Complexity of structures associated with real closed fields

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Abstract

Real closed fields, and structures associated with them, are interesting from the point of view of both model theory and computability. In this paper, we give results on the complexity of value group sections and residue field sections. It is not difficult to show that for any countable real closed field R, there is a value group section that is $\Delta^0_2(R)$. This result is sharp in the sense that there is a computable real closed field for which every value group section codes the halting set. For a real closed field R, there is a residue field section that is $\Pi^0_2(R)$. This result is sharp in the sense that there is a computable real closed field R with no Σ^0_2 value field section. We are also interested in integer parts. Mourgues and Ressayre showed, by a rather complicated construction, that every real closed field has an integer part. The construction becomes canonical once we fix the real closed field R, a residue field section k, and a well ordering of R. The construction involves mapping the elements of R to generalized series, called *developments*, with terms corresponding to elements of the natural value group and coefficients in k. The complexity of the construction is clearly related to the lengths of the developments. We conjecture that for a type ω well ordering on R, the lengths of the developments are less than ω^{ω^ω}. We give an example showing that there is no smaller ordinal bound.

1 Introduction

Tarski [17] showed that the theory of the ordered field of real numbers is decidable. To do this, he gave an elimination of quantifiers, keeping track of the axioms needed. The axioms describe an ordered field in which all polynomials of odd degree have zeroes, and an element is positive precisely when it has a square root. The *real closed fields* are the models of this theory. Tarski's elimination of quantifiers also yields the fact that the theory is *o-minimal*; i.e., in any model, the sets definable by a formula in one free variable, with parameters, are the finite unions of intervals and points—these sets are definable just from the ordering. In Section 2, we give background on value groups and residue fields. Sections 3 and 4 have the results on complexity of value group sections and residue field sections. In Section 5, we give some background on integer parts. In particular, we consider Mourgues and Ressayre's construction of an integer part for an arbitrary real closed field R. This construction, which becomes canonical given a residue field section of R and a well ordering on R, involves embedding R into a field of generalized series of ordinal length. We conjecture that for a type ω well ordering on R, the lengths of the series in the image of this embedding are less than ω^{ω^ω}. In Section 6, we give an example showing that there is no smaller ordinal bound. We construct a countable real closed field R, with a residue field k, and a type ω well ordering of R, such that when we apply the construction of Mourgues and Ressayre, we will define a chain (R_n, δ_n), where R_n is a real closed subfield of R and δ_n is the restriction of the embedding to R_n, and we find that for each $n \geq 1$, δ_n maps some element of R_n to a series of length $\omega^{\omega^{(n-1)}}$.

2 Background on value groups and residue fields

Let R be a real closed field. The *natural value group* of R represents the orders of infinity in R. We begin by defining the "archimedean" equivalence relation on R.

Definition 1 (Archimedean equivalence)**.** *For $x, y \in R^\times := R - \{0\}$, $x \sim y$ iff there exists $n \in \mathbb{N}$ such that $n|x| \geq |y|$ and $n|y| \geq |x|$, where $|x| := \max\{x, -x\}$. We denote the equivalence class of $x \in R^\times$ by $w(x)$.*

Definition 2 (Value group)**.** *The* value group *of R is the set of equivalence classes $w(R^\times) = \{w(x) \mid x \in R^\times\}$ with multiplication on $w(R^\times)$ defined to be $w(x)w(y) = w(xy)$. We endow $w(R^\times)$ with the order*

$$w(x) < w(y) \text{ if } (\forall n \in \mathbb{N})[\, n|x| < |y|\,].$$

By convention, we let $w(0) < w(R^\times)$.

Under the given operation and ordering, $w(R^\times)$ is an ordered abelian group with identity $w(1)$. Moreover, the map $x \mapsto w(x)$ is a *valuation*, i.e. it satisfies the axioms $w(xy) = w(x)w(y)$ and $w(x+y) \leq \max\{w(x), w(y)\}$. If R is a real closed field, then the value group $w(R^\times)$ is divisible (see [9, Theorem 4.3.7]). Under multiplicative notation, an abelian group $(G, \cdot)$ is *divisible* if for all $g \in G$ and $0 \neq n \in \mathbb{N}$, we have $g^{\frac{1}{n}} \in G$. Note that a divisible abelian group $(G, \cdot)$ is a $\mathbb{Q}$-vector space when scalar multiplication by $q \in \mathbb{Q}$ is defined to be g^q.

We are interested in subgroups of $(R^{>0}, \cdot)$ with exactly one element from each $\sim$-equivalence class.

Definition 3 (Value group section)**.** *Let R be a real closed field, and suppose $t : w(R^\times) \to R^{>0}$ taking $w(x)$ to an element of $w(x)$. If t is an embedding of the ordered group $w(R^\times)$ into the ordered group $(R^{>0}, \cdot)$, then we refer to the image of t as a* value group section *of R.*

The next result is well-known (see [12, Theorem 8]). Since we will give an effective version of this result later, we sketch the proof.

Proposition 2.1. *If R is a real closed field, there is a subgroup G of $(R^{>0}, \cdot)$ that is a value group section of R.*

Proof sketch. We consider subgroups H of $(R^{>0}, \cdot)$ with the following features:

1. (Closed under n^{th} roots) $(\forall h \in H)(\forall n \in \mathbb{N}^{>0})[h^{\frac{1}{n}} \in H]$
2. (Unique representatives for $w(R^{>0})$) $(\forall h, h' \in H)[h \neq h' \to w(h) \neq w(h')]$

The trivial subgroup $\{1\}$ has these features. If H is a group with these features, and r is an element of $R^{>0}$ for which $w(r) \neq w(h)$ for all $h \in H$, then the subgroup $H' = \{hr^{\frac{1}{n}} \mid h \in H \ \& \ n \in \mathbb{N}^{>0}\}$ also has these features and properly extends H. The set of groups with these features is closed under unions of chains, so there is a maximal such subgroup G. The group G is a value group section of R, by maximality.

□

If R is archimedean, then the trivial subgroup $\{1\}$ is the unique value group section of R. If R is not archimedean, there are many different value group sections, although they are all isomorphic.

Notation: Once we have fixed a value group section G, we may write $w(x)$ for the element $g \in G$ such that $g \sim x$.

2.1 Residue Fields

For a real closed field R, the *residue field* represents the set of reals present in R. Here is the precise definition.

Definition 4 (Residue field)**.** *Let R be a real closed field. The subring of* finite elements *of R is the set $\mathcal{O} = \{r \in R \mid w(r) \leq 1\}$. The set of* infinitesimals *of R is the set $\mathcal{M} = \{r \in R \mid w(r) < 1\}$. The* residue field *is the quotient of $\mathcal{O}$ by the ideal $\mathcal{M}$.*

Since $\mathcal{M}$ is the maximal ideal of $\mathcal{O}$, the residue field $\mathcal{O}/_{\mathcal{M}}$ is a field, with an ordering induced by that of R. It is Archimedean, so it is isomorphic to a subfield of $\mathbb{R}$. If R is a real closed field, then $\mathcal{O}/_{\mathcal{M}}$ is real closed (see [9, Theorem 4.3.7]).

We want to find ordered subfields of R that represent the reals present.

Definition 5 (Residue field section)**.** *Let R be a real closed field, with residue field $\mathcal{O}/_{\mathcal{M}}$, and suppose ι is a function assigning to each coset in $\mathcal{O}/_{\mathcal{M}}$ an element of that coset. If ι is an embedding of the residue field into R, then we refer to the image as a* residue field section *of R.*

If R is real closed, then there is a residue field section (see [12, Theorem 8]). We again sketch the proof, since we will give an effective version of this result later.

Proposition 2.2. *Every real closed field R has a residue field section k.*

Proof sketch. We take a maximal archimedean real closed subfield of R. The real algebraic elements of R form one such subfield. Given F, an archimedean real closed subfield of R, if there is some $r \in R$ such that $w(r) = 1$, and $w(r - a) = 1$ for all $a \in F$ (so there is no $a \in F$ with $w(r - a) < 1$), then the real closure of $F(r)$ is an archimedean real closed subfield of R that properly extends F. The set of archimedean real closed subfields is closed under unions of chains, and hence there is a maximal such subfield k. By maximality, k is a residue field section.

□

There is, in general, more than one possible residue field section, although all are isomorphic. Since we will only consider value group sections in this paper and residue field sections, we now simply refer to them as value groups and residue fields of R. We now turn to questions of complexity.

Definition 6 (Computable model)**.**
1. *The* atomic diagram *of a model R, denoted $D^a(R)$, is the set of atomic formulas, in the language of the model expanded to include constants for the domain of R, that hold in R.*
2. *A model R is* computable *if $D^a(R)$ is a computable set with respect to some effective enumeration of the atomic formulas in the expanded language.*

3 Complexity of the value group

In this section, we show that for any countable real closed field R, there is a value group section of R that is $\Delta^0_2(R)$. We also show that this result is optimal by giving a computable real closed field for which every value group section codes the halting set.

Theorem 3.1. *For any countable real closed field R, there is a value group section that is $\Delta^0_2(R)$.*

Proof. By Proposition 2.1, there is a subgroup G of $(R^{>0}, \cdot)$, with a unique element from each $\sim$-class, except $\{0\}$. We must produce G using $\Delta^0_2(R)$. Let $(r_n)_{n\in\omega}$ be a list of the elements of $R^{>0}$, with $r_0 = 1$. We can make the list computable in R. We form an increasing sequence $(G_n)_{n\in\omega}$ of divisible subgroups of $(R^{>0}, \cdot)$ such that G_n has representatives for $w(r_k)$ for all $k \leq n$. Let $G_0 = \{1\}$. Given G_n, we see if $w(r_{n+1})$ has a representative in G_n. If so, then $G_{n+1} = G_n$. If not, then G_{n+1} is the group generated by the elements of G_n and r^q_{n+1} for $q \in \mathbb{Q}$. We let $G = \cup_n G_n$. Clearly, G is a divisible abelian group. If R' is the real closure of G, then G is clearly a value group for R'. It is clear from the construction that all $\sim$-classes in R are represented, so G is also a value group for R. We note that $\Delta_2^0(R)$ can determine whether $r_1 \in G_1$, and knowing whether $r_k \in G_k$ for all $k < n$, $\Delta_2^0(R)$ can determine whether $r_n \in G_n$. We just ask whether there exist $g \in G_{n-1}$ and integers k, m such that $kg > r_n$ and $mr_n > g$.

