Abstract. Let \( R \) be a real closed field. An integer part for \( R \) is a subring that sits inside \( R \) the way the integers sit inside the reals. Mourgues and Ressayre [10] showed that every real closed field has an integer part. We may associate with \( R \) a value group \( G \) that is a subgroup of \((R^+,\cdot)\), a residue field \( k \) that is a subfield of \( R \), and an integer part \( I \). The Mourgues and Ressayre construction involves mapping the elements of \( R \) to generalized power series, called “developments,” with terms corresponding to elements of a well-ordered subset of \( G \) and with coefficients in \( k \). We consider the case where \( R \) is countable. We fix a residue field \( k \) and a list \( r_1, r_2, \ldots \) of elements that form a transcendence basis for \( R \) over \( k \). Once these objects are fixed, the Mourgues and Ressayre (MR) procedure is canonical. Let \( R_n \) be the real closure of \( k(r_1, \ldots, r_n) \). We show that the elements of \( R_n \) have developments of length at most \( \omega^{\omega(n-1)} \), so all elements of \( R = \cup_n R_n \) have developments of length less than \( \omega^\omega \). We give an example of a real closed field \( R \), with residue field \( k \) and transcendence basis \( r_1, r_2, \ldots \) for \( R \) over \( k \), such that these bounds are sharp: i.e., for each \( n \), the field \( R_n \) has an element with a development of length \( \omega^{\omega(n-1)} \).

We are interested in effectiveness. We show that for any countable real closed field \( R \), there is a value group \( G \) that is \( \Delta^0_2(R) \) and that this result is sharp. Next, we show that for any countable real closed field \( R \), there is a residue field \( k \) that is \( \Pi^0_2(R) \), and this result is the best possible. Finally, we use our upper bounds on the lengths of developments, together with the results on the value group and the residue field, to show that there is a \( \Delta^0_{\omega}(R) \) integer part, obtained by the MR-procedure.

1. Introduction

Tarski [13] showed that the theory of the ordered field of real numbers is decidable. To do this, he gave an elimination of quantifiers, and in the process, he extracted a computable set of axioms sufficient for the elimination of quantifiers and for settling the truth of the quantifier-free sentences. The axioms describe an ordered field in which all polynomials of odd degree have zeroes, and an element is positive precisely when it has a square root. The real closed fields are the models of this theory. Tarski’s elimination of quantifiers implies that the theory is \( o \)-minimal; i.e., in any model, the sets definable by a formula in one variable, with parameters, are already those definable just from the ordering—finite unions of intervals and points.

We consider integer parts of real closed fields. Roughly speaking, an integer part for a real closed field \( R \) sits in \( R \) the way \( \mathbb{Z} \) sits in \( \mathbb{R} \). Here is the definition.
Definition 1 (Integer part). An integer part for $R$ is a discrete ordered subring $I$ of $R$ such that 1 is the first positive element, and for each $r \in R$, there exists $i \in I$ such that $i \leq r < i + 1$.

For a real closed field $R$, if $R$ is archimedean, then $\mathbb{Z}$ is the unique integer part for $R$. If $R$ is non-archimedean, then there are many possible integer parts. Recall that $I_{\text{Open}}$ is the weak fragment of $PA$ with induction axioms only for open (quantifier-free) formulas. Shepherdson [12] showed that $I$ is an integer part for some real closed ordered field if and only if it is a model of $I_{\text{Open}}$. In [3], D’Aquino, Starchenko, and the second author show that if $R$ is a countable real closed field with an integer part satisfying $PA$ (in fact, the fragment $I_{\Sigma_4}$ is enough), then $R$ is recursively saturated. It follows that $R$ has other integer parts satisfying all computably axiomatizable extensions of $I_{\text{Open}}$.

Mourgues and Ressayre [10] showed that every real closed field has an integer part. Their procedure involves embedding $R$ into a field of generalized power series, in which the terms correspond to elements of a well-ordered subset of the value group $G$, and the coefficients come from the residue field $k$. For $r \in R$, the corresponding power series $\sum_{i<\alpha} a_i g_i$ is called the development of $r$. A priori, if $G$ is countable, the power series may have arbitrary countable ordinal lengths. In §3, we make the Mourgues and Ressayre (MR) procedure canonical, by fixing a residue field $k$ that is a subfield of $R$ and an enumeration $r_1, r_2, \ldots$ of a transcendence basis for $R$ over $k$. In §4, we prove the following.

Theorem 1.1 (Main Result). Let $R$ be a real closed field with residue field $k$, and suppose the elements $r_1, r_2, \ldots$ form a transcendence basis for $R$ over $k$. Let $R_n$ be the real closure of $k(r_1, \ldots, r_n)$. The canonical MR-procedure assigns to all elements of $R_n$ developments of length at most $\omega^{\omega(n-1)}$. Since $R = \bigcup_n R_n$, all elements of $R$ have developments of length less than $\omega^\omega$.

In Section 5, we show that the bounds in Theorem 1.1 are sharp. We describe a countable real closed field such that for each $n$, subfield $R_n$ has some element with a development of length $\omega^{\omega(n-1)}$.

Our motivation was a problem of effectiveness.

Problem 1. What oracle information is needed to find an integer part for a countable real closed field?

We have not yet solved this problem. We do have some effectiveness results related to the construction of Mourgues and Ressayre. In §6, we show that for any countable real closed field $R$, there is a value group $G$, a subgroup of $(R^+, \cdot)$, such that $G$ is $\Delta_2^0(R)$. We also show that this is best possible in the sense that there is a computable real closed field in which every value group codes the halting set. In §7, we show that for any countable real closed field $R$, there is a residue field $k$, a subfield of $R$, such that $k$ is $\Pi_2^0(R)$. We show that this is best possible in the sense that there is a computable real closed field with no $\Sigma_2^0$ residue field. Finally, in §8, we show that for any countable real closed field $R$, there is an integer part that is $\Delta_0^\omega(R)$, obtained by the MR procedure. To prove this, we use our bounds on the lengths of developments, together with the results on complexity of value groups and residue fields. We cannot yet say whether there is a computable real closed field with no $\Delta_0^\omega$ integer part, obtained by some method different from that of Mourgues and Ressayre.
In §2, we give some basic definitions and background.

2. Background and Definitions

Let $R$ be a real closed field.

2.1. Value Groups. The natural value group of $R$ describes the orders of infinity that are represented in $R$. To give a precise definition, we begin by defining the “archimedean” equivalence relation on $R$.

**Definition 2** (Archimedean equivalence). We say that $x$ and $y$ are archimedean equivalent, denoted $x \sim y$, if there exist $m,n \in \mathbb{N}$ such that $m|x| > |y|$ and $n|y| > |x|$.

Note that $\{0\}$ is a $\sim$-equivalence class.

**Definition 3** (Natural valuation map). The natural valuation of $R$ is the function $w$ taking each $r \in R \setminus \{0\}$ to its $\sim$-equivalence class.

**Definition 4** (Natural value group). The natural value group of $R$ is the ordered abelian group consisting of $\sim$-equivalence classes other than $\{0\}$ under the operation inherited from multiplication on $R$. The ordering on the value group is the reverse of the one inherited from $R$.

For any real closed field $R$, the natural value group is a divisible ordered abelian group.

**Proposition 2.1.** There is a subgroup $G$ of the multiplicative group of positive elements of $R$ that is isomorphic to the value group.

Since we will give an effective version of this result later, we sketch the proof.

**Proof sketch.** We consider subgroups $H$ of $(R^+, \cdot)$ with the following features:

- $H$ is a divisible abelian group,
- no two elements of $H$ are $\sim$-equivalent.

The subgroup consisting just of 1 has these features. If $H$ is a group with these features, and $r$ is an element of $R^+$ that is not $\sim$-equivalent to any element of $H$, then we have a larger group $H'$ with these features—the elements of $H'$ have the form $gg'$, where $g \in H$ and $g'$ is a rational power of $r$. The set of groups with these features is closed under unions of chains. We let $G$ be a maximal subgroup with the two features. The group $G$ is isomorphic to the natural value group by maximality.

We use the term value group to refer to any of the groups $G$ as in Proposition 2.1. If $R$ is archimedean, then there is a unique value group, consisting just of 1. If $R$ is not archimedean, there are many different possibilities for the value group, although they are all isomorphic.

**Remarks**

1. Because the value group $G$ is abelian, some people use additive notation, and they write 0 for the identity of $G$. Since our value group is a subgroup of the multiplicative group $(R^+, \cdot)$, we choose not to use additive notation, and we write 1 for the identity of $G$.
The ordering on the group is reversed; we write \( g > 1 \) if \( g \) is infinitesimal. This seems natural when we consider formal series \( \sum_i a_i g_i \), in which the later terms correspond to larger (more infinitesimal) group elements \( g_i \).

2.2. Residue Fields. For a real closed field \( R \), the residue field is essentially the set of reals present in \( R \). Here is the precise definition.

**Definition 5** (Residue field). Let \( R \) be a real closed field. The subring of finite elements of \( R \) is the set \( \mathcal{O} = \{ r \in R \mid w(r) \geq 1 \} \). The set of infinitesimals of \( R \) is the set \( \mathcal{M} = \{ r \in R \mid w(r) > 1 \} \). The residue field of \( R \) is the quotient of \( \mathcal{O} \) by the ideal \( \mathcal{M} \).

Since \( \mathcal{M} \) is the maximal ideal of \( \mathcal{O} \), the residue field is a field. If \( R \) is real closed, then the residue field is also real closed.

**Proposition 2.2.** For a real closed field \( R \), there is subfield \( k \) of \( R \), a section, such that \( k \) is isomorphic to the residue field.

We again sketch the proof, since we will give an effective version of this result later.

**Proof sketch.** We consider archimedean real closed subfields of \( R \). The real algebraic elements of \( R \) form one such subfield. Given \( F \), an archimedean real closed subfield of \( R \), if there is some \( r \in R \) such that \( r \) is finite, and \( r \) is not infinitesimally close to any element of \( F \), then we obtain a larger archimedean real closed subfield \( F' \) by taking the real closure of \( F(r) \). The set of archimedean real closed subfields is closed under unions of chains. Take a maximal such subfield \( k \). This is isomorphic to the natural residue field. 

While there is, in general, more than one possible residue field, all are isomorphic. Moreover, if \( R \) is real closed and \( k \) is a residue field, then \( k \) is also real closed. We refer to any of these sections as a residue field for \( R \).

2.3. The construction of Mourgues and Ressayre. Mourgues and Ressayre [10] showed that every real closed field has an integer part. Suppose \( R \) is a real closed field with value group \( G \) and residue field \( k \). Mourgues and Ressayre defined an embedding of \( R \) into the field \( k\langle\langle G \rangle\rangle \) consisting of formal series in which the terms correspond to members of a well ordered subset of \( G \) and the coefficients come from the residue field \( k \).

**Definition 6** \( k\langle\langle G \rangle\rangle \).

- The field \( k\langle\langle G \rangle\rangle \) consists of formal series \( t = \sum_{g \in S} a_g g \) where \( S \) is a well ordered subset of \( G \), and the coefficients \( a_g \) are in \( k \).
- For an element \( t = \sum_{g \in S} a_g g \) of \( k\langle\langle G \rangle\rangle \), we refer to the set \( \{ g \in S : a_g \neq 0 \} \) as the support of \( t \), and we write \( \text{Supp}(t) \).
- The operations of addition and multiplication on \( k\langle\langle G \rangle\rangle \) are the natural ones, defined in the same way as the operations for ordinary power series. For \( t, t' \in k\langle\langle G \rangle\rangle \), the coefficient of \( g \) in \( t + t' \) is \( a + b \), where \( a, b \) are the coefficients of \( g \) in \( t, t' \). (We make the obvious adjustment if there is a \( g \)-term in only one of \( t, t' \).) The coefficient of \( g \) in \( t \cdot t' \) is the sum of the products \( ab \), where there are terms \( ah \) in \( t \) and \( bh' \) in \( t' \) such that \( hh' = g \).
Remark. The fact that $S$ is well-ordered guarantees that the coefficient of $g$ in a product $t \cdot t'$ is a finite sum.

The construction of Mourgues and Ressayre involves an embedding of $R$ onto a truncation closed subfield of $k\langle\langle G\rangle\rangle$.

Definition 7 (Truncations).
- Let $t \in k\langle\langle G\rangle\rangle$. An element $s \in k\langle\langle G\rangle\rangle$ is a truncation of $t = \Sigma_{a \in S} a_g g$ if $s = \Sigma_{a \in S'} a_g g$, where $S'$ is an initial segment of $S$.
- Let $F$ be a subset of $k\langle\langle G\rangle\rangle$. We say that $F$ is truncation closed if for any $t \in F$, all of the truncations of $t$ are also in $F$.

Mourgues and Ressayre used the following.

Proposition 2.3. If $G$ is a divisible ordered abelian group and $k$ is an archimedean ordered field, then $k\langle\langle G\rangle\rangle$ is real closed.

