial group $\{0\}$, and the full group G.

Suppose that the convex subgroups obtained in this way are $H_0, H_1, \ldots, H_s$, where $H_i \subseteq H_{i+1}$. Thus, H_0 is the trivial group, and H_s is the full group G. By Lemma 2.13, we may adjust the tc-basis so that it respects all the H_i; that is, each r_α has support entirely in $H_{i+1} - H_i$ for some $i < s$, and if $R_{\alpha+1} - R_\alpha$ has an element of the form t^g for some $g \in H_i$, then r_α has this form. We now re-order the basis, putting the elements with support in H_i before those with support outside of H_i, leaving the ordering unchanged otherwise. By Proposition 2.16, the result is a new tc-basis $(\hat{r}_\beta)_{\beta < \beta^*}$. Let $(\hat{R}_\beta)_{\beta \leq \beta^*}$ be the corresponding new canonical sequence. For $i < s$, let β_i be the least ordinal such that $\hat{r}_{\beta_i}$ does not have support in H_i. For $i < s$, let $K_i = \hat{R}_{\beta_i}$. (Note that $\beta_0 = 0$ and $K_0 = K$.) Let G_{i+1} be the subgroup generated by the group elements $g \in H_{i+1} - H_i$ such that some $\hat{r}_\beta$ has the form t^g (here $\beta_i \leq \beta < \beta_{i+1}$).

Suppose $i < s$. We consider the lengths of elements of $\hat{R}_{\beta_{i+1}}$, thought of as elements of $K_i((G_{i+1}))$. The sequence $(\hat{r}_\beta)_{\beta_i \leq \beta < \beta_{i+1}}$ is a tc-basis for $K_{i+1} = \hat{R}_{\beta_{i+1}}$ over K_i. We may suppose that the length of this basis is $\gamma + s_i$ (this is an upper bound on the length), where the initial sequence, of length γ, consists of elements r_α, for $\alpha < \gamma$, and the last s_i terms are elements of form $r_{\gamma+k}$. There are different cases, depending on whether there are elements $r_{\gamma+k}$ with support in $H_{i+1} - H_i$, and, if there are such elements, whether the support has a last $\approx$-class. We consider the cases separately.

1. First, suppose that there is no element $r_{\gamma+k}$ with support in $H_{i+1} - H_i$, so $s_i = 0$. Then our tc-basis for K_{i+1} over K_i has length at most γ. Therefore, by hypothesis, the elements of $K_{i+1} = \hat{R}_{\beta_{i+1}}$, thought of as elements of $K_i((G_{i+1}))$, have length at most ω^{ω^γ}.

2. Next, suppose that there is some $r_{\gamma+k}$ such that $Supp(r_{\gamma+k})$ has a last $\approx$-class and is contained in $H_{i+1} - H_i$, so $s_i \geq 1$. Then all elements of $H_{i+1} - H_i$ are in one $\approx$-class by the definition of the chain, so the group G_{i+1} is Archimedean. In this case, the length of our tc-basis for K_{i+1} over K_i is at most $\gamma + s_i$. Then by Lemma 3.1, the elements of K_{i+1}, thought of as elements of $K_i((G_{i+1}))$, have length at most $\omega^{\omega^{\gamma+s_i}}$.

3. Finally, suppose that there is some $r_{\gamma+k}$, with support in $H_{i+1} - H_i$, having no last $\approx$-class. Then $Supp(r_{\gamma+k})$ must be co-final in G_{i+1}. There may be more than one $r_{\gamma+k}$ with support in $H_{i+1} - H_i$. For all such elements, the support must be co-final in G_{i+1} by definition of the chain. Now, we assumed that $\beta_{i+1} = \beta_i + \gamma + s_i$, so, the tc-basis for $\hat{R}_{\beta_i+\gamma}$ over $K_i = \hat{R}_{\beta_i}$ has length at most γ. Hence, by hypothesis, the elements of $\hat{R}_{\beta_i+\gamma}$ have length less than ω^{ω^γ}, when thought of as elements of $K_i((G_{i+1}))$. By Lemma 3.3, the elements of K_{i+1} all have length at most ω^{ω^γ}, again thought of as elements of $K_i((G_{i+1}))$.

Let ξ_1 be a bound on the lengths of elements of K_1, thought of as elements of $K_0((G_1))$, and let ξ_{i+1} be a bound on the lengths of elements of K_{i+1}, thought of as elements of $K_i((G_{i+1}))$.

Claim: As elements of $K((G))$, the elements of K_i have length at most $\prod_{1\leq i'\leq i}\xi_{i'}$, for $1\leq i\leq s$.