□

Here is the result showing that Theorem 3.1 is optimal.

Theorem 3.2. *There is a computable real closed field R such that any value group section G of R computes the halting set K, i.e., $G \geq_T K$.*

Proof. Our computable real closed field R will have universe ω, which we think of as a set of constants. Let F be a computable real closed field with an infinite sequence of positive elements $(r_n)_{n\in\omega}$ such that $1 << \ldots << r_3 << r_2 << r_1$, where $x << y$ means that for all n, $x^n < y$. At each stage s, we have a finite partial $1-1$ function f_s from ω to F, and we have enumerated a finite part d_s of the atomic diagram of R so that f_s, by interpreting the constants that appear in d_s, makes the sentences true in F. Let $(p_n)_{n\in\omega}$ be the standard sequence of primes. At each stage s, we map the elements of ω that are powers of p_n to $y \in F$ such that either $y \sim 1$ (and $y \neq 1$) or $y \sim r_n^q$ for some $q \in \mathbb{Q}$. At stage 0, we let f_s map 1 (in ω) to 1 (the multiplicative unit in F). At stage $s+1$, we have determined f_s and we have $R_s = dom(f_s)$. Suppose e enters K at stage $s+1$. Let S be the set of $x \in R_s$ such that $F \models f_s(x) \leq r_e \vee \bigvee_{q\in\mathbb{Q}} f_s(x) \sim r_e^q\}$. We define f_{s+1}, preserving d_s, so that for all $x \in S$, if $f_{s+1}(x) \neq 0$, then $f_{s+1}(x) \sim 1$, so $f_{s+1}(x)$ cannot be a group element. We extend the range and domain of f_{s+1} such that for $l \leq s$, p_e^l is mapped to some element of the $\sim$-class of 1, and vowing that for $l \geq s+1$, elements of the form p_e^l not already mapped to the $\sim$-class of 1 will be mapped to the $\sim$-class of $(r_e)^q$ for some q.

Let R be the real closed field with atomic diagram $D(R) = \cup_{s\in\omega} d_s$. By construction, R is computable. We have $R \cong F$ via a function f that is the limit of the f_s. Let G be a value group section of R. We can decide whether $e \in K$ using G. Since $R \cong F$, there is some element $x_e \in R$ such that $f(x_e) = r_e$. By construction, x_e must have the form p_e^l for some l. Moreover, for any q, any $y \in R$ in the $\sim$-class of $(x_e)^q$, has the form p_e^l for some l. We search for (and find) an element y in G of the form $p_e^{l_e}$ for some l_e.

Claim: $e \in K$ iff $e \in K_{l_e}$; i.e., e has entered K by stage l_e.

of Claim. By construction, if y has the form $(p_e)^l$, then either $y \sim 1$ or $y \sim x_e^q$ for some $q \in \mathbb{Q}$. Since $y \in G$, we cannot have $y \sim 1$. Therefore, we must have $y \sim (x_e)^q$, for some q. If $e \in K$, then y has the form $(p_e)^l$ for some l where $e \in K_l$ by construction.

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We can also produce a computable real closed field with no c.e. value group. The construction is straightforward, and we omit it.

4 Complexity of the residue field section

In this section, we show that for any countable real closed field R, there is a residue field section of R that is $\Pi^0_2(R)$. We also show that this result is optimal by giving a computable real closed field with no Σ^0_2 residue field section.

Theorem 4.1. *Given a countable real closed field R, there is a residue field section of R that is $\Pi^0_2(R)$.*

Proof. Recall that all residue field sections for R are isomorphic. We suppose that the universe of R is ω. The case where R is archimedean is trivial—R itself is a residue field section. We suppose that R is not archimedean. If the transcendence degree of the residue field is finite and $\overline{r}$ is a transcendence basis, then the real closure of $\overline{r}$ is c.e. in R, so it is $\Delta^0_2(R)$. Now, suppose the transcendence degree is infinite. As a first step, we locate a transcendence basis for a maximal archimedean subfield of R, using $\Delta^0_3(R)$. Let r_0 be the first element (in the usual ordering of ω) such that the real closure of r_0 is archimedean. Given $r_0, \ldots, r_n$, with real closure k_n, let r_{n+1} be the first element not in k_n, such that the real closure of $k_n(r_{n+1})$ is archimedean, and let k_{n+1} be the real closure of $k_n(r_{n+1})$. The statement that r_{n+1} is not in k_n is $\Pi^0_1(R)$, and the statement that the real closure of $k_n(r_{n+1})$ is archimedean is $\Pi^0_2(R)$. Hence, $\Delta^0_3(R)$ can calculate r_{n+1}. Clearly, $k = \cup_n k_n$ is a residue field section, and the sequence $(r_n)_{n\in\omega}$ is a transcendence basis for k. The field k is $\Delta^0_3(R)$. To see this, note that $n \in k$ iff $n \in k_n$. To decide whether $n \in k$, we first find $r_0, \ldots, r_n$, using $\Delta^0_3(R)$, and then see if n is in the real closure of this tuple, which we can do using $\Delta^0_2(R)$.

We build a $\Pi^0_2(R)$ residue field section k' for R by using $\Delta^0_2(R)$ to (1) build a new sequence $(r'_n)_{n\in\omega}$, which will be the basis for our residue field section k', and (2) enumerate the complement of k'. Since the sequence $(r_n)_{n\in\omega}$ is $\Delta^0_3(R)$, we have a uniform $\Delta^0_2(R)$ procedure for guessing initial segments of the r_n's. Say the stage s guess is $\overline{r}_s$. Since $\Delta^0_2(R)$ can check whether a finite set is algebraically independent, we may suppose that $\overline{r}_s$ is algebraically independent. For each s, the tuple $\overline{r}_{s+1}$ has the form $\overline{r}_t, r$, where $\overline{r}_t$ is an initial segment of $\overline{r}_s$ that has looked correct in our $\Delta^0_2(R)$-approximation since stage t. To account for mistakes caused by incorrect guesses in our $\Delta^0_2(R)$-approximation, we may need to adjust the elements of $\overline{r}_{s+1}$ to form $\overline{r}'_{s+1}$. In particular, we may need to replace r with some r' that fills the same rational cut. We let k'_{s+1} be the real closure of $\overline{r}'_{s+1}$ and $k' = \cup_{s\in\omega} k'_s$.

For any infinitesimal element ϵ, the elements r and $r + \epsilon$ fill the same rational cut in R. Given one infinitesimal element ϵ, the collection of $m\epsilon$ for all $m > 0$ consists of all distinct infinitesimals. Thus, we can obtain many different potential choices for r' by taking $r + m\epsilon$ for some m. (We could do this replacement using any infinite set of distinct infinitesimals.) This is the idea behind the construction that follows.

At stage 0, we let $\overline{r}_0 = \emptyset$, and we let k'_0 be the real closure of $\emptyset$ in R. Suppose at stage s, we have determined $\overline{r}_s$, $\overline{r}'_s$ and k'_s, where for each $r \in \overline{r}_s$, the corresponding element r' of $\overline{r}'_s$ has the form $r + m\epsilon$ for some m. We have enumerated finitely many elements (none of which are in the real closure of $\overline{r}'_s$) into the complement of k'.

At stage $s+1$, we first check whether $\overline{r}_s$ still appears to be an initial segment of the $\Delta^0_3(R)$ sequence of r_n's. If so, then we guess the next element r in this sequence, and we let $\overline{r}'_{s+1} = \overline{r}'_s, r$, checking that this tuple is algebraically independent and that ϵ and the finitely many elements in the complement of k' are not in the real closure of this tuple using $\Delta^0_2(R)$. If these latter checks fail, we follow the procedure outlined below.

Suppose there is some first element of $\overline{r}_s$ that now seems incorrect. We take the greatest $t < s+1$ such that $\overline{r}_t$ has appeared correct since stage t, and we guess at the next r using our $\Delta^0_2(R)$-approximation. We consider $\overline{r}_{s+1} = \overline{r}_t, r$ and $\overline{r}'_{s+1} = \overline{r}'_t, r$, again checking that these tuples are individually algebraically independent and ϵ not in the real closure of either tuple using $\Delta^0_2(R)$. If either tuple fails these tests, we wait until our $\Delta^0_2(R)$-approximation provides us with such $\overline{r}_t, r$ and corresponding $\overline{r}'_t, r$. If no elements of the real closure of $\overline{r}'_t, r$ have been enumerated into the complement of k', then we let $\overline{r}'_{s+1} = \overline{r}'_t, r$; we can check this using $\Delta^0_2(R)$. Otherwise, we replace r by $r' = r + m\epsilon$, where m is chosen so that the real closure of $\overline{r}'_t, r'$ does not include any of the elements already in the complement of k'. Note that $r + m\epsilon$ and $r + n\epsilon$ are not interalgebraic over k'_t, so any element enumerated in the complement is interalgebraic over k'_t with at most one $r + m\epsilon$. Therefore, such an r' exists. We let k'_{s+1} be the real closure of $\overline{r}'_{s+1}$ and we vow not to enumerate into the complement of k' any elements of k'_{s+1} for the time being. We enumerate into the complement of k' all $x \leq s+1$ such that, after searching $s+1$ steps, we find evidence that $k'_{s+1}(x)$ is not archimedean.

Since the $\Delta^0_2(R)$-approximation eventually settles, each r'_i has a limiting value. Consider these values. We can show inductively that that the map taking r_i to r'_i extends to an isomorphism of the respective real closures. Hence, $r'_0, r'_1, r'_2, \ldots$ is a transcendence basis for a maximal archimedean subfield of R, and k' is a residue field section of R that is $\Pi^0_2(R)$.

□

We will show that Theorem 4.1 is optimal, in the following sense.

Theorem 4.2. *There exists a computable real closed field R such that no residue field section k of R is Σ^0_2.*

We consider the following set.

Definition 7. *Let FT be the set of $x \in R$ such that x is finite and not infinitesimally close to any real algebraic number.*

Any residue field section of R must contain the (real) algebraic numbers $\overline{\mathbb{Q}}$, and it cannot contain any element that is infinitesimally close, but not equal, to an algebraic number. The elements of FT are the *other* elements of R that *could* go into a residue field section. The following lemma is not difficult to check.