Mourgues and Ressayre observed that it is easy to find a truncation closed integer part $I$ for $k\langle\langle G\rangle\rangle$. Let $I$ consist of the elements of the form $s + z$, where $s \in k\langle\langle G\rangle\rangle$ such that $\text{Supp}(s)$ consists of infinite elements of $G$ (those of valuation less than 1) and $z \in \mathbb{Z}$. We identify $z$ with $z \cdot 1$. For each element $t \in k\langle\langle G\rangle\rangle$, we calculate the integer part of $t$, denoted by $[t]$, as follows. Say $t = \Sigma_{g \in S} a_g g$. We may suppose that $1 \in S$. Let $S'$ be the initial segment of $S$ consisting of infinite elements, and let $s = \Sigma_{g \in S'} a_g g$—so the element $s$ is a truncation of $t$. Since $k$ is isomorphic to a subfield of $\mathbb{R}$, we calculate $[a_1]$, the integer part of $a_1$, in the usual way. Then the integer part of $t$ is $s + [a_1]$. Notice that if $F$ is a truncation closed subfield of $k\langle\langle G\rangle\rangle$, then $\{[a] | t \in F\}$ is an integer part $I$ of $F$.

To find an integer part for an arbitrary real closed field $R$, with value group $G$ and residue field $k$, Mourgues and Ressayre showed that there is an isomorphism $\delta$ from $R$ onto a truncation closed subfield $F$ of $k\langle\langle G\rangle\rangle$.

The ideas of Mourgues and Ressayre have proved useful in more general settings, in particular, for fields with exponentiation [9]. There are natural questions about integer parts for fields that are not real closed [7], and about ordered rings that do and do not extend to models of $I\text{Open}$ [14]. Berarducci [1] looked at lengths of developments. Without calculating bounds, he showed that length of development is a new valuation function. Erhlich [5] showed that every real closed field can be embedded in the “surreal numbers”.

3. Making the MR-procedure canonical

Suppose $R$ is a real closed field with residue field $k$, and suppose the list $r_1, r_2, \ldots$ forms a transcendence basis for $R$ over $k$. Let $R_n$ be the real closure of $k(r_1, \ldots, r_n)$. In terms of these data, we show how to make the Mourgues and Ressayre construction canonical. We begin by defining, by induction on $n$, a value group $G_n$ appropriate for $R_n$, such that $G_{n+1} \supseteq G_n$. Then $G = \bigcup_n G_n$ is a value group for $R$. We also adjust the list $r_1, r_2, \ldots$. After that, we describe, also by induction on $n$, an isomorphism $\delta_n$ from $R_n$ onto a truncation closed subfield of $k\langle\langle G_n\rangle\rangle$, so that $\delta_{n+1} \supseteq \delta_n$. Then $\bigcup_n \delta_n$ is an isomorphism from $R$ onto a truncation closed subfield of $k\langle\langle G\rangle\rangle$.

1They credit David Marker and Françoise Delon with some help with this.
3.1. **Choosing** \((G_n)_{n \in \omega}\) **and adjusting the list** \(r_1, r_2, \ldots\). For \(n = 0\), \(R_0 = k\), and \(G_0\) consists just of 1. Given \(G_n\) an appropriate value group for \(R_n\), we have two cases.

Case 1 Suppose that for each \(r \in R_{n+1}\), there is some \(g \in G_n\) such that \(r \sim g\).

Then we let \(G_{n+1} = G_n\).

Case 2 Suppose that some \(r \in R_{n+1}\) for which there is no \(g \in G_n\) such that \(r \sim g\). We choose one such \(r\), taking one that is positive and infinitesimal.

We choose \(r = r_{n+1}\) if \(r_{n+1}\) satisfies these conditions. Otherwise, we replace \(r_{n+1}\) by such an \(r\). We let \(G_{n+1}\) consist of the elements \(gg'\), where \(g \in G_n\) and \(g'\) is a rational power of \(r\).

3.2. **Canonical choice of developments.** We have chosen the value groups \(G_n\), and we have modified the list \(r_1, r_2, \ldots\) so that if \(G_{n+1} \neq G_n\), then \(r_{n+1} \in G_{n+1}\).

We define, by induction on \(n\), a chain of functions \((\delta_n)_{n \in \omega}\), where \(\delta_n\) maps \(R_n\) isomorphically onto a truncation closed subfield of \(k(\langle\langle G_n\rangle\rangle)\).

Recall that \(R_0 = k\). For each \(r \in R_0\), we let \(\delta_0(r) = r \cdot 1\). If we have defined \(\delta_n\), we need to extend to \(\delta_{n+1}\). We shall use following result of Mourgues and Ressayre [10].

**Proposition 3.1.** Suppose that \(F\) is a subfield of \(k(\langle\langle G\rangle\rangle)\).

1. If \(F\) is truncation closed, then the real closure of \(F\) is truncation closed.
2. If \(F\) is truncation closed and \(s\) is a further element of \(k(\langle\langle G\rangle\rangle)\) such that all proper initial segments of \(s\) are in \(F\), then \(F(s)\) is truncation closed.

We have \(\delta_n\) mapping \(R_n\) isomorphically onto a truncation closed subfield \(F\) of \(k(\langle\langle G\rangle\rangle)\). We must extend to \(\delta_{n+1}\), mapping \(R_{n+1}\) isomorphically onto a truncation closed subfield \(F'\) of \(k(\langle\langle G\rangle\rangle)\). By Proposition 3.1, it is enough to define \(\delta_{n+1}(r_{n+1})\) such that all of the proper truncations are present in \(F\) and \(\delta_{n+1}(r_{n+1}) = 1 \cdot r_{n+1}\). The non-trivial case is where \(G_{n+1} = G_n\).

We say how to choose the development \(s = \sum a_i g_i\) for \(r = r_{n+1}\). We refer to the initial segment \(\sum_{i < \alpha} a_i g_i\) of \(s\) of length \(\alpha\) as the \(\alpha\)-development of \(r\). We let \(w\) be the valuation function with values in \(G_n\). We are following exactly the procedure of Mourgues and Ressayre.

We let \(g_0 = w(r)\), and we let \(a_0\) be the unique element of \(k\) such that \(r - a_0 g_0 \not\sim g_0\). Then \(a_0 g_0\) is the 1-development of \(r\). Since \(k \subseteq F\), \(a_0 g_0\) is an element of \(F\).

Suppose we have determined the \(\alpha\)-development of \(r\) to be \(s_\alpha = \sum_{i < \alpha} a_i g_i\). If \(\alpha\) is a successor ordinal, say \(\alpha = \beta + 1\), then \(s_\alpha\) is the result of adding a last term \(a_\beta g_\beta\) to \(s_\beta\). If \(s_\beta\) is in \(F\), then \(s_\alpha\) is also in \(F\). If \(\alpha\) is a limit ordinal, then \(s_\alpha\) may or may not be in \(F\). First, suppose that \(s_\alpha\) is in \(F\), say \(s_\alpha = \delta_n(r_\alpha)\). Then we consider \(r - r_\alpha\). Let \(g_\alpha = w(r - r_\alpha)\). Then \(g_\alpha\) is greater (more infinitesimal) than any \(g_i\), for \(i < \alpha\), and there is a unique \(a_\alpha \in k\) such that \((r - r_\alpha) = a_\alpha g_\alpha\) \(\not\sim g_\alpha\). In this case, \(s_{\alpha+1} = \sum_{i < \alpha+1} a_i g_i\) is the \((\alpha + 1)\)-development of \(r\). Of course, \(r_\alpha + a_\alpha g_\alpha\) is in \(R_\alpha\).

Next suppose that \(s_\alpha\) is not in \(F\). Then, we assign \(s_\alpha\) to be the development of \(r\); i.e., \(s = s_\alpha\). It follows from the results of Mourgues and Ressayre that if \(a, b \in R_\alpha\), where \(a < r < b\), then \(\delta_n(a) < s < \delta_n(b)\). In particular, \(a\) and \(b\) cannot have an \(\alpha\)-development that matches \(s_\alpha\). It follows that \(R_{n+1}\) is isomorphic to the real closure of \(F(s)\), under the unique extension of \(\delta_n\) that takes \(r\) to \(s\), and takes
the \( k^{th} \) root of any polynomial over \( R_n[r] \) to the \( k^{th} \) root of the corresponding polynomial over \( F[s] \).

We refer to the specific procedure (for assigning developments) described above as the \( MR \)-procedure. From what we have said so far, it is not clear how to bound the length of the developments. That will be the goal of the next section.

4. Bounding lengths of developments

We want to prove Theorem 1.1. Let \( R \) be a countable real closed field, with residue field \( k \), and let \( r_1, r_2, \ldots \) be a transcendence basis for \( R \) over \( k \). Let \( R_0 = k \), and let \( R_{n+1} \) be the real closure of \( k(r_1, \ldots, r_{n+1}) \). For \( R_0 = k \), it is clear that all elements of \( R_0 \) have developments of length 1. Shepherdson [12] showed that all elements of \( R_1 \) have developments of length at most \( \omega \). We shall give a proof of this result. We generalize this proof to show that for \( n \geq 1 \), all elements of \( R_n \) have developments of length at most \( \omega^{\omega(n-1)} \). The next two lemmas will allow us to make some simplifying assumptions about an element of \( R_{n+1} \).

Lemma 4.1. For any \( t \in R_n \), there is some \( t' \in R_n \) such that \( w(t') = 1 \) and the developments of \( t \) and \( t' \) have the same length.

**Proof.** We let \( t' = \frac{t}{w(t)} \). \( \square \)

Lemma 4.2. If \( t \) is a root of the polynomial \( p(x) = A_0 + A_1x + \cdots + A_kx^k \), where \( A_i \in R_n \), then \( t \) is also a root of a polynomial \( p'(x) = A'_0 + A'_1x + \cdots + A'_kx^k \), where \( A'_i \in R_n \), the developments of \( A_i \) and \( A'_i \) have the same length, and for all \( i \), \( w(A'_i) \geq 1 \).

**Proof.** Suppose \( h \) is the least \( w(A_i) \). We let \( A'_i = \frac{A_i}{h} \). \( \square \)

4.1. Shepherdson’s Theorem. Suppose \( n = 1 \). We suppose that \( R_1 \) has transcendence degree 1 over the residue field \( k \). It follows that \( R_1 \) is non-archimedean. The value group \( G_1 \) is generated as a divisible ordered abelian group by a single infinitesimal element \( g \), and we may suppose that \( r_1 = g \). The development of \( g \) has length 1. Then by Proposition 3.1, the developments of the other elements of \( R_1 \) are uniquely determined. Shepherdson’s result bounds the lengths of these developments.

**Theorem 4.3** (Shepherdson). For all \( t \in R_1 \), the development of \( t \) has length at most \( \omega \).

**Proof.** By Lemma 4.1, it is enough to consider \( t \in R_1 \) with \( w(t) = 1 \). If \( t \in R_1 \), then \( t \) is a root of a polynomial over \( k(g) \)—say \( p(x) = A_0 + A_1x + \cdots + A_kx^k \). We may suppose that the coefficients are in the ring \( k[g] \). Initially, the coefficients are quotients of elements of \( k[g] \), but if we multiply by the product of the denominators, we obtain a polynomial with coefficients \( A_i \) in \( k[g] \). This implies that the development of \( A_i \) has finite length. Applying Lemma 4.2, we may suppose that for all \( i \), \( w(A_i) \geq 1 \). We are giving up the feature that \( A_i \in k[g] \), but we are retaining the important consequence that \( A_i \) has a development of finite length.
**Definition 8.** We say that \( t \in R_1 \) is “well-adjusted” if \( w(t) = 1 \) and \( t \) is a root of a polynomial in which all of the coefficients (elements of \( R_1 \)) are finite or infinitesimal, and all have finite developments.

We have established the following.

**Lemma 4.4.** If all well-adjusted elements of \( R_1 \) have developments of length at most \( \omega \), then all elements of \( R_1 \) have developments of length at most \( \omega \).

Let \( t \) be a well-adjusted element of \( R_1 \). Suppose \( t \) is a root of the polynomial \( p(x) = A_0 + A_1 x + \cdots + A_k x^k \), where for all \( i \), \( w(A_i) \geq 1 \), and the development of \( A_i \) is finite. Let \( X \) be the combined support of the \( A_i \). Then \( X \) is a finite set. Our goal is to show that \( t \) has a development of length at most \( \omega \). The development has the form \( \Sigma_{j<\alpha} b_j h_j \), where \( h_0 = 1 \). We look at the formal expansion of \( p(t) = A_0 + A_1 t + \cdots + A_k t^k \), obtained by substituting \( \Sigma_{j<\alpha} b_j h_j \) for \( t \) and carrying out the indicated operations. Each element of the support of \( t^i \) is a product of \( i \) factors from the support of \( t \). Each element of the support of \( A_i t^i \) is the product of one factor from the support of \( A_i \) and \( i \) factors from the support of \( t \). When we calculate \( p(t) \) formally, we consider terms corresponding to elements of the support of \( A_0 \), plus the products just described, with coefficients in \( k \). The coefficient in \( t^i \) of a product \( h_{j_1} \cdots h_{j_i} \) is \( b_{j_1} \cdots b_{j_i} = c \). We note that if there are \( m \) distinct products resulting from rearranging the same factors \( h_{j_i} \), then the sum of the coefficients is \( mc \).

We let \( S \) be the set of finite tuples of the following forms:

- \((x)\), where \( x \in \text{Supp}(A_0)\),
- \((x, h_{j_1}, \ldots, h_{j_i})\), where \( x \in \text{Supp}(A_i) \) and \( h_{j_1}, \ldots, h_{j_i} \in \text{Supp}(t)\).