Proof of Claim. The statement is true for $i=1$. Supposing that it is true for $i<s$, we prove it for $i+1$. Take $r\in K_{i+1}$. Thought of as an element of $K_i((G_{i+1}))$, r has the form $\sum_{j<\beta} b_j t^{g_j}$, where $\beta\leq\xi_{i+1}$ and $g_j\in G_{i+1}$. Each coefficient b_j is an element of K_i, with support in $G_1\oplus\cdots\oplus G_i$. By the Induction Hypothesis, the length of b_j is at most $\prod_{1\leq i'\leq i}\xi_{i'}$. Thinking of r as an element of $K((G))$, the support, in G, is the union of the sets $\{g_j+h : h\in Supp(b_j)\}$. From this, it is clear that the length is at most $\prod_{1\leq i'\leq i+1}\xi_{i'}$. □

Each ξ_i is bounded by $\omega^{\omega^{\gamma+k_i}}$ where $0\leq k_i\leq s_i\leq n$. We note that the sum of the k_i's is at most n, so if some $k_i=n$, then the others are all 0. By examining the arithmetic, we see that the product $\prod_{1\leq i'\leq i}\xi_{i'}$ is greatest when this happens. Then all $r_{\gamma+k}$ have support in a single $\approx$-class, and s is either 2 or 3. We have the following possibilities.

1. All $r_{\gamma+k}$ have support in H_1. If all r_α have support in H_1, then $R=K_1$, and $\xi_1=\omega^{\omega^{\gamma+n}}$ is a bound on the lengths of elements of R. If some r_α has support outside H_1, then $R=K_2$. In this case, the elements of R all have length at most $\xi_1\cdot\xi_2$, where $\xi_1=\omega^{\omega^{\gamma+n}}$ and $\xi_2=\omega^{\omega^\gamma}$. The product is $\omega^{\omega^{\gamma+n}}\cdot\omega^{\omega^\gamma}$. No element r actually has length equal to this bound, since when we consider r as an element of $K_1((G_2))$, the length is less than ω^{ω^γ}.

2. All $r_{\gamma+k}$ have support in H_2-H_1. If all r_α have support in H_2, then $R=K_2$. The elements of R all have length at most $\xi_1\cdot\xi_2$, where $\xi_1=\omega^{\omega^\gamma}$ and $\xi_2=\omega^{\omega^{\gamma+n}}$. The product is just $\omega^{\omega^{\gamma+n}}$. If some r_α has support outside H_2, then $R=K_3$. In this case, the elements of R all have length at most $\xi_1\cdot\xi_2\cdot\xi_3$, where $\xi_1=\xi_3=\omega^{\omega^\gamma}$ and $\xi_2=\omega^{\omega^{\gamma+n}}$. The product is $\omega^{\omega^{\gamma+n}}\cdot\omega^{\omega^\gamma}$. Again, no element has length equal to this bound.

We have completed the proof of (a) and, hence, of Theorem 1.12. □

4 Sharpness for Theorem 1.12

The result below shows that the bounds on lengths given in Theorem 1.12 are sharp.

Proposition 4.1. *Let γ be a countable limit ordinal, and let n be a positive integer. Let K be a real closed or algebraically closed field. Let G be a divisible ordered Abelian group of rank 2, with positive generators g_1, g_2, where $g_2>ng_1$ for all positive integers n. Then there is a truncation-closed subfield R of $K((G))$, with a tc-basis of length $\gamma+n$, such that R has elements of length $\omega^{\omega^{\gamma+n}}\cdot\omega^{\omega^\alpha}$, for all $\alpha<\gamma$.*

Proof. Let G_i be the subgroup generated by g_i. By the second statement in Theorem 2.4, there is a truncation-closed subfield K_1 of $K((G_1))$ with a *tc*-basis $(r_\alpha)_{\alpha\leq\gamma+n}$ of length $\gamma+n$, and with an element s of length $\omega^{\omega^{\gamma+n}}$, having support in $G_1^{>0}$. There is also a truncation-closed subfield K_2 of $K((G_2))$ with a *tc*-basis $(r'_\alpha)_{\alpha<\gamma}$ of length γ, and with elements s_α of length ω^{ω^α} for all $\alpha<\gamma$, having support in $G_2^{>0}$. We can combine the two *tc*-bases into a single sequence of length $\gamma+n$, alternating terms from the two initial γ-sequences, and keeping the order. The combined sequence is a *tc*-basis of length $\gamma+n$ for a truncation-closed subfield R of $K((G))$. Consider $s\cdot s_\alpha$ as an element of R.

Claim: As an element of $K((G))$, $s\cdot s_\alpha$ has length $\omega^{\omega^{\gamma+n}}\cdot\omega^{\omega^\alpha}$.

Proof. We note that the support of $s\cdot s_n$ consists of all h' such that $h\in Supp(s)$ and $h'\in Supp(s_\alpha)$. We show that there is no possibility of cancellation. Suppose $h_1+h'_1=h_2+h'_2$, where $h_i\in Supp(s)$ and $h'_i\in Supp(s_\alpha)$. Suppose $h'_1\neq h'_2$, say $h'_1<h'_2$. Then $h'_2-h'_1$ is a positive element of G_2. This is equal to h_1-h_2, which is an element of G_1. Since all positive elements of G_1 are smaller than any positive element of G_2, this is a contradiction. Therefore, we must have $h'_1=h'_2$ and $h_1=h_2$. This shows that cancellation is impossible, so the Claim is true. □

□

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