Lemma 4.3. *Let k be a residue field section of R. Then*

$$x \in FT \leftrightarrow w(x) = 1 \;\&\; (\exists y \in k)\,(y \notin \overline{\mathbb{Q}} \;\&\; \bigwedge_n |x-y| < \frac{1}{n}) \;.$$

Using the lemma, we can see that if R is computable and k is a Σ^0_2 residue field section, then FT is also Σ^0_2. Thus, to prove Theorem 4.2, it is enough to prove the following.

Proposition 4.4. *There is a computable real closed field R such that the set FT (consisting of elements that are finite and not infinitesimally close to any algebraic number) is not Σ^0_2.*

Proof. Let S_e be the e^{th} Σ^0_2 set. We have a uniformly computable sequence of approximations $(S_{e,s})_{s\in\omega}$, where $n \in S_e$ iff there exists s_0 such that for all $s \geq s_0$, $n \in S_{e,s}$. We suppose that for any finite set α consisting of pairs $\langle e, n\rangle$ such that $n \notin S_e$ for each $\langle e, n\rangle \in \alpha$, there are infinitely many s such that for all $\langle e, n\rangle \in \alpha$, we have $n \notin S_{e,s}$. We build a computable real closed field R, with universe ω. We satisfy the following requirements:

$\mathbf{R_e}$: $S_e \neq FT$.

Satisfying a single requirement R_e

To satisfy R_e, we choose a witness w. Let F be a computable copy of the field of real algebraic numbers. Our field R will be isomorphic to the real closure of $F(t)$, where w corresponds to t, and $t \in FT$ iff $w \notin S_e$. We use a language with symbols for definable functions, so that all elements of R will have names $f(w)$. If we can enumerate the diagram using these symbols, then we can also make the domain of R equal to ω. We do not specify t in advance, but determine it by a nested sequence of intervals $(I_s)_{s\in\omega}$ with endpoints in F. At stage s, we choose an interval I_s, and we determine a finite part d_s of the atomic diagram of our R. We make sure that the sentences of d_s are *valid* on I_s; i.e., for all $x \in I_s$, if we assign w value x, then the sentences in d_s are true in F. Let $(\varphi_s)_{i\in\omega}$ be an effective enumeration of all atomic sentences involving just w (plus symbols from the language of fields and for the definable functions). We ensure that φ_i or $\neg\varphi_i$ is in d_s for all $i \leq s$.

We start with $I_0 = (0, 1)$ and $d_0 = \emptyset$. At stage $s > 0$, we choose $I_s = (a, b)$ with the following features.

1. $(a, b) \subseteq I_{s-1}$,
2. $b - a \leq \frac{1}{s}$,
3. If $w \notin S_{e,s}$, then for any x among the first s algebraic numbers in a fixed enumeration of $\overline{\mathbb{Q}}$, either $x < a$ or $x > b$, and if $w \in S_{e,s}$, then a is the left boundary of I_{s-1},

4. the sentences in d_s are all valid on I_s.

Starting with I_{s-1}, we first change the right endpoint to make 1 hold. In the case where $w \notin S_{e,s}$, we reduce the interval further so that the first x algebraic numbers lie outside the closure. Now, we want to make φ_s or $\neg\varphi_s$ valid. If $w \in S_{e,s}$, then we want to make φ_s or $\neg\varphi_s$ valid, while preserving the left endpoint. The following lemma, an easy consequence of o-minimality, says that we can do this.

Lemma 4.5. *For any open interval I and any sentence φ involving just w and symbols for functions definable from w, there is an open interval $J \subseteq I$ on which φ or $\neg\varphi$ is valid. Moreover, we may choose J to have the same left boundary as I.*

If $w \notin S_e$, then all algebraic numbers are eventually excluded from the closure of the intervals I_s. Therefore, $w \in FT$. If $w \in S_e$, then there is an algebraic number a such that for all sufficiently large s, the left endpoint of I_s is a. Then w is infinitesimally close to a, so it is not in FT.

Satisfying Requirements R_0 and R_1

For R_0, we have a single witness w_0, and we act on R_0 as described above. For R_1, we have one witness w_1^0 for the case where $w_0 \notin S_0$, and further possible witnesses $w_{1,k}^1$, for the case where $w_0 \in S_0$. At stage s, we have either two or three "live" witnesses. We determine w_0 and w_1^0 at stage 0. If $w_0 \notin S_{0,s}$ and $w_0 \in S_{0,s+1}$, then at stage $s+1$, we determine a new witness $w_{1,k}^1$. The witness w_1^0 is still live since we may return to it later. If $w_0 \in S_{0,s}$ and $w_0 \notin S_{0,s+1}$, then at stage $s+1$, we abandon the stage s witness $w_{1,k}^1$. It would be enough to make it definable from w_0 and w_1^0, but in fact, we may choose it to be an algebraic number. It is no longer live.

Our language includes symbols for definable functions, and at various stages, we specify which constants $w_{1,k}^1$ we wish to use as witnesses. It may be that we will use only finitely many. At each stage s, we have determined a finite part d_s of the diagram of R, and we have a cell C_s on which what we have said in d_s about the live witnesses is valid. For an abandoned witness $w_{1,k}^1$, the statements in d_s will include a sentence saying that $w_{1,k}^1$ is equal to a particular algebraic number. The cells C_s will have a certain form.

Definition 8. *A* special n-cell *has the form*

$$C = \{(x_1, \ldots, x_n) \mid a_1 < x_1 < b \;\&\; a_2 < x_2 < f_2(x_1) \;\&\; \ldots \;\&\; a_n < x_n < f_n(x_1, \ldots, x_{n-1})\}.$$

The point $(a_1, \ldots, a_n)$ is called the left boundary *of C. The cell C is* ordered *if it has the further property that for all $(x_1, \ldots, x_n) \in C$*

$$x_1 - a_1 > x_2 - a_2 > \ldots > x_n - a_n.$$

We use special ordered 2-cells and 3-cells, always definable in F. If at stage s, there are just two live witnesses, w_0 and w_1^0, then both are active, and C_s will be a special ordered 2-cell $\{(x,y) : a < x < b \;\&\; c < y < f(x)\}$. Any assignment of (w_0, w_1^0) to $(x,y) \in C_s$ makes the sentences in d_s true. The fact that we have only these live witnesses means that our approximation guesses that $w_0 \notin S$ so w_0 will move away from a, and then w_1^0 is free to either approach c or move away from it. If we have three live witnesses w_0, w_1^0, and $w_{1,k}^1$, then w_1^0 is inactive. The cell C_s has the form $\{(z,x,y) : a < z < b \;\&\; c < x < f(z) \;\&\; d < y < g(z,x)\}$, where any assignment of $(w_{1,k}^1, w_0, w_1^0)$ to $(z,x,y) \in C_s$ makes the sentences in d_s true. The fact that $w_{1,k}^1$ is live means that our approximation guesses that w_0 approaches c. Since the cell is ordered, w_1^0 is forced to approach d, while $w_{1,k}^1$ is free to either approach or move away from a.

To start off, C_0 is an ordered 2-cell, and we suppose that $w_0 \notin S_{0,0}$. In general, if $w_0 \notin S_{0,s}$, our intention is to move w_0 away from the first s algebraic numbers. If $w_1^0 \in S_{1,s}$, then we move w_1^0 toward the left endpoint of its interval, and otherwise, we move it away from the first s algebraic numbers. If $w_0 \in S_{0,s}$, then at stage s, we create a new witness $w_{1,k}^1$, and we pass to an ordered 3-cell C_s, with first variable z corresponding to $w_{1,k}^1$. At stage s, we decide the first s atomic sentences in the constants used so far.

Below, we give the main lemmas saying that the appropriate cells exist. We give the lemmas for n-cells since we will use them in the general construction. Lemma 4.6 generalizes Lemma 4.5.

Lemma 4.6. *Let C be a special n-cell. For any formula $\varphi(x_1, \ldots, x_n)$, there is a special n-cell $C' \subseteq C$ on which φ is valid or $\neg\varphi$ is valid. Moreover, we may take C' to have the same left boundary as C. In this case, if C is ordered, then C' is ordered.*

The next lemma lets us start with an ordered special n-cell and insert a new variable y into the cell. Note that the new cell preserves the left boundary of the original cell.

Lemma 4.7. *Let C be the ordered special n-cell in variables $(x_1, \ldots, x_n)$. There is an ordered special $(n+1)$-cell C' in variables $(x_1, \ldots, x_k, y, x_{k+1}, \ldots, x_n)$ such that the left boundary of x_i in C is preserved in C'. We may also insert the new variable y before x_1 or after x_n. Then, all formulas $\varphi(x_1, \ldots, x_n)$ that are valid on C are valid on C'.*

The following lemma says that certain variables can move away from the left boundary of their interval while other variables can move toward the left boundary of their interval or be set equal to algebraic numbers.

Lemma 4.8. *Let C be a special ordered n-cell with left boundary $(a_1, \ldots, a_n)$. For any $1 \leq k \leq n$, there is a special ordered n-cell $C' \subseteq C$ with left boundary $(b_1, \ldots, b_k, a_{k+1}, \ldots, a_n)$, where $b_i > a_i$ for $1 \leq i \leq k$.*

Moreover, for any $1 \leq l \leq k \leq n$, there is a special ordered $n - (k - l + 1)$-cell C'' and numbers $b_i' > a_i$ for $1 \leq i \leq k$ such that C'' has left boundary

$(b'_1, \ldots, b'_{l-1}, a_{k+1}, \ldots a_n)$ *and*

$$\{(x_1, \ldots, x_{l-1}, b'_l, \ldots, b'_k, x_{k+1}, \ldots, x_n) \mid (x_1, \ldots, x_{l-1}, x_{k+1}, \ldots, x_n) \in C''\} \subset C.$$

Proof. Since C is an (open) ordered cell of the form

$$\{(x_1, \ldots, x_n) \mid a_1 < x_1 < b \ \& \ a_2 < x_2 < f_2(x_1) \ \& \ \ldots \ \& \ a_n < x_n < f_n(x_1, \ldots, x_{n-1})\},$$

there is an open box $B \subseteq C$ with left boundary $(b_1, \ldots, b_n)$ and right boundary $(c_1, \ldots, c_n)$. We can take $b_i > a_i$ for all $1 \leq i \leq k$.