We consider \( h_j \) as a name for an unknown element of \( \text{Supp}(t) \), with a value to be assigned in the group. What we assume initially about these names is that \( h_0 = 1 \), and that \( h_i < h_j \) if \( i < j \). To determine \( h_1 \), we imagine the least group element \( u \) that can be obtained as the product of a tuple in \( S \) in which \( h_1 \) appears. Let \( g^* \) be the least \( w(A_i) \), for \( i > 0 \). Then \( u \) must be equal to the product of \( g^* \) and an \( i \)-tuple with one entry \( h_1 \) and the other entries all \( h_0 = 1 \). Thus, \( u = g^* h_1 \). Since \( u \) has coefficient 0 in the expansion of \( p(t) \), we must also obtain \( u \) as the product of a second tuple in \( S \), in which \( h_1 \) and greater \( h_j \)'s do not appear. This second tuple has first term \( x \), where \( x \in \text{Supp}(A_i) \) for some \( i \). If \( i = 0 \), there is nothing more to the tuple. We have \( u = x \). If \( i > 0 \), then there are \( i \) further terms, but all of them must be \( h_0 = 1 \). Again, we get \( u = x \). Then \( g^* h_1 = x \), and \( h_1 = \frac{x}{g^*} \). Since \( h_1 > 1 \), we must have \( x > g^* \).

We can show, by induction on \( n \), that \( h_n \) can be expressed as a finite product of factors of the form \( \frac{x}{g^*} \), where \( x \in X \) and \( x \geq g^* \). We have seen that this is true for \( h_0, h_1 \). Supposing that it is true for all \( m < n \), we consider \( h_n \). Let \( u \) be the least group element that can be obtained as the product of a tuple in \( S \) in which \( h_n \) appears. Again, we see that the tuple starts with \( g^* \), and the other entries, apart from a single \( h_n \), are all \( h_0 = 1 \). Therefore, \( u \) can be expressed in the form \( g^* h_n \). Since the coefficient of \( u \) in the expansion of \( p(t) \) is 0, \( u \) is also the product of another tuple in \( S \), with first term \( x' \in A_j \) and \( j \) further terms of the form \( h_m \), for \( m < n \). If \( j = 0 \), there are no factors \( h_m \). In this case, we have \( u = x' \), so \( h_n = \frac{x'}{g^*} \), and \( x' > g^* \) just because \( h_n > 1 \). If \( j > 0 \), then \( u \) is the product of \( x' \) and \( j \) factors \( h_m \), so \( h_n \) is the product of \( \frac{x'}{g^*} \) and the \( j \) factors \( h_m \). In this case \( x' > g^* \) just by our choice of \( g^* \).
We have $h_0 = 1$ and for $0 < m < n$, our induction hypothesis says that $h_m$ is a finite product of factors $\frac{x}{g_i}$, where $x \geq g^*$.

Dropping the factors that are equal to 1, we may suppose that for $n > 0$, $h_n$ is a product of finitely many factors $\frac{x}{g^*}$, where $x \geq g^*$. We must show that there is no term $h_*$. Suppose there is such a term. Again, we consider the least group element $u$ that can be expressed as the product of some tuple in $S$ in which $h_*$ appears. The tuple must start with $g^*$, and the other entries, apart from a single $h_*$, must be $h_0 = 1$. Then $u = g^*h_*$. Since the coefficient of $u$ in $p(t)$ is 0, $u$ can also be expressed as the product of a second tuple in $S$, with first term $x \in A_j$ and $j$ further terms $h_m$, for $m < \omega$. If $j > 0$, then there are no factors $h_m$, and $h_* = \frac{x}{g^*}$. We must have $x > g^*$. If $j > 0$, then $x \geq g^*$ by our choice of $g^*$, and $h_*$ is the product of $\frac{x}{g^*}$ and finitely many $h_m$, each of which is either 1 or a finite product of factors $\frac{x'}{g'}$, for $x' > g^*$. In either case, $h_*$ is a finite product of factors of the form $\frac{x}{g^*}$, where $x \in X$ and $x > g^*$.

**Lemma 4.5.** Let $P$ consist of 1 and the set of finite products of elements of the form $\frac{x}{g^*}$, where $x \in X$ and $x > g^*$. For each $p \in P$, there are only finitely many $q \in P$ such that $q < p$.

**Proof.** Let $z$ be the least element of $X$ such that $z > g^*$. (Such a $z$ must exist if $t$ has a development of length greater than 1.) For some $k$, $(\frac{z}{g})^k \geq p$. Fix $p > 1$ in $P$. Then for $q \in P$ such that $q < p$, either $q = 1$, or $q$ is a product of at most $k$ factors $\frac{x}{g^*}$ such that $v > g^*$. Since $X$ is finite, there are only finitely many possibilities for $q$.

We have seen that $h_*$ and all $h_n$ are elements of $P$. By Lemma 4.5, there are only finitely many elements of $P$ less than $h_*$. This is a contradiction, so there is no $h_*$.

In the proof above, we ignored the fact that the polynomial $p(x)$ may have several roots $t$ such that $w(t) = 1$. The proof shows that for any such $t$, $\text{Supp}(t)$ consists of 1 and finite products of elements $\frac{x}{g^*}$, where $x \in X$, and $x^* > g^*$. The set $X$ was derived from the polynomial.

The bound in Shepherdson’s Theorem is sharp.

**Proposition 4.6.** In $R_1$ there is an element $t$ whose development has length $\omega$.

**Proof.** We are supposing that $R_1$ is the real closure of $k(g)$, where $g$ is infinitesimal. Then $t = \frac{1}{1-g}$ has the obvious development $1 + g + g^2 + \ldots$ of length $\omega$.

Our Theorem 1.1 generalizes Theorem 4.3. In the proof, we shall need some results on products.

### 4.2. Products

Let $G$ be a divisible ordered abelian group. Let $X$ be a well ordered set of infinitesimals in $G$, and let $P_X$ be the set of finite products of elements of $X$. The next lemma is very simple.

**Lemma 4.7.** Let $X$ be a well ordered set of infinitesimals in $G$. For all $m$, the set of products of $m$ elements of $X$ is well-ordered.
Proof. We can give a soft proof. Let $s = \sum_{x \in X} x$. This is an element of $\mathbb{R}(\langle G \rangle)$. Then $\text{Supp}(s) = X$, and $\text{Supp}(s^m)$ is the set of products of $m$ elements of $X$. It follows that the set is well ordered.

It follows from 4.7 that the set of products of $m$ or fewer elements of $X$ is a well ordered. The next lemma requires more work.

**Lemma 4.8.** Let $X$ be a well ordered set of infinitesimals in $G$. Then $P_X$ is well ordered.

**Proof.** Earlier, we defined the archimedean equivalence relation $\sim$. Now, we define another equivalence relation, $\approx$. As before, we consider only non-negative elements.

**Definition 9** ($\approx$). Suppose $x, y > 0$. Let $x \approx y$ if there exist $m, n$ such that $x^m > y$ and $y^n > x$.

Suppose $P_X$ is not well ordered, expecting a contradiction. Consider the set of strictly decreasing sequences $(p_n)_{n \in \omega}$ in $P_X$. Let $(p_n)_{n \in \omega}$ be the sequence such that the first term $p_0$ belongs to the least possible $\approx$-class.

**Case 1:** Suppose that all elements of $X$ that occur as factors of the $p_n$ belong to a single $\approx$-class.

Take the least $x \in X$ such that $x$ occurs as a factor in some $p_n$. There is some $m$ such that $x^m \geq p_0$. Since the sequence $(p_n)_{n \in \omega}$ is decreasing, $x^m \geq p_n$ for all $n$. Then no $p_n$ can have more than $m$ factors in $X$. By Lemma 4.7, the set of elements of $P_X$ with at most $m$ factors is well ordered. This contradicts the fact that the sequence $(p_n)_{n \in \omega}$ is decreasing.

**Case 2:** Suppose the elements of $X$ that occur as factors of the $p_n$ represent more than one $\approx$-class.

Since the sequence $(p_n)_{n \in \omega}$ is decreasing, the $\approx$-class of $p_0$ is the greatest represented by elements of $X$ that are factors of the $p_n$. For each $n$, we have $p_n = p'_n q_n$, where $q_n$ is the product of the factors in the greatest $\approx$-class, and $p'_n$ is the product of the remaining factors. Using Ramsey’s Theorem, and replacing the given sequence by an appropriate subsequence, we may suppose that the sequence $(q_n)_{n \in \omega}$ is strictly decreasing, strictly increasing, or constant. By Case 1, it cannot be strictly decreasing, so it is either strictly increasing or constant. Then the sequence $(p'_n)_{n \in \omega}$ must be strictly decreasing. However, the $\approx$-class of $p'_n$ lies to the left of that of $p_0$, contradicting our choice of the original sequence $(p_n)_{n \in \omega}$.

It is tempting to try to give a soft proof of Lemma 4.8, similar to the one we gave for Lemma 4.7. Let $s = \sum_{g \in X} g$, and consider $t = \frac{1}{1 + s}$. We would like to claim that $t = 1 + s + s^2 + s^3 + \ldots$, and that the support of $t$ consists of 1 and the elements of $P_X$. To flesh out this argument, we need to know that $1 + s + s^2 + s^3 + \ldots$ is well defined. We need the following lemma that can be found in ([8], VIII.5) and uses ideas similar to those above.

**Lemma 4.9.** Each $p$ occurs in $\text{Supp}(s^m)$ for only finitely many $m$. 

4.3. **Lengths of sums and products.** We need to consider lengths developments of sums and products. Suppose \( s, t \in k\langle\langle G\rangle\rangle \), where \( \text{Supp}(s) \) has order type at most \( \alpha \) and \( \text{Supp}(t) \) has order type at most \( \beta \). We want to set an upper bound on the order type of \( \text{Supp}(s + t) \) and \( \text{Supp}(s \cdot t) \). We shall need commutative sum and product of ordinals.

**Definition 10** (Commutative Sum). For ordinals \( \alpha \) and \( \beta \), the **commutative sum**, denoted by \( \alpha \oplus \beta \), is the ordinal whose Cantor normal form is obtained by expressing \( \alpha \) and \( \beta \) in Cantor normal form, and summing the coefficients of like terms.

**Definition 11** (Commutative Product). For ordinals \( \alpha \) and \( \beta \), the **commutative product**, denoted by \( \alpha \otimes \beta \), is the ordinal whose Cantor normal form is obtained by expressing \( \alpha \) and \( \beta \) in Cantor normal form, multiplying as for polynomials, using commutative sum in the exponents, and then combining like terms. If \( \alpha = \sum_i \omega^{\alpha_i} m_i \) and \( \beta = \sum_j \omega^{\beta_j} n_j \), then the product of the terms \( \omega^{\alpha_i} m_i \) and \( \omega^{\beta_j} n_j \) is \( \omega^{\alpha_i \oplus \beta_j} (m_i \cdot n_j) \).

For example, \( \omega \otimes \omega = \omega^2, \omega^2 \otimes \omega^2 = \omega^4, \omega^\omega \otimes \omega^\omega = \omega^{\omega^2}, \omega^\omega \otimes \omega^\omega = \omega^{\omega^2}, \) etc.

The following Lemma and Proposition give bounds on the lengths developments of sums and products. These results (or the ideas behind them) have been shown in [6], [2]. To set a bound on the length of \( s \cdot t \), one can use a result of De Jongh and Parikh (see [4], [11], [12]). We are grateful to Andreas Weiermann for describing (at the CiE meeting in Heidelberg in 2009), the results behind this, and for giving us the appropriate references.

**Lemma 4.10.** If \( X \) and \( Y \) are well ordered subsets of \( G \), of order type \( \alpha \), \( \beta \), respectively, then \( X \cup Y \) has order type at most \( \alpha \oplus \beta \). Hence, if \( s, t \in k\langle\langle G\rangle\rangle \), where \( s \) has length \( \alpha \) and \( t \) has length \( \beta \), then \( s + t \) has length at most \( \alpha \oplus \beta \).

**Proposition 4.11.** Suppose \( X \) and \( Y \) are well-ordered subsets of \( G \), of order types at most \( \alpha \) and \( \beta \), respectively. Then \( X \cdot Y = \{ x \cdot y : x \in X \land y \in Y \} \) has order type at most \( \alpha \otimes \beta \). Hence, if \( s, t \in k\langle\langle G\rangle\rangle \), where \( s \) has length \( \alpha \) and \( t \) has length \( \beta \), then \( s \cdot t \) has length at most \( \alpha \otimes \beta \).

Recall that \( P_X \) denotes the set of finite products of elements of \( X \). In proving Shepherdson’s Theorem, we showed that if \( X \) is a finite set of infinitesimals, in a value group generated by a single element, then \( P_X \) has order type at most \( \omega \). The next lemma generalizes this.

**Lemma 4.12.** Suppose \( X \subseteq G \) is a set of infinitesimals, all in one \( \approx \)-class. If \( X \) has order type \( < \omega^m \), then \( P_X \) has order type at most \( \omega^{\omega^m} \).

**Proof.** Let \( p \in P_X \), and consider \( \text{pred}(p) = \{ q \in P : q < p \} \). We must show that \( \text{pred}(p) \) has order type less than \( \omega^{\omega^m} \). Let \( x \) be the least element of \( X \). There is some \( m \) such that \( x^m > p \). Then all elements of \( \text{pred}(p) \) have at most \( m \) factors in \( X \). Using Proposition 4.11, we see that the order type of \( \text{pred}(p) \) is less than \( \omega^{\omega^m} \). \( \square \)

4.4. **Inductive proof of Theorem 1.1.** To prove Theorem 1.1, we prove three statements by simultaneous induction on \( n \).

1. If \( R_n \) is a real closed field of transcendence degree \( n \) over its residue field, then all elements of \( R_n \) have developments of length at most \( \omega^{\omega^{(n-1)}} \).
(2) Suppose $R_n$ is a real closed field of transcendence degree $n$ over its residue field. Let $X = \text{Supp}(s)$, where $s$ is an infinitesimal in $R_n$. Then, $P_X$ has order type at most $\omega^{(n-1)}$.