For $2 \leq i \leq k$, let $g_i(x) = \min(x + b_i - b_{i-1}, c_i)$, and let

$$\begin{gathered} C' = \{(x_1, \ldots, x_n) \mid \\ b_1 < x_1 < c_1 \ \& \ b_2 < x_2 < g_2(x_1) \ \& \ \ldots \ \& \ b_k < x_k < g_k(x_{k-1}) \\ \& \ a_{k+1} < x_{k+1} < \min(f_{k+1}(x_1, \ldots, x_k), x_k + a_{k+1} - b_k) \ \& \\ a_{k+2} < x_{k+2} < f_{k+2}(x_1, \ldots, x_{k+1}) \ \& \ \ldots \ \& \ a_n < x_n < f_n(x_1, \ldots, x_{n-1})\}. \end{gathered}$$

The set C' is an ordered special n-cell contained in C that satisfies the inequalities

1. $x_i - b_i > x_{i+1} - b_{i+1}$ for all $1 \leq i \leq k-1$,
2. $x_k - b_k > x_{k+1} - a_{k+1}$, and
3. $x_i - a_i > x_{i+1} - a_{i+1}$ for all $k+2 \leq i \leq n$.

Moreover, C' has left boundary $(b_1, \ldots, b_k, a_{k+1}, \ldots a_n)$.

The argument to construct C'' is very similar. Take the open box $B \subseteq C$ and define $g_i(x)$ as above for $2 \leq i \leq k$. For $i < l$, let $b'_i = b_i$. For i satisfying $l \leq i \leq k$, let b'_i be some algebraic number in (b_i, c_i). We let

$$\begin{gathered} C'' = \{(x_1, \ldots, x_{l-1}, x_{k+1}, \ldots, x_n) \mid \\ b_1 < x_1 < c_1 \ \& \ b_2 < x_2 < g_2(x_1) \ \& \ \ldots \ \& \ b_{l-1} < x_{l-1} < g_{l-1}(x_{l-2}) \\ \& \ a_{k+1} < x_{k+1} < \min(f_{k+1}(x_1, \ldots, x_{l-1}, b'_l, \ldots, b'_k), x_{l-1} + a_{k+1} - b_{l-1}) \ \& \\ a_{k+2} < x_{k+2} < f_{k+2}(x_1, \ldots, x_{l-1}, b'_l, \ldots, b'_k, x_{k+1}) \ \& \ \ldots \ \& \\ a_n < x_n < f_n(x_1, \ldots, x_{l-1}, b'_l, \ldots, b'_k, x_{k+1}, \ldots, x_{n-1})\}. \end{gathered}$$

It is straightforward to check that C'' has the desired properties. □

Satisfying R_e for all e

We organize our strategies on the full binary tree $2^{<\omega}$. For any $\sigma \in 2^{<\omega}$, node σ is associated with requirement R_e for $e = |\sigma|$, and node σ may have a witness w_e^σ associated with it. We will see that the superscript σ for a witness w_e^σ will help us describe the ordering on the various witnesses. For any string $\sigma \in 2^{<\omega}$ and $i \leq |\sigma|$, we let $\sigma \restriction i$ denote the substring of σ of length i.

Definition 9 (Active nodes and σ_s)**.**

1. *A node σ is* active at stage s *if $|\sigma| \leq s$ and for each $i < |\sigma|$, there is a witness $w_i^{\sigma \restriction i}$ associated with strategy R_i such that $\sigma(i) = 0$ if and only if $w_i^{\sigma \restriction i} \notin S_{i,s}$.*

2. Let σ_s be the active node of length s on the tree at stage s.

The node σ_s gives the stage s approximation to whether the requirement R_i is trying to build $w_i^{\sigma\restriction i}$ infinitesimally close to an algebraic number (the guess if $\sigma_s(i) = 1$) or whether R_i is be working to put $w_i^{\sigma_s\restriction i}$ into FT (the guess if $\sigma_s(i) = 0$) for any $i < |\sigma_s|$. At another stage t, the active node σ_t may have different guesses about what strategy R_i is trying to enact for $i < |\sigma_t|$.

If a node σ is active at stage s, and there is no current (or "live") witness associated with σ, we create a new witness w_e^σ for $e < |\sigma|$. This witness, as well as witnesses associated with predecessors of σ, are live and active. Witnesses associated with nodes to the right of σ are not live. Witnesses associated with nodes to the left of σ are live but inactive. An inactive witness becomes active again when the active witnesses to the right of it are abandoned.

We describe the construction at stage $s+1$. We are given σ_s the node of length s that is active at stage s, and let σ_{s+1} be the active node of length $s+1$ at stage $s+1$. Let τ be the maximal initial substring such that $\sigma_s, \sigma_{s+1} \succ \tau$, and let $e = |\tau|$. In other words, we guess that all the strategies associated with τ continue to act in the same manner at stages s and $s+1$. We have an ordered special cell C_s that locates all live witnesses. We refine this special cell in several steps to form C_{s+1}. We use the following ordering on $2^{<\omega}$.

Definition 10. *Let $<_S$ denote the ordering of $2^{<\omega}$ that results from squashing the binary tree.*

Here is the ordering of all $\sigma \in 2^{<\omega}$ with $|\sigma| \leq 3$ where λ is the empty string.

$$000 <_S 00 <_S 001 <_S 0 <_S 010 <_S 01 <_S 011 <_S \lambda$$

$$\lambda <_S 100 <_S 10 <_S 101 <_S 1 <_S 110 <_S 11 <_S 111$$

Step 1: Update witnesses.

First, suppose $\sigma_s \succeq \tau 1$ and, hence, $\sigma_{s+1} \succeq \tau 0$. We abandon the witnesses associated with nodes to the right of τ (i.e., nodes extending $\tau 1$) by defining them to be algebraic numbers while maintaining all other witnesses. Assume we have ordered the live witnesses as $(w_1, \ldots, w_n)$ in our given special cell
$C_s = \{(x_1, \ldots, x_n) : a_1 < x_1 < b_1 \ \& \ a_2 < x_2 < f_2(x_1) \ \& \ \ldots \ \& \ a_n < x_n < f_n(x_1, \ldots, x_{n-1})\}$.
Moreover, if witnesses w_i and w_j are associated with nodes γ_i and γ_j, then $w_i - a_i < w_j - a_j$ if and only if $\gamma_i <_S \gamma_j$ if and only if $j < i$. Because of this ordering, the witnesses associated with nodes to the right of τ are $w_l, w_2, \ldots w_k$ for some l and k with $1 \leq l \leq k \leq n$. If $l \neq 1$, the nodes γ associated with witnesses $w_1, \ldots, w_{l-1}$ satisfy $\gamma \prec \tau$ and $\tau <_S \gamma$. Hence, for such γ, we have that $\tau = \gamma 0 \sigma$ for some $\sigma \in 2^{<\omega}$. So, we currently are guessing that the witnesses $w_1, \ldots, w_{l-1}$ are moving away from the left boundary. By the second half of Lemma 4.8, there is a special ordered $n - (k - l + 1)$-cell that sets each witness $w_l, w_2, \ldots w_k$ equal to an algebraic number, moves the witnesses $w_1, \ldots, w_{l-1}$ away from the left boundary of the original special cell, and is contained in the

original cell (when we consider an expanded version of the new cell that includes the definitions of $w_l, w_2, \dots w_k$). Moreover, any other witnesses are maintained under this construction. Hence, all formulas included in the atomic diagram are preserved. Second, suppose that $\sigma_s \succeq \tau 0$ and, hence, $\sigma_{s+1} \succeq \tau 1$. Since our approximation to the true path moves to the right, we do not need to make any witnesses definable.

In both the first and second cases, we must ensure that all witnesses associated with $\gamma \preceq \sigma_s$ remain live. We also must ensure that each $\gamma \prec \sigma_{s+1}$ properly extending τ has a witness. We show this inductively. Suppose we have γ, an initial segment of σ_{s+1} extending τ. We assume we have a special cell of live witnesses associated with nodes preceding and to the left of γ. As before, we assume that we have live witnesses $(w_1, \dots, w_n)$ corresponding to elements in the ordered special n-cell with left boundary $(a_1, \dots, a_n)$. In other words, $w_1 - a_1 > \dots > w_n - a_n$. Again, if witnesses w_i and w_j are associated with nodes γ_i and γ_j, then $w_i - a_i < w_j - a_j$ if and only if $\gamma_i <_S \gamma_j$ if and only if $j < i$. Some of these witnesses are active. Among the active witnesses, some are supposed to be moving away from their left boundary, while others are supposed to be moving toward their left boundary, according to our current guesses. By construction, if w_i is moving away from a_i and w_j is moving toward a_j, then $i < j$ and the formula $x_i - a_i > x_j - a_j$ is valid on the current cell. Let w_j denote the witness corresponding to the node γ.

By definition of σ_{s+1}, the node $\gamma^+ = \sigma_{s+1} \upharpoonright (|\gamma| + 1)$ is $\gamma 0$ if $w_j \notin S_{|\gamma|, s+1}$ and $\gamma 1$ otherwise. We must ensure that γ^+ has an active witness that can satisfy its associated requirement. Specifically, γ^+ needs a witness w so that $w - a$ is between the $w_i - a_i$ that are moving towards a_i and the $w_i - a_i$ moving away, so that it can move either toward or away from its left boundary. If γ^+ has a live witness that is currently inactive, then we reactivate it. By construction, this witness already has the desired properties.

Suppose γ^+ does not have a live witness. We construct a live witness w for γ^+ and an associated special $(n+1)$-cell that preserves what has been included in the atomic diagram so far and allows our strategy for R_l for $l = |\gamma^+|$ to succeed. By assumption, γ has a live and active witness w_j in our current collection of witnesses $(w_1, \dots, w_n)$. By induction, all live witnesses w_i for $i < j$ are moving away from a_i, and all witnesses w_i for $i > j$ are moving towards a_i at stage $s+1$. At stage $s+1$, the strategy for R_{l-1} calls for witness w_j to move away from a_j if and only if $\gamma^+(l-1) = 0$.

If $\gamma^+(l-1) = 0$, we take the ordered special cell for witnesses $(w_1, \dots, w_n)$ (with variables $x_1, \dots, x_n$) and use Lemma 4.7 to build a new special cell with a variable y (corresponding to a new witness w) between x_j and x_{j+1}. As before, everything already included in the atomic diagram is preserved. Since all witnesses w_i for $i \leq j$ are moving away from a_i and all witnesses w_i for $i > j$ are moving towards a_i, we may move w according to the strategy for R_l in the next step. Also, the ordering of the special $(n+1)$-cell continues to respect the linear ordering $<_S$.

If $\gamma^+(l-1) = 1$, we take the ordered special cell for $(w_1, \dots, w_n)$ (with variables $x_1, \dots, x_n$) and use Lemma 4.7 to build a new ordered special cell with a variable y, corresponding to w, between x_{j-1} and x_j. Since all formulas valid on the original ordered special cell are valid on the new ordered special cell, all formulas already included in the atomic diagram are preserved. Moreover, the ordering of the $(n+1)$-cell continues to respect the linear ordering $<_S$. Since all witnesses w_i for $i < j$ are moving away from a_i and all witnesses w_i for $i \geq j$ are moving towards a_i, we may move w according to the strategy for R_l in the next step.

We may repeat this process so that each initial substring of σ_{s+1} has a live and active witness, and we have an ordered special cell C of live witnesses for nodes above and to the left of σ_{s+1}. We have preserved the validity of all formulas already included in the atomic diagram.

Step 2: Avoiding algebraic numbers in open intervals.

In this step, we make sure that each live witness w_i is located in an interval that contains none of the first $s+1$ algebraic numbers in the fixed enumeration of $\overline{\mathbb{Q}}$. We have a special cell of live witnesses associated with nodes above and to the left of σ_{s+1}. We again assume we have the live witnesses $(w_1, \dots, w_n)$, and the ordered special cell has the form:

$$\{(x_1, \dots, x_n) : a_1 < x_1 < b \ \& \ a_2 < x_2 < f_2(x_1) \ \& \ \dots \& \ a_n < x_n < f_n(x_1, \dots, x_{n-1})\}.$$

For each i, choose $b_i > a_i$ such that none of the first $s+1$ algebraic numbers are in the interval (a_i, b_i). Let $g_i(x_1, \dots, x_{i-1}) = \min(b_i, f_i(x_1, \dots, x_{i-1}))$ for $1 \leq i \leq n$. The refined ordered special cell

$$\{(x_1, \dots, x_n) : a_1 < x_1 < b_1 \ \& \ a_2 < x_2 < g_2(x_1) \ \& \ \dots \& \ a_n < x_n < g_n(x_1, \dots, x_{n-1})\}$$

has the desired features. Since this cell is a subcell of the original, we have preserved the validity of all formulas already included in the atomic diagram.

Step 3: Enacting the strategy.

Assume we have ordered the live witnesses as $(w_1, \dots, w_n)$ in our given ordered special cell. There is a greatest j such that w_j is associated with an active node $\gamma \preceq \sigma_{s+1}$ and $\gamma(|\gamma| - 1) = 0$, i.e., the strategy for $R_{|\gamma|}$ is to move w_j away from a_j. By Step 1, any witness w_i for $i \leq j$ is either not active or is associated with a node whose strategy is to move w_i away from a_i. By the first half of Lemma 4.8, there is an ordered special cell $C' \subseteq C$, that moves all active witnesses w_i for $i \leq j$ away from a_i and preserves the left boundary of the witnesses w_i for $i > j$. Since we simply refined the cell, we again have preserved the validity of all formulas already included in the atomic diagram. Note that Step 2 already naturally moves the active witnesses w_i for $i > j$ towards a_i at stage $s+1$ (as long as we preserve the left boundary of the associated part of the cell).

Step 4: Adding to the atomic diagram.

Let φ be the first sentence involving constants used so far such that neither φ nor $\neg\varphi$ are already in d_s. By Lemma 4.6, there is a special subcell of the cell we have on which φ or $\neg\varphi$ is valid. Moreover, the boundaries of this subcell remain unchanged. We include in the diagram d_{s+1} the sentence that is valid on this subcell. Let C_{s+1} be the resulting ordered special n-cell. This ends stage $s+1$.

Verification

We define the *true path* $f \in 2^\omega$ to be $f(i) = 0$ if $\lim_{s\to\infty} \sigma_s(i)$ equals 0 or does not exist and $f(i) = 1$ if $\lim_{s\to\infty} \sigma_s(i) = 1$. We now verify that R_e is satisfied for all e. We suppose by induction that, for each $i < e$, requirement R_i is satisfied by a witness $w_i^{f\restriction i}$ that does not change after some stage t^*. Moreover, at each stage in the construction, we have the live witnesses $(w_1, \ldots, w_n)$ chosen from the ordered special cell C_s with left boundary $(a_1, \ldots, a_n)$ such that if witnesses w_i and w_j are associated with nodes γ_i and γ_j, then $w_i - a_i < w_j - a_j$ if and only if $\gamma_i <_S \gamma_j$ if and only if $j < i$.

We show R_e is satisfied by witness $w_e^{f\restriction e}$, which does not change after some stage. By definition of f, we may take a stage $t > e, t^*$ such that if $f(i-1) = 1$ for any $0 < i < e$, then $\sigma_s(i-1) = 1$ for all $s \geq t$. Then, after stage t, any approximation to the true path $\sigma_s \restriction e$ cannot be to the left of $f \restriction e$. Let t' be the first stage $t' \geq t$ such that R_e has a witness $w_e^{f\restriction e}$ associated with it. (Such a stage exists by construction and the definition of f). By the last remark, this witness remains a live witness throughout the rest of the construction (i.e., it is never defined to be an algebraic number). Let $\gamma = f \restriction (e-1)$.

First, if $f(e-1) = 0$, then $f \restriction e = \gamma 0$. By construction, if the left endpoints of the intervals associated with witnesses $w_e^{\gamma 0}$ and w_{e-1}^{γ} are a and a' respectively at any stage $s > t'$, then $w_e^{\gamma 0} - a < w_{e-1}^{\gamma} - a'$. At any stage $s > t'$ such that all the substrings of $\gamma 0$ are active, witness $w_e^{\gamma 0}$ moves away from the left boundary a of its associated interval in Step 3 if $w_e^{\gamma 0} \notin S_{e,s}$. If $w_e^{\gamma 0} \notin S_e$, there are infinitely many such stages, so over the construction, $w_e^{\gamma 0}$ moves away from all algebraic numbers. (There are infinitely many such stages by our assumption that for any finite set α consisting of pairs $\langle e, n\rangle$ such that $n \notin S_e$ for each $\langle e, n\rangle \in \alpha$, there are infinitely many s such that for all $\langle e, n\rangle \in \alpha$, we have $n \notin S_{e,s}$.) Hence, $w_e^{\gamma 0}$ is in FT, demonstrating that S_e's prediction is incorrect.

If $w_e^{\gamma 0} \in S_e$, there is a stage $t'' > t'$ such that $w_e^{\gamma 0} \in S_{e,s}$ for all $s > t''$. Then, at each stage $s > t''$ such that all the substrings of $\gamma 0$ are active, the left endpoint of the interval associated with $w_e^{\gamma 0}$ is preserved. This endpoint is also preserved at any other stage $s > t''$ since no nodes to the left of $\gamma 0$ are ever active (and these are the only nodes that could change the left endpoint of the interval associated with $w_e^{\gamma 0}$). Then, $w_e^{\gamma 0}$ is defined to be infinitesimally close to this fixed left endpoint, an algebraic number. Hence, $w_e^{\gamma 0} \notin FT$, again showing that S_e's prediction is wrong.

On the other hand, if $f(e-1) = 1$, then $f \restriction e = \gamma 1$. As before, if the left endpoints of the intervals associated with witnesses $w_e^{\gamma 1}$ and w_{e-1}^{γ} are a and a'

respectively at any stage $s > t'$, then $w^{\gamma}_{e-1} - a' < w^{\gamma 1}_{e} - a$. The argument that R_e is satisfied is symmetric.

□

Note that any real closed field that satisfies the conditions in Proposition 4.4 must be non-archimedean. An element r is in FT for a fixed archimedean real closed R field if r is not any real algebraic number. Hence, FT for this R is $\Pi^0_1(R)$ and hence, also $\Sigma^0_2(R)$.

5 Background on integer parts

Definition 11 (Integer part)**.** *For a real closed field R, an* integer part *is a discrete ordered subring I such that for each $r \in R$, there is some $i \in I$ with $i \leq r < i + 1$.*

Remark 1. *If R is archimedean, then $\mathbb{Z}$ is the unique integer part.*

Shepherdson [16] characterized the discrete ordered rings that can serve as integer parts for real closed fields.

Theorem 5.1 (Shepherdson)**.** *For a discrete ordered ring I, the following are equivalent:*

1. *there is a real closed field for which I is an integer part,*
2. *I is a model of $IOpen$.*

Recall that $IOpen$ is the fragment of PA with induction axioms just for quantifier-free (open) formulas. In [5], D'Aquino, Starchenko, and the first author showed that if R is a countable real closed field with an integer part satisfying PA, then R is recursively saturated. It follows that R has other integer parts satisfying all computably axiomatizable extensions of $IOpen$.

In [13], Mourgues and Ressayre proved the following.

Theorem 5.2 (Mourgues-Ressayre)**.** *Every real closed field R has an integer part.*

Mourgues and Ressayre considered $k((G))$, a field of generalized series, defined below. They observed that If F is a "truncation closed" subfield of $k((G))$, then F has an integer part I_F that is simple to describe. They defined an isomorphism δ from R onto a truncation closed $F \subseteq k((G))$. Then $\delta^{-1}(I_F)$ is the integer part for R.