(3) Suppose $R_n$ is a real closed field of transcendence degree $n$ over its residue field. Let $X = \text{Supp}(s)$, where $s$ is an element of $R_n$ with $w(s) \geq 1$ and $X$ has order type less than $\omega^{(n-1)}$. Then $P_X$ has order type at most $\omega^{(n-1)}$.

The first statement is the one we are most interested in. However, it is clear from the proof of Shepherdson’s Theorem that the three statements are related. First, suppose $n = 1$. The first statement holds, by Theorem 4.3. The third statement was proved along the way—see the comment before Lemma 4.12. The second statement is easily verified. Let $s$ be an infinitesimal in $R_1$, and let $X = \text{Supp}(s)$. We must show that for $p \in P_X$, \text{pred}(p)$ is finite. Let $x$ be least in $X$. There is some $m$ such that $x^m \geq p$. Then for all $q \in P_X$ with $q < p$, $q$ has at most $m$ factors in $X$. All factors must be less than $p$. Since $X = \text{Supp}(s)$, $X$ has order type at most $\omega$, by Theorem 4.3. If $X$ has order type equal to $\omega$, then $X$ is co-final in $G_1$, for if there were an upper bound $g$, then $s + g$ would have a development of length greater than $\omega$. Therefore, only finitely many elements of $X$ are less than $p$. Since each $q < p$ is a product of at most $m$ elements from a finite set, there are only finitely many possible $q$.

Proceeding by induction, we suppose that all three statements hold for $n$, and we prove all three for $n + 1$. First, we show that $\text{III}_{n+1}$ and $I_n$ imply $I_{n+1}$. Next, we show that $\text{III}_{n+1}$ implies $II_{n+1}$. Finally, we prove $\text{III}_{n+1}$, using $II_n$.

4.4.1. Proof of $I_{n+1}$ from $\text{III}_{n+1}$ and $I_n$. We must show that all elements of $R_{n+1}$ have developments of length at most $\omega^{(n)}$. Recall that $R_{n+1}$ is the real closure of $R_n(r_{n+1})$, and we are modifying the given transcendence basis, if necessary, so as to include a basis for the value group. We may suppose that $r = r_{n+1}$ satisfies one of the following.

(1) $r$ is a new group element, so that $G_{n+1}$ consists of the products $gr^q$, for $g \in G_n$ and $q \in \mathbb{Q}$.

(2) $G_{n+1} = G_n$, and for some limit ordinal $\alpha \leq \omega^{(n-1)}$, the $\alpha$-development of $r$ is not assigned to any element of $R_n$, in which case, this is the development assigned to $r$.

In either case, we can see the following.

Lemma 4.13. The development of $r = r_{n+1}$ has length at most $\omega^{(n-1)}$.

Apart from $r$, the other elements of $R_{n+1}$ are all roots of polynomials over the field $R_n(r)$, or over the ring $R_n[r]$. Let $t \in R_{n+1}$. By Lemma 4.1, we may suppose that $w(t) = 1$. Suppose $t$ is a root of the polynomial $p(x) = A_0 + A_1x + \ldots + A_kx^k$, where $A_i \in R_n[r]$. Let us suppose that $A_i = a_{i,0} + a_{i,1}r + \ldots + a_{i,n}r^n$, where $a_{i,j} \in R_n$. By $I_n$, the $a_{i,j}$ all have developments of length at most $\omega^{(n-1)}$. Using the results on lengths of sums and products, we see that all $A_i$ have developments of length less than $\omega^{(n)}$. By Lemma 4.2, $t$ is a root of a polynomial $p'(x) = A'_0 + A'_1x + \ldots + A'_kx^k$, where $w(A'_i) \geq 1$, and the developments of $A_i$ and $A'_i$ have the same length. For simplicity, we suppose that $A_i = A'_i$. Let $Y = \bigcup_i \text{Supp}(A_i)$. By Lemma 4.1, this has order type less than $\omega^{(n)}$. Let $g^*$ be the least $w(A_i)$ for $i > 0$, and let $X = \{ \frac{y}{g^*} : y \in Y \ \& \ y \geq g^* \}$. The order type of $X$ is bounded by that of $Y$, so it is
also less than \( \omega^n \). To put ourselves in a position to apply \( III_{n+1} \), we must show that \( \text{Supp}(t) \subseteq P_X \) and \( X = \text{Supp}(s) \) for some \( s \).

**Lemma 4.14.** \( \text{Supp}(t) \subseteq P_X \).

*Proof.* Let \( \Sigma_i b_i h_i \) be the development of \( t \), where \( h_i \in G_{n+1} \), \( b_i \in k \). We show by induction that \( h_i \in P_X \). Suppose \( h_j \in P_X \) for all \( j < i \), and we also have a term \( h_i \). As in the proof of Shepherdson’s Theorem, we expand \( p(t) \), and we use the fact that when we combine like terms, all coefficients should be 0. Let \( S \) be the set of finite tuples consisting of \( a \in \text{Supp}(A_m) \) for some \( m \), followed by terms \( h_j \). We use the fact that \( h_0 = 1 \) and the \( h_j \)'s are increasing. Let \( g^* \) be the least \( w(A_m) \) for \( m > 0 \). Let \( u \) be the least group element that can be expressed as a product of a tuple in \( S \) in which \( h_i \) appears. Since \( u \) is least, our tuple consists of \( g^* \), a single term \( h_i \), and the other \( i-1 \) terms \( h_0 \). We have \( u = g^* h_i \). Since the coefficient of \( u \) is 0, there is another tuple in \( S \), also with product \( u \), where the second tuple consists of some \( y \in A_m \), followed by \( m \) terms \( h_j \), for \( j < i \). Say \( p \) is the product of these \( h_j \). We have \( g^* h_j = y \cdot p \), so \( h_i = \frac{g^*}{p} \cdot p \). If \( m = 0 \), then \( h_i = \frac{g^*}{p} \cdot 1 \), so \( y \geq g^* \). If \( m > 0 \), then \( y \geq g^* \) by our choice of \( g^* \). We have shown that \( h_i \in P_X \).

\( \square \)

Recall that \( Y = \cup_i \text{Supp}(A_i) \). The set \( X \) is the union of the supports of elements \( t_i \), where \( t_i \) is obtained from \( A_i \) in the following way. First, let \( B_i = \frac{A_i}{p^i} \). The development of \( B_i \) may have an initial segment of infinite terms. Let \( C_i \) have development equal to this initial segment. Then \( t_i = B_i - C_i \). So far, we have shown the following, using \( I_n \).

**Lemma 4.15.** If \( t \in R_{n+1} \), where \( w(t) = 1 \), there is a set \( X \) of order type less than \( \omega^n \) such that \( \text{Supp}(t) \subseteq P_X \), and \( X \) is the union of the supports of finitely many elements \( t_i \).

The next lemma says that there is an \( s \) such that \( \text{Supp}(s) = \cup_i \text{Supp}(t_i) \). We cannot simply take \( s = \sum_i t_i \). It might be that when we combine like terms, there will be some cancellation. We want a linear combination \( s = \sum_{i \leq k} b_i t_i \), with coefficients \( b_i \) in the residue field, such that \( \text{Supp}(s) = \cup_i \text{Supp}(t_i) \). It is not clear that the residue field \( k \) actually has the coefficients we want.

**Lemma 4.16.** Suppose \( t_1, \ldots, t_k \in R_{n+1} \). There is a real closed field \( R'_{n+1} \), extending \( R_{n+1} \), such that the residue field \( k' \) for \( R'_{n+1} \) extends \( k \), the elements \( r_1, \ldots, r_{n+1} \) form a transcendence basis for \( R'_{n+1} \) over \( k' \), and there is some \( s \in R'_{n+1} \) such that \( \text{Supp}(s) = \cup_i \text{Supp}(t_i) \).

*Proof.* We note that \( k(\langle G_{n+1} \rangle) \) is isomorphic to a subfield of \( \mathbb{R}(\langle G_{n+1} \rangle) \). Since \( \mathbb{R} \) is uncountable, it contains elements \( b_1, \ldots, b_k \) that are algebraically independent over \( R_{n+1} \). The elements \( r_1, \ldots, r_{n+1} \) are algebraically independent over \( k \). They are also algebraically independent over \( k \) together with the new \( b_i \). Let \( k' \) be the real closure of \( k \) together with these \( b_i \), and let \( R'_{n+1} \) be the corresponding extension of \( R_{n+1} \), the real closure of \( k'(r_1, \ldots, r_{n+1}) \). Clearly, \( k' \) is the residue field of \( k(\langle G_{n+1} \rangle) \). The embedding \( \delta_{n+1} \) maps \( R_{n+1} \) onto a truncation closed subfield of \( k(\langle G_{n+1} \rangle) \), and this is also a truncation closed subfield of \( k'(\langle G_{n+1} \rangle) \). Using Proposition 3.1, we have a unique extension of \( \delta_{n+1} \) to \( R'_{n+1} \) that acts as the identity on \( k' \).
Now, \( s = \sum_{i \leq k} b_i t_i \) is an element of \( R'_{n+1} \). Let \( X = \cup_i \text{Supp}(t_i) \). Suppose \( a_i \) is the coefficient of \( x \) in \( t_i \). Then \( \sum_{i \leq k} b_i a_i \) is the coefficient of \( x \) in \( s \). The coefficient cannot be 0, since the \( b_i \)'s are algebraically independent over the \( a_i \)'s. Therefore, \( X = \text{Supp}(s) \).

\[ \square \]

Now, we can apply \( III_{n+1} \) to the set \( X = \cup_i \text{Supp}(t_i) \). We get the fact that \( P_X \) has order type at most \( \omega^{\omega^n} \). We had already seen that this is enough to complete the proof of \( I_{n+1} \).

4.4.2. Proof of \( II_{n+1} \) from \( III_{n+1} \) and \( I_n \). Suppose \( X = \text{Supp}(s) \), where \( s \) is infinitesimal in \( R_{n+1} \). We must show that \( P_X \) has order type at most \( \omega^{\omega^n} \). First, we replace \( s \) by \( 1 + s \) so that the valuation is 1. Using Lemma 4.15 and \( I_n \), we get a set \( Y \) with the following properties.

1. for all \( y \in Y \), \( w(y) \geq 1 \),
2. \( Y \) has order type less than \( \omega^{\omega^n} \),
3. there is a finite set of elements \( t_i \) such that \( Y = \cup_i \text{Supp}(t_i) \),

Using Lemma 4.16, we get a single element \( s' \) such that \( Y = \text{Supp}(s') \). The element \( s' \) may lie not in \( R_{n+1} \) itself, but in an extension with the same transcendence basis over its residue field. Now, \( P_X \subseteq P_Y \). Applying \( III_{n+1} \) to \( Y \), we get the fact that \( P_Y \) has order type at most \( \omega^{\omega^n} \), so the same is true for \( P_X \). This completes the proof of \( II_{n+1} \) from \( III_{n+1} \) and \( I_n \).

4.4.3. Proof of \( III_{n+1} \) from \( I_n \). Let \( X = \text{Supp}(s) \), where \( w(s) \geq 1 \), and suppose that \( X \) has order type less than \( \omega^{\omega^n} \). We must show that \( P_X \) has order type at most \( \omega^{\omega^n} \). We use the equivalence relation \( \approx \). The group \( G_{n+1} \) is generated (as an ordered divisible abelian group) by some of the elements \( r_1, \ldots, r_{n+1} \). The number of \( \approx \)-classes in \( X \) is bounded by the number of generators for the group. We proceed by induction on the number of \( \approx \)-classes in \( X \).

First, suppose that \( X \) lies entirely in one \( \approx \)-class. Using Lemma 4.12, we obtain the fact that \( P_X \) has order type at most \( \omega^{\omega^n} \). Now, suppose that there are \( (k+1) \approx \)-classes in \( X \). Let \( Y \) consist of the elements of \( X \) in the first \( k \approx \)-classes, and let \( Z \) consist of those in the last class. We can express \( s \) as \( s' + s'' \), where \( s' \) has support \( Y \) and \( s'' \) has support \( Z \). If \( \delta \) is the development function, then \( \delta(s') \) is the truncation of \( \delta(s) \) with the terms in \( Y \), and \( \delta(s'') \) has the remaining terms.

**Lemma 4.17.** There is a real closed subfield \( R' \) of \( R_{n+1} \) such that

1. \( s' \in R' \),
2. \( R' \) has transcendence degree at most \( n \) over \( k \),
3. \( \delta \) maps \( R' \) onto a truncation closed subfield of \( k((G_{n+1})) \).

**Proof.** We describe the transcendence basis for \( R' \) over \( k \). Recall that our transcendence basis for \( R_{n+1} \) over \( k \) includes the generators of the value group \( G_{n+1} \). Let \( g_1, \ldots, g_t \) be the group generators that belong to the first \( k \approx \)-classes, and let \( g \) be a generator in the last \( \approx \)-class. We put \( g_1, \ldots, g_t \) into \( R' \). We consider the element \( s' \) and its “initial segments”, where \( t \) is an initial segment of \( s \) if \( \delta(t) \) is a truncation of \( \delta(s') \). We order the initial segments by length of development.