Definition 12 ($k((G))$)**.** *Let k be an archimedean ordered field and let G be an ordered abelian group.*

1. *The field $k((G))$ of generalized series is the set of formal sums $s = \sum_{g \in G} a_g g$ with $a_g \in k$ and $Supp(s) := \{g \in G : a_g \neq 0\}$ is an anti-well ordered subset of G.*

2. *The* length *of s, denoted length(s), is the order type of Supp(s) under the reverse ordering. Later, we may write* $s = \sum_{i<\alpha} a_i g_i$*, where* $g_i \in G$ *with* $g_i > g_j$ *for* $i < j < \alpha$*, and* $a_i \in k - \{0\}$*. Under this notation, the length of s is* α*.*

3. *For* $s = \sum_{g \in G} a_g g$ *and* $t = \sum_{g \in G} b_g g$ *in* $k((G))$*, the* sum $s + t$ *and the* product $s \cdot t$ *are defined as for ordinary power series.*

   (a) *In* $s + t$*, the coefficient of* g *is* $a_g + b_g$*.*

   (b) *In* $s \cdot t$*, the coefficient of* g *is the sum of the products* $a_{g'} b_{g''}$*, where* $g = g' \cdot g''$*.*

4. $k((G))$ *is ordered anti-lexicographically by setting* $s > 0$ *if* $a_g > 0$ *where* $g =: max(Supp(s))$*.*

Definition 13 (Truncation closed subfield)**.** *Let* $s \in k((G))$ *with* $s = \sum_{i<\alpha} a_i g_i$*. The* truncation *of s with respect to* $\beta < \alpha$ *is* $s_\beta = \sum_{i<\beta} a_i g_i$*.*

A subfield $F \subset k((G))$ *is* truncation closed *if for all* $s \in F$*,*

$$(\forall \beta < length(s))[s_\beta \in F]$$

Proposition 5.3 (Mourgues-Ressayre)**.** *If* $F \subset k((G))$ *is truncation closed subfield, then* $I_F = \{t + z \mid Supp(t) \subset G^{>1} \ \& \ z \in \mathbb{Z}\}$ *is an integer part for F where* $G^{>1} = \{g \in G \mid g > 1\}$*.*

To obtain an integer part for a real closed field R using Proposition 5.3, Mourgues and Ressayre defined a *development function* δ mapping R isomorphically onto a truncation closed subfield F of $k((G))$. For any $r \in R$, we call $\delta(r)$ the *development* of r. Once we fix R, a residue field section k, and a well ordering of R, the procedure is canonical, producing both a value group section G and the desired embedding δ.

We briefly describe the inductive step in Mourgues and Ressayre's construction. Let $A \subset R$ be a real closed subfield field of R containing k, let H be a value group section for A, and let $r \in R - A$. Let $\phi : A \to k((H))$ be an embedding of A onto a truncation closed subfield of $k((H))$, where ϕ is the identity on $k \cup H$, and let $A' = RC(A(r))$. Mourgues and Ressayre specify a value group section $H' \supseteq H$ for A' and an embedding $\phi' \supseteq \phi$ of A' into $k((H'))$, where ϕ' is the identity on H' and $\phi'(A')$ is truncation closed.

Definition 14. *The* partial development of r over (A, H, ϕ) *is the element* $s = \sum_{i<\alpha} a_i g_i$ *of* $k((H))$ *of greatest length* α *such that*

1. $g_0 = w(r)$ *and* a_0 *is the unique element of k such that* $w(r - a_0 g_0) < g_0$*. Equivalently,* $w(\frac{r}{g_0} - a_0) < 1$*, or* $\frac{r}{g_0}$ *is infinitesimally close to* a_0*.*

2. *For* $\beta < \alpha$*, there exists* $\hat{r}_\beta \in A$ *such that*

   (a) $\phi(\hat{r}_\beta) = \sum_{j<\beta} a_j g_j$ *with* $w(r - \hat{r}_\beta) = g_\beta$*, and*

(b) *a_β is the unique element of k such that $w(r - \hat{r}_\beta - a_\beta g_\beta) < g_\beta$. Equivalently, a_β is the unique element of k such that $w(\frac{r-\hat{r}_\beta}{g_\beta} - a_\beta) < 1$, or $\frac{r-\hat{r}_\beta}{g_\beta}$ is infinitesimally close to a_β.*

Let s be the partial development of r over (A, H, ϕ). We are in one of two cases.

Case 1 Suppose $s \notin \phi(A)$. In this case, we let $\phi'(r) = s$. We let $H' = H$ and we extend ϕ' to A', the real closure of $A(r)$, in the only way possible.

Case 2 Suppose $s \in \phi(A)$. This means that there exists $\hat{r} \in A$ with $\phi(\hat{r}) = s$ and there is no $h \in H$ with $w(r - \hat{r}) = h$. In this case, we let H' be the divisible closure of H and the new group element $g = |r - \hat{r}|$. We note that g and r are interalgebraic over A, so A' is the real closure of $A(g)$. We let $\phi'(g) = g$, and we extend ϕ' to A' in the only way possible. Of course, $\phi'(r) = \phi'(\hat{r}) + (r - \hat{r})$, where $r - \hat{r} = \pm g$.

We must be sure that ϕ' maps A' onto a truncation closed subfield of $k((H'))$. This is the main step in the proof of Mourgues and Ressayre, and they acknowledge some help from Dave Marker and Françoise Delon.

Proposition 5.4 (Mourgues-Ressayre, Delon, Marker)**.** *If F is a truncation closed subfield of $k((G))$ and t is an element of $k((G)) - F$ with all proper initial segments in F, then the real closure of $F(t)$ is also truncation closed.*

To show that $\phi'(A')$ is truncation closed, we apply Proposition 5.4, letting $F = \phi(A)$, and letting t be s, in Case 1, or g in Case 2.

The ideas of Mourgues and Ressayre have proved useful in more general settings, in particular, for fields with exponentiation [14], [3]. There are natural questions about integer parts for fields that are not real closed [10], and about ordered rings that do and do not extend to models of *IOpen* [18]. Berarducci [2] looked at lengths of developments. Without calculating bounds, he showed that length of development is a new valuation function. Erhlich [7] showed that every real closed field can be embedded in the "surreal numbers" as an initial subfield.

6 Conjecture on lengths of developments

We focus on the case where R is countable and we have fixed a residue field section k and a well ordering $\prec$ of R of order type ω. In order to describe the construction of δ, we write $R = \cup_{i \in \omega} R_n$ as follows using $\prec$. Let $R_0 = k$. Assuming that $R_n \neq R$, let r_{n+1} be the $\prec$-first element of $R - R_n$, and let R_{n+1} be the real closure of $R_n(r_{n+1})$. Note that $\{r_i : i \in \omega\}$ is a transcendence basis for R over k.

We obtain a sequence (R_n, G_n, δ_n) where G_n is a value group section of R_n and δ_n is an embedding from R_n onto a truncation closed subfield of $k((G_n))$, with $G_{n+1} \supseteq G_n$, and $\delta_{n+1} \supseteq \delta_n$. We let G_0 be the trivial group $\{1\}$ and $\delta_0 : R_0 = k \to k((G_0))$ is the identity, and we use the inductive step described above to pass from (R_n, G_n, δ_n) to $(R_{n+1}, G_{n+1}, \delta_{n+1})$. Then, $\delta = \cup_{i\in\omega}\delta_i$ is the desired embedding from R onto a truncation closed subfield of $k((G))$ where $G = \cup_{i\in\omega}G_i$.

Conjecture 1. *For $n \geq 1$, the developments of elements of R_n have length at most $\omega^{\omega^{n-1}}$. Hence, the developments of elements of R have length at most ω^{ω^ω}.*

If the conjecture holds, then we can show that the whole Mourgues and Ressayre construction is $\Delta^0_{\omega^\omega}(R)$. We thought that we had proven the conjecture, but we found a mistake in our proof.

The conjecture is true for the case where $n = 1$. Suppose that R is a nonarchimedean real closed field. For any $r \in R - k$, the real closure of $k(r)$ is not archimedean. Hence, R_1 is not archimedean. We may suppose that R_1 is the real closure of $k(g)$, where g is infinitesimal. The group $G_1 = \{g^q : q \in \mathbb{Q}\}$ is a value group section of R_1. We calculate developments as Mourgues and Ressayre do. The development function is trivial on g and the elements of k. By Proposition 5.4, the developments of the other elements of R_1 are uniquely determined. The next result bounds the lengths of these developments.

Theorem 6.1 (Newton-Puiseux, Shepherdson)**.** *For all $r \in R_1$, the development of r has length at most ω.*

Remark 2. *If r has the development $\sum_{i<\omega} b_i h_i$ of length ω, then for each n, there is some i such that $h_i \leq g^n$, i.e., there are arbitrarily small h_j relative to G_1.*

Proof. If for some n we had $g^n < h_i$ for all i, then $r + g^n$ would have a development of length $\omega + 1$.

□

7 Example

In this final section, our goal is to give an example that shows Conjecture 1 is sharp. Moreover, the construction of this example provides insight into the tools that would be used to provide a proof of Conjecture 1 or to construct a counterexample.

Theorem 7.1. *There is a countable real closed field R, with a residue field section k and a well ordering $\prec$ of type ω, such that for each $n \geq 1$, the real closed subfield R_n of R, produced by the Mourgues and Ressayre procedure on inputs R, k, and $\prec$, has an element t_n with a development of length $\omega^{\omega^{(n-1)}}$.*

We construct the example first and then show that it has the required features.

7.1 Construction of example

Let k be a fixed countable archimedean real closed field—we may use the real algebraic numbers $\overline{\mathbb{Q}}$. Let G be a value group generated by a single infinitesimal g. Our R will be a truncation closed subfield of $k((G))$. We will obtain R as the union of a chain of truncation closed subfields $(R_n)_{n\in\omega}$. For $n \geq 1$, we describe special elements $r_n, t_n \in R_n$. We choose a type ω well ordering $\prec$ of R, so that the Mourgues and Ressayre procedure considers the same subfields R_n, and the development function on each R_n is the identity. Specifically, we ensure:

1. $R_0 = k$.
2. r_{n+1} is the $\prec$-first element of $R - R_n$.
3. $r_1 = g$, and for $n \geq 1$, r_{n+1} is an infinitesimal of length $\omega^{\omega^{(n-1)}}$ and with all positive coefficients. Moreover, there is a $g \in G$ such that for all $g' \in Supp(r_{n+1})$, we have $g' > g$, i.e., $Supp(r_{n+1})$ is bounded away from 0.
4. R_{n+1} is the real closure of $R_n(r_{n+1})$ in $k((G))$, and $t_{n+1} \in R_{n+1}$.
5. t_{n+1} has length ω^{ω^n} with all positive coefficients. Furthermore, $w(t_{n+1}) = 1$ and $Supp(t_{n+1})$ is co-initial in G, i.e, for all $g \in G$ there is a $g' \in Supp(t_{n+1})$ such that $g' < g$.