If \( s' \) is in the real closure of \( k(g_1, \ldots, g_t) \), then we do not add anything more to the transcendence basis—\( R' \) is the real closure of \( k(g_1, \ldots, g_t) \). Otherwise, let \( s_0 \) be the first initial segment of \( s' \) not in the real closure of \( k(g_1, \ldots, g_t) \). We
add $s_0$ to our transcendence basis for $R'$. In general, suppose we have put into our transcendence basis the group generators $g_1, \ldots, g_\ell$ plus finitely many initial segments of $s'$, say $s_i$, for $i < m$, where the developments of the $s_i$ are of increasing length. If $s'$ is not in the real closure of $k$ and the elements so far put into the transcendence basis, then we let $s_m$ be the first initial segment of $s'$ that is not included, and we add $s_m$ to our transcendence basis for $R'$.

**Claim:** There is some $m$ such that $s'$ is in the real closure of the set \( \{g_1, \ldots, g_\ell\} \cup \{s_i : i < m\} \) over $k$, so this set is the transcendence basis for $R'$. Moreover, $\ell + m < n + 1$.

To see why this is so, note that since $R'$ is a subfield of $R_{n+1}$, the transcendence degree of $R'$ over $k$ cannot be greater than that of $R_{n+1}$. Let $G'$ be the divisible abelian group generated by $g_1, \ldots, g_\ell$. Using Proposition 3.1, we can show that the development function $\delta$ maps $R'$ onto a truncation closed subfield of $k(\langle G' \rangle)$. We have $\delta$ mapping the real closure of $k(r_1, \ldots, r_{n+1})$ onto a truncation closed subfield of $k(\langle G_{n+1} \rangle)$. For $1 \leq i \leq \ell$, let $R'_i$ be the real closure of $k'(g_1, \ldots, g_i)$. For $0 \leq m$, let $R'_{\ell + m}$ be the real closure of $\{g_1, \ldots, g_\ell\} \cup \{s_i : i < m\}$ over $k$. We can show that for all $i \leq \ell + m$, $\delta$ maps $R'_i$ onto a truncation closed subfield of $k(\langle G' \rangle)$. We use Proposition 3.1, noting that in each case, the image of the new generator $g_j$ or $s_j$ has a development with all of its proper initial segments already in the range.

Also, $\delta$ maps $R_{n+1}$ onto a truncation closed subfield of $k(\langle G_{n+1} \rangle)$, including $g$. Therefore, the transcendence degree of $R'$ over $k$ must be smaller. We have $\ell + m < n + 1$.

Using Lemma 4.17, we complete the proof of $III_{n+1}$ as follows. Let $p \in P_X$. We show that $\text{pred}(p)$ has order type less than $\omega^n$. Each $q < p$ is a product of some $y(q) \in P_Y$ and some $z(q) \in P_Z$. By Lemma 4.17, $s'$ lies in a field of lower transcendence degree over the residue field $k$. We can apply $II_n$ to get the fact that $P_Y$ has order type at most $\omega^{(n-1)}$.

By Lemma 4.12, $Z$ has order type at most $\omega^n$. The set $\{z(q) : q < p\}$ is a proper initial segment of $Z$. This is clear since for $z \in Z$, there is some $k$ such that $z^k > p$. It follows that $\{z(q) : q < p\}$ has order type less than $\omega^n$. By Proposition 4.11, $\text{pred}(p)$ has order type less than $\omega^n$. Therefore, $X$ has order type at most $\omega^n$.

We have finished the inductive proof of Theorem 1.1.

5. Examples

The following examples show that our bounds on lengths of developments are sharp; i.e., it is possible to choose $r_1, \ldots, r_n$ such that some element of the real closure $R_n$ actually has a development of length $\omega^{(n-1)}$. We start by including in $R_1$ an infinitesimal $g$. The group $G$ generated by this $g$ will suffice for all of our $R_n$.

**Example for $R_1$.** We have seen that the elements of $R_1$ all have developments of length at most $\omega$. Let $t = \frac{1}{1-g}$. We noted earlier that this has the development $1 + g + g^2 + g^3 + \ldots$, of length $\omega$.
Example for $R_2$. Let $(q_i)_{i \in \omega}$ be an increasing sequence of rationals with limit $e$. Let $g_i = g^q_i$. Let $r = r_q$ have the $\omega$-development $\Sigma_i g_i$. Shepherdson showed that in $R_1$, if an element has a development of length $\omega$, then the terms converge to 0. It follows that the $\omega$-development of our $r$ is not assigned to any element of $R_1$—$r$ has a development of length $\omega$, but the terms are bounded away from 0. The developments of all other elements of $R_2$ are now uniquely determined. We shall consider the element $t = \frac{1}{1+r}$. We want to show that the development of $t$ has length $\omega^\omega$. We need the following.

Facts

(a) The development of $r^m$ has length $\omega^m$.

(b) Each group element appears in the support of $t^m$ for only finitely many $m$, so $t = 1 + r + r^2 + r^3 + \ldots$ makes sense.

We have argued for (b) in Lemma 4.9. The support of $t$ consists of 1 and the elements of $\text{Supp}(r^m)$, for various $m$. Assuming (a), we can see that the development of $t$ has length $\omega^\omega$. We shall prove (a) and (b) in greater generality. We turn to an inductive method for obtaining further examples, for all $n$. Assuming that $R_n$ has an element $s$ with a development of length $\omega^{\omega^{(n-1)}}$, we put into $R_{n+1}$ an element $r$ with a development of the same length, but with special features allowing us to prove that $t = \frac{1}{1+r}$ has a development of length $\omega^{\omega^n}$.

Passing from $R_n$ to $R_{n+1}$. Take an element $s$ from $R_n$ with a development of length $\omega^{\omega^{(n-1)}}$. The first step is to “scale” $s$ as in the lemma below.

Lemma 5.1. Suppose $s$ is an element of $R_n$ with development of length $\omega^{\omega^{(n-1)}}$. There is a development of length $\omega^{\omega^{(n-1)}}$, with proper initial segments all represented by elements of $R_n$, such that the terms are bounded away from 0.

Proof. The support of $s$ corresponds to a sequence of rationals of order type $\omega^{\omega^{(n-1)}}$. We may suppose that these are all positive. Let $x_0$ be the first, let $x_1$ be the limit of the initial segment of type $\omega^{\omega^{(n-2)}}$, and in general, let $x_i$ be the limit of the initial segment of type $\omega^{\omega^{(n-2)}}$. We may suppose that the sequence $(x_i)_{i \in \omega}$ is unbounded in $\mathbb{R}$.

Let $(q_i)_{i \in \omega}$ be an increasing sequence of rationals with a bound. Choose $(\epsilon_i)_{i \in \omega}$ such that $x_i + \epsilon_i$ is rational, and $\epsilon_i < \frac{q_i}{2^i}, \frac{(q_{i+1} - q_i)}{2^i}$. For $x < x_1$ in our unbounded sequence, we replace $x$ by $x' = q_1 + x - (x_1 + \epsilon_1)$. We drop the initial segment and keep just the positive terms. What remains has type $\omega^{\omega^{(n-2)}}$, and the limit is $q_1 - \epsilon_1$. We can show that it is the support of some element of $R_n$. Let $s_1$ be the result of truncating the development of our given $s$ to the first $\omega^{\omega^{(n-2)}}$ terms. This is in $R_n$. We multiply by $g^{q_1 + (x_1 + \epsilon_1)}$ and then drop the initial segment consisting of $g^{x'}$, where $x'$ is not positive. This is in $R_n$. For $x_1 < x < x_2$, we replace $x$ by $x' = q_2 + x - (x_2 + \epsilon_2)$. We drop any terms less than $q_1$ from the beginning of the sequence. What remains still has type $\omega^{\omega^{(n-2)}}$, and the limit is $q_2 - \epsilon_2$. Again, it is the support of an element of $R_n$. We continue in this way. The new sequence of rationals has type $\omega^{\omega^{(n-1)}}$, and it is bounded. For each initial segment $I$, there is an element of $R_n$ with support consisting of $g^{x'}$ for $x' \in I$. 

\[\square\]
We put into $R_{n+1}$ the element $r$ with the development we have given. The next step is to show that for all $m$, the element $r^m$ has a development of length $\omega^{(n-1)-m}$. The next lemma does this. It also shows that, in the case of $R_2$ above, our element $r^m$ has a development of length $\omega^m$.

**Lemma 5.2.** Suppose $\beta$ has the form $\omega^\gamma$, or $\omega^{\omega^\gamma}$, or $\omega^{\omega^{(k)}}$, for some $k$. Let $\alpha = \beta^m$, for some $m$. Suppose $s$ has a development of length $\alpha$, and $t$ has a development of length $\beta$. Suppose $\text{Supp}(t)$ is bounded away from 0. Then $s \cdot t$ has a development of length $\alpha \cdot \beta = \beta^{(m+1)}$.

**Proof.** We are assuming that the value group consists of $g^\gamma$, where $g$ is a fixed positive infinitesimal, and $q$ is rational. Let $q^*$ be the least upper bound of the $q$ such that $g^\gamma$ is in $\text{Supp}(t)$. Then $mq^*$ is the least upper bound of the $q$ such that $g^\gamma$ is in $\text{Supp}(s)$. We ignore those elements $g^\gamma$ of $\text{Supp}(t)$ that are limits, where this means that there is an increasing sequence $q_i$ such that $q = \sup q_i$ and $g^{q_i}$ is also in $\text{Supp}(t)$. Ignoring the limits does not change the order type. It means that for each $h = g^{q_i}$ in $\text{Supp}(t)$, there is some $\epsilon > 0$ such that for all $h' < h$ in $\text{Supp}(t)$, $h' = g^{q_i}$ for some $q - q_i > \epsilon$.

Take $h' \in \text{Supp}(t)$, let $h$ be the successor in $\text{Supp}(t)$, and let $\epsilon$ be as above. There is some $k \in \text{Supp}(s)$, of the form $g^\ell$, where $mq^* - \ell < \epsilon$. Then for $k' > k$ in $\text{Supp}(s)$, $k'$ has the form $g^{\ell'}$, where $mq^* > \ell' > \ell$, and $\ell' - \ell < \epsilon$. Since $\ell' - \ell < q - q^*$, we have $\ell' + q' < \ell + q$, so $h'k' < hk$. Our assumption on $\beta$ means that terminal segments of $\beta^m$ have the same order type as $\beta^m$. The set of $k' > k$ in $\text{Supp}(s)$ and the set of products $k'h'$, for our fixed $h'$, both have order type $\alpha = \beta^m$. Then the order type of the full set of products is equal to $\alpha \cdot \beta$.

We have shown in Lemma 4.9 that if $r$ is infinitesimal, then $t = r^{1+1}$ can be expressed as the sum $1 + r + r^2 + \ldots$, and $\text{Supp}(t)$ consists of 1 and the finite products of elements of $\text{Supp}(r)$.

The following example shows that the sequence of elements forming a transcendence basis (finite or of length $\omega$) is important to the MR-procedure.

**Example.** Let $G$ be the divisible ordered abelian group generated by a single element $g$, which we think of as infinite—$G = \{g^q : q \in \mathbb{Q}\}$. Let $k$ be the field of real algebraic numbers. Let $\alpha$ be an arbitrary countable ordinal. Choose a decreasing sequence of negative rationals $g_\beta$, for $\beta < \alpha$. Let $g_\beta = g^{q_\beta}$. Then for all $\beta < \alpha$, $g_\beta$ is infinitesimal, and the sequence $(g_\beta)_{\beta < \alpha}$ is strictly increasing under the reverse ordering—becoming more infinitesimal. Let $r_0 = 1$, and for each $\beta < \alpha$, let $r_\beta = \sum_{\gamma < \beta} g_\gamma$. Let $R$ be the real closure of $\{r_\beta : \beta < \alpha\}$. This is a truncation closed subfield of $k(\langle G\rangle)$. For $\beta \geq 1$, $r_\beta$ has a development of length $\beta$.

We have completed our results on lengths of developments. In the remaining sections, we turn to effectiveness.

6. **Complexity of the value group**

Let $R$ be a countable real closed field. By Proposition 2.1, there is a subgroup of $(R^+, \cdot)$ isomorphic to the natural group of $R$. We refer to any such group as a value group for $R$. Since $R$ is real closed, $G$ is a divisible ordered abelian group.
In this section, we show that for any countable real closed field $R$, there is a value group for $R$ that is $\Delta_0^0(R)$. We also show that this result is optimal by giving a computable real closed field in which every value group codes the halting set.

**Theorem 6.1.** For any countable real closed field $R$, there is a value group that is $\Delta_0^0(R)$.

**Proof.** By Proposition 2.1, there is a subgroup $G$ of $(R^+, \cdot)$, with a unique element from each $\sim$-class, except $\{0\}$. We must produce $G$ using $\Delta_0^0(R)$. Let $(r_n)_{n \in \omega}$ be a list of the elements of $R^+$, with $r_0 = 1$. We can make the list computable in $R$.

We form an increasing sequence $(G_n)_{n \in \omega}$ of divisible subgroups of $(R^+, \cdot)$ such that $G_n$ has representatives for $w(r_k)$ for all $k \leq n$. Let $G_0 = \{1\}$. Given $G_n$, we see if $w(r_{n+1})$ has a representative in $G_n$. If so, then $G_{n+1} = G_n$. If not, then $G_{n+1}$ is the group generated by the elements of $G_n$ and $r_{n+1}^q$ for $q \in \mathbb{Q}$. We let $G = \bigcup_n G_n$. Clearly, $G$ is a divisible abelian group. If $R'$ is the real closure of $G$, then $G$ is clearly a value group for $R'$. It is clear from the construction that all $\sim$-classes in $R$ are represented, so $G$ is also a value group for $R$. We note that $\Delta_0^0(R)$ can determine whether $r_1 \in G_1$, and knowing whether $r_k \in G_k$ for all $k < n$, $\Delta_0^0(R)$ can determine whether $r_n \in G_n$. We just ask whether there exist $g \in G_{n-1}$ and integers $k, m$ such that $kg > r_n$ and $mr_n > g$.

Here is the result showing that Theorem 6.1 is optimal.

**Theorem 6.2.** There is a computable real closed field $R$ such that for any value group $G$ for $R$, $K \leq_T G$.

**Proof.** Our computable real closed field $R$ will have universe $\omega$, which we think of as a set of constants. Let $F$ be a computable real closed field with an infinite sequence of positive elements $(r_n)_{n \in \omega}$ such that $1 << \ldots << r_3 << r_2 << r_1$, where $x << y$ means that for all $n$, $x^n < y$. At each stage $s$, we have a finite partial $1 \rightarrow 1$ function $f_s$ from $\omega$ to $F$, and we have enumerated a finite part $d_s$ of the atomic diagram of $R$ so that $f_s$, by interpreting the constants that appear in $d_s$, makes the sentences true in $F$. Let $(p_n)_{n \in \omega}$ be the standard sequence of primes.

At each stage $s$, we map the elements of $\omega$ that are powers of $p_n$ to $y \in F$ such that either $y \sim 1$ (and $y \neq 1$) or $y \sim r_q^s$ for some $q \in \mathbb{Q}$. At stage 0, we let $f_0$ map 1 (in $\omega$) to 1 (the multiplicative unit in $F$). At stage $s+1$, we have determined $f_s$ and we have $R_s = dom(f_s)$. Suppose $e$ enters $K$ at stage $s+1$. Let $S$ be the set of $x \in R_s$ such that $F \models f_s(x) \leq r_e \vee \bigvee_{q \in \mathbb{Q}} f_s(x) \sim r_q^s$. We define $f_{s+1}$, preserving $d_s$, so that for all $x \in S$, if $f_{s+1}(x) \neq 0$, then $f_{s+1}(x) \sim 1$, so $f_{s+1}(x)$ cannot be a group element. We extend the range and domain of $f_{s+1}$ such that for $l \leq s$, $p_l^s$ is mapped to something of $\sim$-class 1, and vowing that for $l \geq s+1$, elements of the form $p_l^s$ not already mapped to the $\sim$-class 1 will be mapped to the $\sim$-class of $(r_e)^q$ for some $q$.

Let $R$ be the real closed field with atomic diagram $D(R) = \cup_{s \in \omega} d_s$. By construction, $R$ is computable. We have $R \cong F$ via a function $f$ that is the limit of the $f_s$. Let $G$ be a value group for $R$. We can decide whether $e \in K$ using $G$. Since $R \cong F$, there is some element $x_e \in R$ such that $f(x_e) = r_e$. By construction, $x_e$ must have the form $p_{l_e}^e$ for some $l_e$. Moreover, for any $q$, any $y \in R$ $y \in R$ in the $\sim$-class of $(x_e)^q$, has the form $p_l^e$ for some $l$. We search for (and find) an element $y$ in $G$ of the form $p_l^e$ for some $l_e$. 


Claim: \( e \in K \) iff \( e \in K_l \); i.e., \( e \) has entered \( K \) by stage \( l \).

Proof of Claim. By construction, if \( y \) has the form \((p_n)^l\), then either \( y \sim 1 \) or \( y \sim x_q^q \) for some \( q \in \mathbb{Q} \). Since \( y \in G \), we cannot have \( y \sim 1 \). Therefore, we must have \( y \sim (x_q)^q \) for some \( q \). If \( e \in K \), then \( y \) has the form \((p_n)^l\) for some \( l \). Moreover, the construction guarantees that \( e \in K_l \).

We can also produce a computable real closed field with no c.e. value group. The construction is straightforward, and we omit it. In the next section, we turn to the complexity of the residue field.

7. Complexity of the residue field

Recall that for a real closed field \( R \), the residue field is the quotient of the ring of finite and infinitesimal elements by the ideal \( M \) of infinitesimals. By Proposition 2.2, there is a subfield of \( R \) isomorphic to the residue field—the idea is to take a maximal archimedean subfield. We refer to any such \( k \) as a residue field.

Theorem 7.1. For a countable real closed field \( R \), there is a residue field that is \( \Pi^0_2(R) \).

Proof. Recall that all residue fields for \( R \) are isomorphic. We suppose that the universe of \( R \) is \( \omega \). The case where \( R \) is archimedean is trivial, so we suppose that \( R \) is not archimedean. There is another trivial case, where the transcendence degree of the residue field is finite. If \( \tau \) is a transcendence basis, then the real closure of \( \tau \) is \( \Delta^0_2(R) \), even c.e. in \( R \). Now, suppose the transcendence degree is infinite. As a first step, we locate a transcendence basis for a maximal archimedean subfield of \( R \), using \( \Delta^0_2(R) \). Let \( r_0 \) be the first element (in the usual ordering of \( \omega \)) such that the real closure of \( r_0 \) is archimedean. Given \( r_0, \ldots, r_n \), with real closure \( k_n \), let \( r_{n+1} \) be the first element not in \( k_n \), such that the real closure of \( k_n(r_{n+1}) \) is archimedean, and let \( k_{n+1} \) be the real closure of \( k_n(r_{n+1}) \). The statement that \( r_{n+1} \) is not in \( k_n \) is \( \Pi^1_1(R) \), and the statement that the real closure of \( k_n(r_{n+1}) \) is archimedean is \( \Pi^0_2(R) \). Hence, \( \Delta^0_2(R) \) can calculate \( r_{n+1} \). Clearly, \( k = \cup_n k_n \) is a residue field, and the sequence \((r_n)_{n \in \omega}\) is a transcendence basis for \( k \). The field \( k \) is \( \Delta^0_2(R) \). To see this, note that \( n \in k \) iff \( n \in k_n \). To decide whether \( n \in k \), we first find \( r_0, \ldots, r_n \), using \( \Delta^0_2(R) \), and then see if \( n \) is in the real closure of this tuple, which we can do using \( \Delta^0_2(R) \).

We build a \( \Pi^0_2(R) \) residue field \( k' \) for \( R \) by using \( \Delta^0_2(R) \) to (1) build a new sequence \((r'_n)_{n \in \omega}\), which will be the basis for our residue field \( k' \), and (2) enumerate the complement of \( k' \). Since the sequence \((r_n)_{n \in \omega}\) is \( \Delta^0_2(R) \), we have a uniform \( \Delta^0_2(R) \) procedure for guessing initial segments of the \( r_n \)'s. Say the stage \( s \) guess is \( \tau_s \). Since \( \Delta^0_2(R) \) can check whether a finite set is algebraically independent, we may suppose that \( \tau_s \) is algebraically independent. For each \( s \), the tuple \( \tau_{s+1} \) has the form \( \tau_t, r \), where \( \tau_t \) is an initial segment of \( \tau_s \) that has looked correct in our \( \Delta^0_2(R) \)-approximation since stage \( t \). To account for mistakes caused by incorrect guesses in our \( \Delta^0_2(R) \)-approximation, we may need to adjust the elements of \( \tau_{s+1} \) to form \( \tau'_{s+1} \). In particular, we may need to replace \( r \) with some \( r' \) that fills the same rational cut. We let \( k'_{s+1} \) be the real closure of \( \tau'_{s+1} \) and \( k' = \cup_{s \in \omega} k'_s \).
For any infinitesimal element $\epsilon$, the elements $r$ and $r + \epsilon$ fill the same rational cut in $R$. Given one infinitesimal element $\epsilon$, the collection of $me$ for all $m > 0$ consists of all distinct infinitesimals. Thus, we can obtain many different potential choices for $r'$ by taking $r + me$ for some $m$. (We could do this replacement using any infinite set of distinct infinitesimals.) This is the idea behind the construction that follows.

At stage 0, we let $\mathcal{R}_0 = \emptyset$, and we let $k_0'$ be the real closure of $\emptyset$ in $R$. Suppose at stage $s$, we have determined $\mathcal{R}_s$, $\mathcal{R}_s'$ and $k_s'$, where for each $r \in \mathcal{R}_s$, the corresponding element $r'$ of $\mathcal{R}_s'$ has the form $r + me$ for some $m$. We have enumerated finitely many elements (none of which are in the real closure of $\mathcal{R}_s'$) into the complement of $k'$.

At stage $s + 1$, we first check whether $\mathcal{R}_s$ still appears to be an initial segment of the $\Delta^0_3(R)$ sequence of $r_n$'s. If so, then we guess the next element $r$ in this sequence, and we let $r'_{s+1} = r'_s, r$, checking that this tuple is algebraically independent and that none of the finitely many elements in the complement of $k'$ are in the real closure of this tuple using $\Delta^0_3(R)$.

Suppose there is some first element of $\mathcal{R}_s$ that now seems incorrect. We take the greatest $t < s + 1$ such that $\mathcal{R}_t$ has appeared correct since stage $t$, and we guess at the next $r$ using our $\Delta^0_3(R)$-approximation. We consider $\mathcal{R}_{s+1} = \mathcal{R}_t, r$, again checking that this is algebraically independent. If no elements of the real closure of $\mathcal{R}_t, r$ have been enumerated into the complement of $k'$, then we let $\mathcal{R}_{s+1} = \mathcal{R}_t', r$; we can check this using $\Delta^0_3(R)$. Otherwise, we replace $r$ by $r' = r + me$, where $m$ is chosen so that the real closure of $\mathcal{R}_t, r'$ does not include any of the forbidden elements. Note that $r + me$ and $r + ne$ are not interalgebraic over $k'_t$, and each $r + me$ is interalgebraic over $k'_t$ with at most one element enumerated in the complement. Therefore, such an $r'$ exists. We may suppose that none are in $k'_t$. We let $k'_{s+1}$ be the real closure of $\mathcal{R}'_{s+1}$ and we vow not to enumerate into the complement of $k'$ any elements of $k'_{s+1}$ for the time being. We enumerate into the complement of $k'$ all $x \leq s + 1$ such that, after searching $s + 1$ steps, we find evidence that $k'_{s+1}(x)$ is not archimedean.

Since the $\Delta^0_3(R)$-approximation eventually settles, each $r'_i$ has a limiting value. Consider these values. We can show inductively that that the map taking $r_i$ to $r'_i$ extends to an isomorphism of the respective real closures. Hence, $r'_0, r'_1, r'_2, \ldots$ is a transcendence basis for a maximal archimedean subfield of $R$.

\[\Box\]

We will show that Theorem 7.1 is optimal, in the following sense.

**Theorem 7.2.** There exists a computable real closed field $R$ such that no residue field $k$ of $R$ is $\Sigma^0_2$.

We consider the following set.

**Definition 12.** Let $FT$ be the set of $x \in R$ such that $x$ is finite and not infinitesimally close to any algebraic number.

Any residue field for $R$ must contain the algebraic numbers, and it cannot contain any element that is infinitesimally close, but not equal, to an algebraic number. The elements of $FT$ are the other elements of $R$ that could go into a residue field. The following lemma is not difficult to check.
Lemma 7.3. Let \( k \) be a residue field for \( R \). Then
\[
x \in FT \iff w(x) = 1 \& (\exists y \in k) \left( y \notin \mathbb{Q} \& \bigwedge_n |x - y| < \frac{1}{n} \right).
\]

Using the lemma, we can see that if \( R \) is computable and \( k \) is a \( \Sigma^0_2 \) residue field, then \( FT \) is also \( \Sigma^0_2 \). Thus, to prove Theorem 7.2, it is enough to prove the following.

Proposition 7.4. There is a computable real closed field \( R \) such that the set \( FT \) (consisting of elements that are finite and not infinitesimally close to any algebraic number) is not \( \Sigma^0_2 \).

Proof. Let \( S_e \) be the \( e \)-th \( \Sigma^0_2 \) set. We have a uniformly computable sequence of approximations \((S_{e,s})_{s \in \omega} \), where \( n \in S_e \) iff there exists \( s_0 \) such that for all \( s \geq s_0 \), \( n \in S_{e,s} \). We suppose that for any finite set \( \alpha \) consisting of pairs \((e,n)\) such that \( n \notin S_e \) for each \((e,n) \in \alpha \), there are infinitely many \( s \) such that for all \((e,n) \in \alpha \), \( n \notin S_{e,s} \). We build a computable real closed field \( R \), with universe \( \omega \). We satisfy the following requirements:

\( R_e : S_e \neq FT \).

Satisfying a single requirement \( R_e \)

To satisfy \( R_e \), we choose a witness \( w \). Let \( F \) be a nice copy of the field of real algebraic numbers. Our field \( R \) will be isomorphic to the real closure of \( F(t) \), where \( w \) corresponds to \( t \), and \( t \in FT \) iff \( w \notin S_e \). We use a language with symbols for definable functions, so that all elements of \( R \) will have names \( f(w) \). If we can enumerate the diagram using these symbols, then we can also make the universe \( \omega \).

We do not specify \( t \) in advance, but determine it by a nested sequence of intervals \((I_s)_{s \in \omega} \) with endpoints in \( F \). At stage \( s \), we choose an interval \( I_s \), and we determine a finite part \( d_s \) of the atomic diagram of our \( R \). We make sure that the sentences of \( d_s \) are valid on \( I_s \); i.e., for all \( x \in I_s \), if we assign \( w \) value \( x \), then the sentences in \( d_s \) are true in \( F \). Let \((\varphi_s)_{s \in \omega} \) be an effective enumeration of all atomic sentences involving just \( w \) (plus symbols from the language of fields and for the definable functions). We ensure that \( \varphi_i \) or \( \neg \varphi_i \) is in \( d_s \) for all \( i \leq s \).