We let $R_0 = k$, as required. The development function determined by Mourgues and Ressayre is essentially the identity on k—it takes a to $a \cdot 1$. In the ordering $\prec$ we are constructing, we put the first element of k at the beginning, and then we set $r_1 = g$. This assures that in the Mourgues and Ressayre construction, $R_1 = RC(k(g))$ and g goes into the value group section. Then the development function will map g to g, and, in fact, the development function will be the identity on all of $R_1 = RC(k(g)) \subset k((G))$. By Proposition 5.4 of Mourgues and Ressayre, R_1 is truncation closed. We let $t_1 = \frac{1}{1-g}$. In $k((G))$, this is equal to $1 + g + g^2 + \ldots$, which has has length ω. We have $w(t_1) = 1$, and the support of t_1 is co-initial in G.

We turn to r_2. Let $(q_i)_{i\in\omega}$ be an increasing sequence of positive rationals, with an upper bound—we may take $\lim_{i\to\infty} q_i = e$. Then $\sum_i g^{q_i}$ is an element of $k((G))$. We let this be r_2. The ordering $\prec$ has the first element of k, followed by r_1. Next, we put r_2, then the first element of k not already included, and then the first element of R_1 not already included. When we run the Mourgues and Ressayre construction, the development function will map r_2 to r_2, and it will be the identity on $R_2 = RC(R_1(r_2))$. Again by Proposition 5.4, R_2 is truncation closed. We let $t_2 = \frac{1}{1-r_2}$. Showing that t_2 actually has length ω^ω requires some work, and we will also see that $w(t_2) = 1$ and the support of t_2 is co-initial in G.

We continue by induction. Suppose we have r_n, R_n, t_n, and a finite initial segment of the ordering $\prec$ (defined on a subset of R_n) with the properties described above. We obtain r_{n+1} by "scaling" t_n, as in the lemma below.

Lemma 7.2. *Suppose t_n is an element of R_n of length $\omega^{\omega^{(n-1)}}$, such that $Supp(t_n)$ is co-initial in G, $w(t_n) = 1$, and all coefficients of t_n are positive. There is some $r_{n+1} \in k((G))$ of length $\omega^{\omega^{(n-1)}}$, such that r_{n+1} is an infinitesimal, the proper initial segments of r_{n+1} are all in R_n, the support of r_{n+1} is bounded away from 0, and the coefficients of r_{n+1} are all positive.*

Proof. The set $X = \{q \in \mathbb{Q} \mid g^q \in Supp(t_n)\}$ forms an increasing sequence in $\mathbb{Q}$, starting with 0, and having order type $\omega^{\omega^{(n-1)}}$. Let x_0 be the first element of the sequence. Thus, $x_0 = 0$. Let x_1 be the least upper bound in $\mathbb{R}$ of the initial sequence of rationals in X of type $\omega^{\omega^{(n-2)}}$, and, in general, let x_i be the least upper bound in $\mathbb{R}$ of the initial sequence of rationals in X of type $\omega^{\omega^{(n-2)} \cdot i}$. The sequence $(x_i)_{i\in\omega}$ is strictly increasing and unbounded in $\mathbb{R}$ since $Supp(t_n)$ is co-initial in G. For $i > 0$, x_i may not be rational.

Let $(q_i)_{i\in\omega}$ be an increasing sequence of positive rationals with an upper bound. Choose a sequence of reals $(\epsilon_i)_{i\in\omega}$ such that $x_i + \epsilon_i$ is rational, and not too far from x_i—we choose $0 < \epsilon_i < \min(\frac{q_i}{2^i}, \frac{(q_{i+1}-q_i)}{2^i})$. For $x \in X$ such that $x < x_1$, we replace x by $x' = q_1 + x - (x_1 + \epsilon_1)$. Note that as x approaches x_1 from below, x' approaches $q_1 - \epsilon_1$ from below. An initial segment of these x' may be nonpositive. We drop these x'. Let X_0 be the subset of $\mathbb{Q}$ that remains. Then, X_0 has order type $\omega^{\omega^{(n-2)}}$.

Claim 1: $\{g^q : q \in X_0\}$ is the support of some element of R_n.

Proof of Claim. Let s_1 be the result of truncating the development of t_n to the first $\omega^{\omega^{(n-2)}}$ terms. This is in R_n. We multiply s_1 by $g^{q_1-(x_1+\epsilon_1)}$ and then drop the initial segment consisting of $g^{x'}$, where x' is nonpositive. This is an element of R_n with the desired support and all positive coefficients.

□

The sequence of x such that $x_1 \leq x < x_2$ has order type $\omega^{\omega^{(n-2)} \cdot 2}$, the same as the sequence of all $x < x_2$. For $x_1 \leq x < x_2$, we replace x by $x' = q_2 + x - (x_2 + \epsilon_2)$. As x approaches x_2 from below, x' approaches $q_2 - \epsilon_2$ from below. There may an initial segment of these $x' < q_1$. We drop these x'. The subset of $\mathbb{Q}$ that remains, call it X_1, still has order type $\omega^{\omega^{(n-2)} \cdot 2}$, and the limit is $q_2 - \epsilon_2$. In the same way that we proved Claim 1, we can show that $\{g^q : q \in X_1\}$ is the support of some element s_2 of R_n. We continue in this way. For each i, we have X_i, a set of rationals q such that $q_{i-1} \leq q < q_i$, where the order type of X_i is $\omega^{\omega^{(n-2)} \cdot i}$. For each X_i, there is an element s_i of R_n with support $\{g^q : q \in X_i\}$.

Consider the full set $X = \cup_i X_i$. This set is bounded since the set of q_i is bounded. For each n, $\cup_{i\leq k} X_i$ is an initial segment of X of order type $\omega^{\omega^{(n-2)} \cdot k}$.

Therefore, X has order type $\omega^{\omega^{(n-1)}}$. For each $x \in X$, there is an element of R_n with support $\{g^{x'} : x' < x\}$ obtained by taking a truncation of some finite sum $s_0 + s_1 + s_2 + \ldots + s_n$.

We let r_{n+1} be the element of $k((G))$ with the development given by the limit of our finite sums of s_i's, i.e., the development $s_0 + s_1 + s_2 + \ldots$. By construction, r_{n+1} has the desired properties. $\square$

We continue to construct the type ω well ordering $\prec$. So far, $\prec$ is defined on finitely many elements. We put r_{n+1} next after the finitely many elements already ordered by $\prec$, then the first element of R_0 not already included, then the first element of R_1 not included, then the first element of R_2 not included, and so on, up to the first element of R_n not yet included. Then the development function will map r_{n+1} to r_{n+1}, and will be the identity on all of $R_{n+1} = RC(R_n(r_{n+1}))$. By the results of Proposition 5.4, R_{n+1} is truncation closed. We will let $t_{n+1} = \frac{1}{1-r_{n+1}}$. We must show that t_{n+1} has length ω^{ω^n}. We will see also that $w(t_{n+1}) = 1$ and the coefficients of t_{n+1} are all positive.

7.2 Example has desired properties

To show that the example we have constructed has the desired properties, it is enough to prove the following.

Proposition 7.3. *Suppose $r \in k((G))$ is an infinitesimal and let $t = \frac{1}{1-r}$. Suppose r has length α, where α is ω or ω^{ω^k} for some k, $Supp(r)$ is bounded away from 0 and all coefficients of r are positive. Then t has length α^ω, $w(t) = 1$, $Supp(t)$ is co-initial in G, and all coefficients of t are positive.*

For example, in the case where $r = r_2$, we have $\alpha = \omega$, and $\alpha^\omega = \omega^\omega$. In the case where $r = r_{n+1}$, for $n > 2$, we have $\alpha = \omega^{\omega^{(n-1)}}$ and $\alpha^\omega = \omega^{\omega^n}$.

To prove Proposition 7.3, we consider the set $X = \{q \in \mathbb{Q} : g^q \in Supp(r)\}$. Since r is infinitesimal, X is a set of positive rationals. Since r has length α and g is infinitesimal, X has order type α, under the usual ordering. Since $Supp(r)$ is bounded away from 0, X has an upper bound, so there is a least upper bound in $\mathbb{R}$, say e. The fact that α has the form ω or ω^{ω^k} means that for any proper initial segment Y of X, the "tail" $X - Y$ has the same order type as X. For finite $m \geq 1$, let $S_m(X)$ be the set of sums of m elements of X. Let $S(X) = \{0\} \cup \bigcup_{m \geq 1} S_m(X)$.

To prove Proposition 7.3, we want to identify $t = \frac{1}{1-r}$ with the infinite sum $1 + r + r^2 + \ldots$. We first show that the infinite sum makes sense—that it represents some element of $k((G))$. For this, we need some lemmas.

Lemma 7.4. *For all m, $S_m(X)$ is well ordered.*

Proof. We can give a soft proof. The support of r^m is the set of products of m factors g^q, for $q \in X$, which is the same as the set of g^r for $r \in S_m(X)$.

$\square$

The support of r^m is $\{g^q : q \in S_m(X)\}$, and the coefficients or r^m are all positive.

Lemma 7.5. *$S(X)$ is well ordered.*

Proof. Suppose not. Let m_0 be the least $m \geq 1$ such that there is an infinite decreasing sequence with first term $x_0 \in S_{m_0}(X)$. Take the least possible x_0 with this property. Let m_1 be least such that there is a decreasing sequence that starts with $x_0 > x_1$, for x_0 already chosen, and for $x_1 \in S_{m_1}(X)$. Necessarily, $m_1 > m_0$. Take the least possible x_1. Let m_2 be least such that there is a decreasing sequence that starts with $x_0 > x_1 > x_2$, for x_0, x_1 already chosen, and for $x_2 \in S_{m_2}(X)$. Necessarily, $m_2 > m_1$. Take the least possible x_2. Continue in this way. Let q be the first element of X. There is some n such that $nq > x_0$. Then for $m \geq n$, $S_m(X)$ cannot contain any element of our sequence x_i. Since the sequence of m_i is strictly increasing, this is a contradiction.