We start with \( I_0 = (0,1) \) and \( d_s = \emptyset \). At stage \( s > 0 \), we choose \( I_s = (a,b) \) with the following features:

1. \((a,b) \subseteq I_{s-1}\).
2. \( b - a \leq \frac{1}{s} \).
3. If \( w \notin S_{e,s} \), then for any \( x \) among the first \( s \) algebraic numbers, either \( x < a \) or \( x > b \), and if \( w \in S_{e,s} \), then \( a \) is the left boundary of \( I_{s-1} \).
4. The sentences in \( d_s \) are all valid on \( I_s \).

Starting with \( I_{s-1} \), we first change the right endpoint to make 1 hold. In the case where \( w \notin S_{e,s} \), we reduce the interval further so that the first \( x \) algebraic numbers lie outside the closure. Now, we want to make \( \varphi_s \) or \( \neg \varphi_s \) valid. If \( w \in S_{e,s} \), then we want to make \( \varphi_s \) or \( \neg \varphi_s \) valid, while preserving the left endpoint. The following lemma, an easy consequence of \( \alpha \)-minimality, says that we can do this.

Lemma 7.5. For any open interval \( J \) and any sentence \( \varphi \) involving just \( w \) and symbols for functions definable from \( w \), there is an open interval \( J \subseteq I \) on which \( \varphi \) or \( \neg \varphi \) is valid. Moreover, we may choose \( J \) to have the same left boundary as \( I \).
If \( w \notin S_c \), then all algebraic numbers are eventually excluded from the closure of the intervals \( I_s \). Therefore, \( w \in FT \). If \( w \in S_c \), then there is an algebraic number \( a \) such that for all sufficiently large \( s \), the left endpoint of \( I_s \) is \( a \). Then \( w \) is infinitesimally close to \( a \), so it is not in \( FT \).

**Satisfying Requirements \( R_0 \) and \( R_1 \)**

For \( R_0 \), we have a single witness \( w_0 \), and we act on \( R_0 \) as described above. For \( R_1 \), we have one witness \( w_1^0 \) for the case where \( w_0 \notin S_0 \), and further possible witnesses \( w_1^{1,k} \), for the case where \( w_0 \in S_0 \). At stage \( s \), we have either two or three “live” witnesses. We determine \( w_0 \) and \( w_1^0 \) at stage 0. If \( w_0 \notin S_{0,s} \) and \( w_0 \in S_{0,s+1} \), then at stage \( s + 1 \), we determine a new witness \( w_1^{1,k} \). The witness \( w_1^0 \) is still live since we return to it later. If \( w_0 \in S_{0,s} \) and \( w_0 \notin S_{0,s+1} \), then at stage \( s + 1 \), we abandon the stage \( s \) witness \( w_1^{1,k} \). It would be enough to make it definable from \( w_0 \) and \( w_1^0 \), but in fact, we may choose it to be an algebraic number. It is no longer live.

Our language includes symbols for definable functions, and at various stages, we specify which constants \( w_1^{1,k} \) we wish to use as witnesses. It may be that we will use only finitely many. At each stage \( s \), we have determined a finite part \( d_s \) of the diagram of \( R \), and we have a cell \( C_s \) on which what we have said in \( d_s \) about the live witnesses is valid. For an abandoned witness \( w_1^{1,k} \), the statements in \( d_s \) will include a sentence saying that \( w_1^{1,k} \) is equal to a particular algebraic number. The cells \( C_s \) will have a certain form.

**Definition 13.** A special \( n \)-cell has the form

\[
C = \{ (x_1, \ldots, x_n) \mid a_1 < x_1 < b \& a_2 < x_2 < f_2(x_1) \& \ldots \& a_n < x_n < f_n(x_1, \ldots, x_{n-1}) \}.
\]

The point \( (a_1, \ldots, a_n) \) is called the left boundary of \( C \). The cell is ordered if it has the further property that

\[
x_1 - a_1 > x_2 - a_2 > \ldots > x_n - a_n.
\]

We use special ordered 2-cells and 3-cells, always definable in \( F \). If at stage \( s \), there are just two live witnesses, \( w_0 \) and \( w_1^0 \), then both are active, and \( C_s \) will be a special ordered 2-cell \( \{ (x, y) : a < x < b \& c < y < f(x) \} \). Any assignment of \( (w_0, w_1^0) \) to \( (x, y) \in C_s \) makes the sentences in \( d_s \) true. The fact that we have only these live witnesses means that our approximation guesses that \( w_0 \notin S \) so \( w_0 \) will move away from \( a \), and then \( w_0^0 \) is free to either approach \( c \) or move away from it.

If we have three live witnesses \( w_0, w_1^0, \) and \( w_1^{1,k} \), then \( w_1^0 \) is inactive. The cell \( C_s \) has the form \( \{ (z, x, y) : a < z < b \& c < x < f(z) \& d < y < g(z, x) \} \), where any assignment of \( (w_1^{1,k}, w_0, w_1^0) \) to \( (z, x, y) \in C_s \) makes the sentences in \( d_s \) true. The fact that \( w_1^{1,k} \) is active means that our approximation guesses that \( w_0 \) approaches \( c \).

Since the cell is ordered, \( w_1^0 \) is forced to approach \( d \), while \( w_1^{1,k} \) is free to either approach or move away from \( a \).

To start off, \( C_0 \) is an ordered 2-cell, and we suppose that \( w_0 \notin S_{0,0} \). In general, if \( w_0 \notin S_{0,s} \), our intention is to move \( w_0 \) away from the first \( s \) algebraic numbers. If \( w_1^0 \in S_{1,s} \), then we move \( w_1^0 \) toward the left endpoint of its interval, and otherwise, we move it away from the first \( s \) algebraic numbers. If \( w_0 \in S_{0,s} \), then at stage \( s \), we create a new witness \( w_1^{1,k} \), and we pass to an ordered 3-cell \( C_s \), with first variable \( z \) corresponding to \( w_1^{1,k} \). At stage \( s \), we decide the first \( s \) atomic sentences in the constants used so far.
Lemma 7.6. Let \( n \) be a special \( n \)-cell. For any formula \( \varphi(x_1, \ldots, x_n) \), there is a special \( n \)-cell \( C' \subseteq C \) on which \( \varphi \) is valid or \( \neg \varphi \) is valid. Moreover, we may take \( C' \) to have the same left boundary as \( C \). In this case, if \( C \) is ordered, then \( C' \) is ordered.

The next lemma lets us start with an ordered special \( n \)-cell and insert a new variable \( y \) into the cell. Note that the new cell preserves the left boundary of the original cell.

Lemma 7.7. Let \( C \) be the ordered special \( n \)-cell in variables \( (x_1, \ldots, x_n) \). There is an ordered special \( (n + 1) \)-cell \( C' \) in variables \( (x_1, \ldots, x_k, y, x_{k+1}, \ldots, x_n) \) such that the left boundary of \( x_i \) in \( C \) is preserved in \( C' \). We may also insert the new variable \( y \) before \( x_1 \) or after \( x_n \). Then, all formulas \( \varphi(x_1, \ldots, x_n) \) that are valid on \( C \) are valid on \( C' \).

The following lemma says that certain variables can move away from the left boundary of their interval while other variables can move toward the left boundary of their interval.

Lemma 7.8. Let \( C \) be a special ordered \( n \)-cell with left boundary \( (a_1, \ldots, a_n) \). For any \( 1 \leq k \leq n \), there is a special ordered \( n \)-cell \( C' \subseteq C \) with left boundary \( (b_1, \ldots, b_k, a_{k+1}, \ldots, a_n) \), where \( b_i > a_i \) for \( 1 \leq i \leq k \).

Proof. Since \( C \) is an (open) ordered cell of the form
\[
(\{x_1, \ldots, x_n\} \mid a_1 < x_1 < b & a_2 < x_2 < f_2(x_1) & \ldots & a_n < x_n < f_n(x_1, \ldots, x_{n-1}))
\]
there is an open box \( B \subseteq C \) with left boundary \( (b_1, \ldots, b_n) \) and right boundary \( (c_1, \ldots, c_n) \). We can take \( b_i > a_i \) for all \( 1 \leq i \leq k \).

For \( 2 \leq i \leq k \), let \( g_i(x) = \min(x + b_k - b_{k-1}, c_k) \), and let
\[
C' = \{(x_1, \ldots, x_n) \mid b_1 < x_1 < c_1 & b_2 < x_2 < g_2(x_1) & \ldots & b_k < x_k < g_k(x_{k-1}) & a_{k+1} < x_{k+1} \leq \min(f_{k+1}(x_1, \ldots, x_k), x_k + a_{k+1} - b_k) & a_{k+2} < x_{n} < f_{k+2}(x_1, \ldots, x_{k+1}) & \ldots & a_n < x_n < f_n(x_1, \ldots, x_{n-1})\}.
\]

The set \( C' \) is an ordered special \( n \)-cell contained in \( C \) that satisfies the inequalities
- \( x_i - b_i > x_{i+1} - b_{i+1} \) for all \( 1 \leq i \leq k - 1 \),
- \( x_k - b_k > x_{k+1} - a_k \), and
- \( x_i - a_i > x_{i+1} - a_{i+1} \) for all \( k + 2 \leq i \leq n \).

Moreover, \( C' \) has left boundary \( (b_1, \ldots, b_k, a_{k+1}, \ldots, a_n) \).

\[\square\]

Satisfying \( R_e \) for all \( e \)

We organize our strategies on the full binary tree \( 2^{\omega} \). For any \( \sigma \in 2^{\omega} \), node \( \sigma \) is associated with requirement \( R_e \) for \( e = |\sigma| \), and node \( \sigma \) may have a witness \( w_\sigma^e \) associated with it. We will see that the superscript \( \sigma \) for a witness \( w_\sigma^e \) will help us describe the ordering on the various witnesses. For any string \( \sigma \in 2^{\omega} \) and \( i \leq |\sigma| \), we let \( \sigma \upharpoonright i \) denote the substring of \( \sigma \) of length \( i \).
Definition 14 (Active nodes and \( \sigma_s \)).

(1) A node \( \sigma \) is active at stage \( s \) if \( |\sigma| \leq s \) and for each \( i < |\sigma| \), there is a witness \( w^\sigma_{|\sigma|} \) associated with strategy \( R_i \) such that \( \sigma(i) = 0 \) if and only if \( w^\sigma_{|\sigma|} \notin S_{i,s} \).

(2) Let \( \sigma_s \) be the active node of length \( s \) on the tree at stage \( s \).

The node \( \sigma_s \) gives the stage \( s \) approximation to whether the requirement \( R_i \) is trying to build \( w^\sigma_{|\sigma|} \) infinitely close to an algebraic number (the guess if \( \sigma_s(i) = 1 \)) or whether \( R_i \) is being put working to put \( w^\sigma_{|\sigma|} \) into \( FT \) (the guess if \( \sigma_s(i) = 0 \)) for any \( i < |\sigma_s| \). At another stage \( t \), the active node \( \sigma_t \) may have different guesses about what strategy \( R_i \) is trying to enact for \( i < |\sigma_t| \).

If a node \( \sigma \) is active at stage \( s \) and there is no current (or "live") witness associated with \( \sigma \), we create a new witness \( w^\sigma_S \). This witness, as well as witnesses associated with predecessors of \( \sigma \), are live and active. Witnesses associated with nodes to the right of \( \sigma \) are not live. Witnesses associated with nodes to the left of \( \sigma \) are live but inactive. An inactive witness becomes active again when the active witnesses to the right of it are abandoned.

We describe the construction at stage \( s+1 \). We are given \( \sigma_s \) the node of length \( s \) that is active at stage \( s \), and let \( \sigma_{s+1} \) be the active node of length \( s+1 \) at stage \( s+1 \). Let \( \tau \) be the maximal initial substring such that \( \sigma_s, \sigma_{s+1} \supseteq \tau \), and let \( e = |\tau| \). In other words, we guess that all the strategies associated with \( \tau \) continue to act in the same manner at stages \( s \) and \( s+1 \). We have an ordered special cell \( C_s \) that locates all live witnesses. We refine this special cell in several steps to form \( C_{s+1} \).

We use the following ordering on \( 2^\omega \).

**Definition 15.** Let \( <_\lambda \) denote the ordering of \( 2^\omega \) that results from squashing the binary tree.

Here is the ordering of all \( \sigma \in \omega \) with \( |\sigma| \leq 2 \) where \( \lambda \) is the empty string.

\[
00 <_\lambda 0 <_\lambda 01 <_\lambda \lambda <_\lambda 10 <_\lambda 1 <_\lambda 11
\]

**Step 1:** Update witnesses.

First, suppose \( \sigma_s \supseteq \tau 1 \) and, hence, \( \sigma_{s+1} \supseteq \tau 0 \). We abandon the witnesses associated with nodes to the right of or above \( \tau \). Assume we have ordered the live witnesses as \( w_1 > \ldots > w_n \) in our given special cell

\[
C_s = \{(x_1, \ldots, x_n) : a_1 < x_1 < b \land a_2 < x_2 < f_2(x_1) \land \ldots \land a_n < x_n < f_n(x_1, \ldots, x_{n-1})\}.
\]

Moreover, if witnesses \( w_i \) and \( w_j \) are associated with nodes \( \gamma_i \) and \( \gamma_j \), then \( w_i - a_i < w_j - a_j \) if and only if \( \gamma_i <_\lambda \gamma_j \).