□

Lemma 7.6. *Each rational r is in $S_m(X)$ for only finitely many m.*

Proof. Let q be the least element of X. There is some n such that $nq > r$. If $r \in S_m(X)$, then m must be less than n

□

The lemmas above show that we can identify the sum $1+r+r^2+r^3+\ldots$ with an element of $k((G))$. We want to identify this element with $t = \frac{1}{1-r}$. Using Taylor's Theorem, we can see that for each m, $\frac{1}{1-r} = 1 + r + r^2 + \ldots + r^m + R$, where $w(R) = w(r^{(m+1)})$, so the element of $k((G))$ that represents the infinite sum $1 + r + r^2 + r^3 + \ldots$ is t. The support of r^m is $\{g^q : q \in S_m(X)\}$, and the support of t is $\{g^q : q \in S(X)\}$. Since $S(X)$ is unbounded in $\mathbb{R}$, the support of t is co-initial in G.

Since the support of t is $\{g^q : q \in S(X)\}$, to show that t has length α^ω, it is enough to show that $S(X)$ has order type α^ω. This will complete the proof of Proposition 7.3. We will also show that $S_m(X)$ has order type α^m. We first show that these are lower bounds on the order types of $S_m(X)$ and $S(X)$.

Lemma 7.7. *The order type of $S_m(X)$ is at least α^m.*

Proof. We proceed by induction. For $m = 1$, X has order type α. Supposing that the statement holds for m, we prove it for $m+1$. Let Y be the initial segment of $S_m(X)$ of order type α^m. For each $x \in X$, let x' be the successor in X. The set Y is bounded above by me, so there is a least upper bound y^* of Y in $\mathbb{R}$. For each $x \in X$, take $y_x \in Y$ such that $y^* - y_x < x' - x$. Let $Y_x = \{y \in Y : y \geq y_x\}$. Then, for all $y \in Y_x$, we have $x + y_x \leq x + y < x + y^* < x' + y_x$. Since Y_x has order type α^m, the set $x + Y_x = \{x + y : y \in Y_x\}$ does as well for a fixed $x \in X$. Also note that $x' + y_x < x' + y^* < x'' + y_{x'}$. Since $\{x' + y_x : x \in X\}$ has order type α, the subset of $S_{m+1}(X)$ consisting of $\{x + y : x \in X \ \& \ y \in Y_x\}$ has order type $\alpha^m \cdot \alpha = \alpha^{m+1}$.

□

Since $S_m(X) \subseteq S(X)$, the order type of X is at least as great as that of $S_m(X)$, for all m. Hence, we have the following.

Lemma 7.8. *The order type of $S(X)$ is at least α^ω.*

We must show that the order type of $S_m(X)$ is at most α^m and the order type of $S(X)$ is at most α^ω. We need a special kind of ordinal sum and product.

Definition 15 (Commutative Sum)**.** *For ordinals γ and β, the* commutative sum*, denoted by $\gamma \oplus \beta$, is the ordinal whose Cantor normal form is obtained by expressing γ and β in Cantor normal form, and summing the coefficients of like terms.*

Definition 16 (Commutative Product)**.** *For ordinals γ and β, the* commutative product*, denoted by $\gamma \otimes \beta$, is the ordinal whose Cantor normal form is obtained by expressing γ and β in Cantor normal form, multiplying as for polynomials, using commutative sum in the exponents, and then combining like terms. If $\gamma = \sum_i \omega^{\gamma_i} m_i$ and $\beta = \sum_j \omega^{\beta_j} n_j$, then the product of the terms $\omega^{\gamma_i} m_i$ and $\omega^{\beta_j} n_j$ is $\omega^{\gamma_i \oplus \beta_j}(m_i \cdot n_j)$.*

Examples:

- $\omega \otimes \omega = \omega^2$,
- $(\omega \cdot 3) \otimes (\omega^\omega \cdot 2) = \omega^{\omega+1} \cdot 6$,
- $(\omega^\omega) \otimes (\omega^{\omega^2}) = \omega^{\omega^2+\omega}$,
- $(\omega^{\omega^2}) \otimes (\omega^{\omega^2}) = \omega^{\omega^2 \cdot 2}$

The next two lemmas are not really new. The ideas are found in [8], [4]. For $s, t \in k((G))$, we may set a bound on the length of $s \cdot t$ using a result of De Jongh and Parikh (see [6], [15], [16]). We are grateful to Andreas Weiermann for describing (at the CiE meeting in Heidelberg in 2009) the basic ideas and for giving us the appropriate references. We state the two lemmas in terms of well ordered subsets of $\mathbb{Q}$.

Lemma 7.9. *If X and Y are well ordered subsets of $\mathbb{Q}$, of order type γ, β, respectively, then $X \cup Y$ has order type at most $\gamma \oplus \beta$.*

The proof will also show when the maximum is and is not achieved.

Proof. We show by induction on δ that if $\gamma, \beta \leq \omega^\delta$, then the order type of $X \cup Y$ is at most $\gamma \oplus \beta$. We start with $\delta = 1$. Then $\gamma, \beta \leq \omega$.

If γ, β are both finite, then the statement is clear. Suppose $\gamma = \omega$ and β is finite. If no element of Y is greater than all elements of X, then $X \cup Y$ has order type ω. Suppose some elements of Y are greater than all elements of X; say m is the number of these elements. Then the order type of $X \cup Y$ is $\omega + m \leq \gamma \oplus \beta$. The argument is the same if $\beta = \omega$ and γ is finite. Now, suppose $\gamma = \beta = \omega$. Suppose some elements of Y are greater than all elements of X, and let Y' be the set of such elements. Then, $X \cup (Y - Y')$ has order type ω, and $X \cup Y$ has order type $\omega + \omega = \omega \oplus \omega$. (The argument is the same if some elements of X are greater than all elements of Y.) If X and Y are both cofinal in $X \cup Y$, then $X \cup Y$ has order type $\omega < \omega \oplus \omega$.

Suppose the inductive statement holds for $\delta' < \delta$. Suppose $\gamma = \omega^\delta$, and β is smaller. If no element of Y is greater than all elements of X, then, using the inductive hypothesis, we can see that for each $x \in X$, the set of predecessors of x in $X \cup Y$ has order type less than ω^δ. Therefore, $X \cup Y$ has order type $\omega^\delta \leq \omega^\delta \oplus \beta$. Suppose some elements of Y are greater than all elements of X, and let Y' be the set of such elements, with order type β' (less than ω^δ). Then $X \cup (Y - Y')$ has order type ω^δ, and $X \cup Y$ has order type $\omega^\delta + \beta' \leq \omega^\delta \oplus \beta$. The argument is the same if $\beta = \omega^\delta$ and γ is smaller. Now, suppose $\gamma = \beta = \omega^\delta$. Suppose some elements of Y are greater than all elements of X, and let Y' be the set of such elements. Then $X \cup (Y - Y')$ has order type ω^δ, and $X \cup Y$ has order type $\omega^\delta + \omega^\delta = \gamma \oplus \beta$. The same is true if some elements of X are greater than all elements of Y. Suppose X and Y are both cofinal in $X \cup Y$. Let $(x_n)_{n \in \omega}$ be a sequence of elements of X that is cofinal in $X \cup Y$. Let U_n consist of those $u \in X \cup Y$ such that $u \leq x_n$. Then U_n has order type less than ω^δ. Therefore, $X \cup Y$ has order type $\omega^\delta < \gamma \oplus \beta$.

□

Definition 17. *A* well partial ordering *is a partial ordering with no infinite decreasing sequence, and no infinite sequence of incomparable elements.*

Lemma 7.10. *Suppose X and Y are well ordered subsets of $\mathbb{Q}$, of order types at most γ and β, respectively. Then $X + Y = \{x + y : x \in X \ \& \ y \in Y\}$ has order type at most $\gamma \otimes \beta$.*

Proof. We define $\trianglelefteq$ on $X \times Y$ such that $(x, y) \trianglelefteq (x', y')$ iff $x + y \leq x' + y'$.

Claim: This is a well partial ordering.

Proof. First, we show that there is no infinite decreasing sequence. Suppose $(x_n, y_n)_{n \in \omega}$ is strictly decreasing. Using Ramsey's Theorem and replacing the given sequence by an appropriate subsequence, we may suppose that each of the sequences $(x_n)_{n \in \omega}$, $(y_n)_{n \in \omega}$ is strictly decreasing, strictly increasing, or constant. Since X and Y are well ordered, the sequences cannot be strictly decreasing. Then the sequence $(x_n + y_n)_{n \in \omega}$ is nondecreasing, a contradiction.

There is no infinite sequence of incomparable elements since by definition of $\trianglelefteq$ no two elements are incomparable.

□

We may extend any partial ordering to a linear ordering, and for a well partial ordering, any linear extension is a well ordering (see Proposition 4.2 of [15]). By a result of de Jongh and Parikh (Theorem 4.8 of [15]), the maximum order type of such a linear extension is $\gamma \otimes \beta$. Hence, $X + Y$ has order type at most $\gamma \otimes \beta$.

□

Lemma 7.11.

1. *The order type of $S_m(X)$ is at most α^m*
2. *The order type of $S(X)$ is at most α^ω.*

Proof. Recall that X has order type α, where α is ω or $\omega^{\omega^{n-1}}$ for some $n \geq 2$. For (1), suppose $S_m(X)$ has order type at most α^m by induction. Then, $S_{m+1}(X)$ has order type at most $\alpha^m \otimes \alpha$, which is α^{m+1} by Lemma 7.10.

For (2), suppose $S(X)$ has order type greater than α^ω. Let $y \in S(X)$ so that $pred(y) = \{u \in S(X) : u < y\}$ has order type α^ω. Now, $y \in S_m(X)$, for some m. Let q be the first element of X. For some n, $nq > y$. Then $pred(y) \subseteq \cup_{k<n} S_k(X)$. By Lemmas 7.9 and 7.11, $\cup_{k<n} S_k(X)$ has order type at most $\alpha \oplus \alpha^2 \oplus \ldots \oplus \alpha^{n-1} < \alpha^n$, a contradiction.

□

We have now established that $S(X)$ has order type α^ω. This completes the proof of Proposition 7.3, showing that our example has the desired properties. Thus, we have completed Theorem 7.1, which shows that Conjecture 1 is sharp.

Clearly, the main remaining question is whether Conjecture 1 is valid. We believe any proof of the Conjecture or counterexample would require using ideas in the lemmas found in §7.2 as well as a close analysis of Newton's method in the context of $k((G))$.

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