Because of this ordering, the witnesses associated with nodes to the right or above \( \tau \) are \( w_1, w_2, \ldots, w_k \) for some \( k \). We recursively define \( w_i \) for \( 1 \leq i \leq k \) to be some point in the appropriate interval. First, we let \( w_1 \in (a_1, b) \), then \( w_2 \in (a_2, f_2(w_1)) \), etc. Finally, we let \( w_k \in (a_k, f_k(w_1, \ldots, w_{k-1})) \). We could take the midpoint of the interval at each step. Hence, all the witnesses associated with nodes to the right of or above \( \tau \) have been abandoned and assigned algebraic values, and we have a new special cell for witnesses \( w_{k+1}, \ldots, w_n \). Second, suppose that \( \sigma_s \supseteq \tau 0 \) and, hence, \( \sigma_{s+1} \supseteq \tau 1 \). Since our approximation to the true path moves to the right, we do not need to make any witnesses definable. In either case, we must ensure that all witnesses associated with \( \gamma \subseteq \sigma_s \) remain live. We also must ensure that each \( \gamma \subset \sigma_{s+1} \) properly extending \( \tau \) has a witness. We show this inductively.
Suppose we have \( \gamma \), an initial segment of \( \sigma_{s+1} \) extending \( \tau \). We assume we have a special cell of live witnesses associated with nodes above and to the left of \( \gamma \). As before, we assume we have the live witnesses \( (w_1, \ldots, w_n) \) corresponding to elements in the ordered special \( n \)-cell with left boundary \( (a_1, \ldots, a_n) \). In other words, \( w_1 - a_1 > \ldots > w_n - a_n \). Again, if witnesses \( w_i \) and \( w_j \) are associated with nodes \( \gamma_i \) and \( \gamma_j \), then \( w_i - a_i < w_j - a_j \) if and only if \( \gamma_i \leq \gamma_j \). Some of these witnesses are active. Among the active witnesses, some are supposed to be moving away from their left boundary, while others are supposed to be moving towards their left boundary, according to our current guesses. By construction, if \( w_i \) is moving away from \( a_i \) and \( w_j \) is moving toward \( a_j \), then \( i < j \) and the formula \( x_i - a_i > x_j - a_j \) is valid on the current cell. Let \( w_j \) denote the witness corresponding to the node \( \gamma \).

By definition of \( \sigma_{s+1} \), the node \( \gamma^+ = \sigma_{s+1} \upharpoonright (|\gamma| + 1) \) is \( \gamma 0 \) if \( w_j \notin S_{|\gamma|+1} \) and \( \gamma 1 \) otherwise. We must ensure that \( \gamma^+ \) has an active witness that can satisfy its associated requirement. Specifically, \( \gamma^+ \) needs a witness \( w \) so that \( w - a \) is between the \( w_i - a_i \) that are moving towards \( a_i \) and the \( w_i - a_i \) moving away, so that it can move either toward or away its left boundary. If \( \gamma^+ \) has a live witness that is currently inactive, then we reactivate it. By construction, this witness already has the desired properties.

Suppose \( \gamma^+ \) does not have a live witness. We construct a live witness \( w \) for \( \gamma^+ \) and an associated special \( (n+1) \)-cell that preserves what has been included in the atomic diagram so far and allows our strategy for \( R_l \) for \( l = |\gamma^+| \) to succeed. By assumption, \( \gamma \) has a live and active witness \( w_j \) in our current collection of witnesses \((w_1, \ldots, w_n)\). By induction, all live witnesses \( w_i \) for \( i < j \) are moving away from \( a_i \), and all witnesses \( w_i \) for \( i > j \) are moving towards \( a_i \) at stage \( s + 1 \). At stage \( s + 1 \), the strategy for \( R_{l-1} \) calls for witness \( w_j \) to move away from \( a_j \) if and only if \( \gamma^+(l-1) = 0 \).

If \( \gamma^+(l-1) = 0 \), we take the ordered special cell for \( (w_1, \ldots, w_n) \) (with variables \( x_1, \ldots, x_n \)) and use Lemma 7.7 to build a new ordered special cell with a variable \( y \), corresponding to \( w \), between \( x_{j-1} \) and \( x_j \). Since all formulas valid on the original ordered special cell are valid on the new ordered special cell, all formulas already included in the atomic diagram are preserved. Moreover, the ordering of the \( (n+1) \)-cell continues to respect the linear ordering \( <_S \). Since all witnesses \( w_i \) for \( i < j \) are moving towards \( a_i \) and all witnesses \( w_i \) for \( i \geq j \) are moving away from \( a_i \), we may move \( w \) according to the strategy for \( R_l \) in the next step.

If \( \gamma^+(l-1) = 1 \), we take the ordered special cell for witnesses \( (w_1, \ldots, w_n) \) (with variables \( x_1, \ldots, x_n \)) and use Lemma 7.7 to build a new special cell with a variable \( y \) (corresponding to a new witness \( w \)) between \( x_{j+1} \) and \( x_j \). As before, everything already included in the atomic diagram is preserved. Since all witnesses \( w_i \) for \( i \leq j \) are moving towards \( a_i \) and all witnesses \( w_i \) for \( i > j \) are moving away from \( a_i \), we may move \( w \) according to the strategy for \( R_l \) in the next step. Again, the ordering of the special \( (n+1) \)-cell continues to respect the linear ordering \( <_S \).

We may repeat this process each initial substring of \( \sigma_{s+1} \) has a live and active witness, and we have an ordered special cell \( C \) of live witnesses for nodes above and to the left of \( \sigma_{s+1} \). We have preserved the validity of all formulas already included in the atomic diagram.

**Step 2:** Avoiding algebraic numbers in open intervals.
In this step, we make sure that each live witness \( w_i \) is located in an interval that contains none of the first \( s + 1 \) algebraic numbers. We have a special cell of live witnesses associated with nodes above and to the left of \( \sigma_{s+1} \). We again assume we have the live witnesses \( (w_1, \ldots, w_n) \), and the ordered special cell has the form:

\[
\{(x_1, \ldots, x_n) : a_1 < x_1 < b & a_2 < x_2 < f_2(x_1) & \ldots & a_n < x_n < f_n(x_1, \ldots, x_{n-1})\}.
\]

For each \( i \), choose \( b_i > a_i \) such that none of the first \( s+1 \) algebraic numbers are in the interval \((a_i, b_i)\). Let \( g_i(x_1, \ldots, x_{i-1}) = \min(b_i, f_i(x_1, \ldots, x_{i-1})) \) for \( 1 \leq i \leq n \).

The refined ordered special cell

\[
\{(x_1, \ldots, x_n) : a_1 < x_1 < b' & a_2 < x_2 < g_2(x_1) & \ldots & a_n < x_n < g_n(x_1, \ldots, x_{n-1})\}
\]

has the desired features. Since this cell is a subcell of the original, we have preserved the validity of all formulas already included in the atomic diagram.

**Step 3:** Enacting the strategy.

Assume we have ordered the live witnesses as \( (w_1, \ldots, w_n) \) in our given ordered special cell. There is a greatest \( j \) such that \( w_j \) is associated with an active node \( \gamma \subseteq \sigma_{s+1} \) and \( \gamma(\gamma - 1) = 0 \), i.e., the strategy for \( R_{\gamma} \) is to move \( w_j \) away from \( a_j \).

By Step 1, any witness \( w_i \) for \( i < j \) is either not active or is associated with a node whose strategy is to move \( w_i \) away from \( a_i \). By Lemma 7.8, there is an ordered special cell \( C' \subseteq C \), that moves all active witnesses \( w_i \) for \( i < j \) away from \( a_i \) and preserves the left boundary of the witnesses \( w_i \) for \( i > j \). Since we simply refined the cell, we again have preserved the validity of all formulas already included in the atomic diagram. Note that Step 2 already naturally moves the active witnesses \( w_i \) for \( i > j \) towards \( a_i \) at stage \( s + 1 \) (as long as we preserve the left boundary of the associated part of the cell).

**Step 4:** Adding to the atomic diagram.

Let \( \varphi \) be the first sentence in constants used so far such that neither \( \varphi \) nor \( \neg \varphi \) are already in \( d_s \). By Lemma 7.6, there is a special subcell of the cell we have on which \( \varphi \) or \( \neg \varphi \) is valid. Moreover, the boundaries of this subcell remain unchanged. We include in the diagram \( d_{s+1} \) the sentence that is valid. Let \( C_{s+1} \) be the resulting ordered special \( n \)-cell. This ends stage \( s + 1 \).

**Verification**

We define the **true path** \( f \in 2^\omega \) to be \( f(i) = 0 \) if \( \lim_{s \to \infty} \sigma_s(i) \) equals 0 or does not exist and \( f(i) = 1 \) if \( \lim_{s \to \infty} \sigma_s(i) = 1 \). We now verify that \( R_e \) is satisfied for all \( e \). We suppose by induction that, for each \( i < e \), requirement \( R_i \) is satisfied by a witness \( w_{f\upharpoonright l}^{f\upharpoonright l} \) that does not change after some stage \( t^* \). Moreover, at each stage in the construction, we have the live witnesses \( (w_1, \ldots, w_n) \) chosen from the ordered special cell \( C_s \) with left boundary \( (a_1, \ldots, a_n) \) such that if witnesses \( w_i \) and \( w_j \) are associated with nodes \( \gamma_i \) and \( \gamma_j \), then \( w_i - a_i < w_j - a_j \) only if \( \gamma_i < s \gamma_j \).

We show \( R_e \) is satisfied by witness \( w_{f\upharpoonright e}^{f\upharpoonright e} \), which does not change after some stage. By definition of \( f \), we may take a stage \( t > e, t^* \) such that if \( \sigma(i - 1) = 1 \) for any \( 0 < i < e \), then \( \sigma_s(i - 1) = 1 \) for all \( s \geq t \). Then, after stage \( t \), any approximation to the true path \( \sigma_e \upharpoonright e \) cannot be to the left of \( f \upharpoonright e \). Let \( t' \) be the first stage \( t' \geq t \) such that \( R_e \) has a witness \( w_{f\upharpoonright e}^{f\upharpoonright e} \) associated with it. (Such a stage exists by
construction and the definition of \( f \). By the last remark, this witness remains a live witness throughout the rest of the construction (i.e., it is never defined to be a specific algebraic number). Let \( \gamma = f \upharpoonright (e - 1) \).

First, if \( f(e - 1) = 0 \), then \( f \upharpoonright e = \gamma 0 \). By construction, if the left endpoints of the intervals associated with witnesses \( w^0_e \) and \( w^1_{e-1} \) are \( a \) and \( a' \) respectively at any stage \( s > t' \), then \( w^0_e - a < w^1_{e-1} - a' \). At any stage \( s > t' \) such that all the substrings of \( \sigma 0 \) are active, witness \( w^0_e \) moves away from the left boundary \( a \) of its associated interval in Step 3 if \( w^0_e \not\in S_{e,s} \). If \( w^0_e \not\in S_e \), there are infinitely many such stages, so over the construction, \( w^0_e \) moves away from all algebraic numbers. (There are infinitely many such stages by our assumption that for any finite set \( \alpha \) consisting of pairs \( \langle e, n \rangle \) such that \( n \not\in S_e \) for each \( \langle e, n \rangle \in \alpha \), there are infinitely many \( s \) such that for all \( \langle e, n \rangle \in \alpha, n \not\in S_{e,s} \).) Hence, \( w^0_e \) is in \( FT \), demonstrating that \( S_e \)'s prediction is incorrect.

If \( w^0_e \in S_e \), there is a stage \( t'' > t' \) such that \( w^0_e \in S_{e,s} \) for all \( s > t'' \). Then, at each stage \( s > t'' \) such that all the substrings of \( \sigma 0 \) are active, the left endpoint of the interval associated with \( w^0_e \) is preserved. This endpoint is also preserved at any other stage \( s > t'' \) since no nodes to the left of \( \sigma 0 \) are ever active (and these are the only nodes that could change the left endpoint of the interval associated with \( w^0_e \)). Then, \( w^0_e \) is defined to be infinitesimally close to this fixed left endpoint, an algebraic number. Hence, \( w^0_e \not\in FT \), again showing that \( S_e \)'s prediction is wrong.

On the other hand, if \( f(e - 1) = 1 \), then \( f \upharpoonright e = \gamma 1 \). As before, if the left endpoints of the intervals associated with witnesses \( w^1_e \) and \( w^1_{e-1} \) are \( a \) and \( a' \) respectively at any stage \( s > t' \), then \( w^1_{e-1} - a' < w^1_e - a \). The argument that \( R_e \) is satisfied is symmetric.

8. Bounding the oracle needed

We want to set a bound on the oracle needed to compute an integer part for a real closed field \( R \). We obtain the following theorem.

**Theorem 8.1.** Any countable real closed field \( R \) has a \( \Delta^0_\omega(R) \) integer part.

Let \( R \) be real closed field. For simplicity, we assume that \( R \) is computable, but the argument relativizes to \( R \). We first fix a residue field \( k \) for \( R \). We showed in Proposition 7.1, that there exists a \( \Pi^0_2 \) residue field \( k \) of \( R \). Hence, \( k \) is certainly \( \Delta^0_\omega \).

Next we fix a transcendence basis \( B = \{ r_1, r_2, \ldots \} \) for \( R \). (We assume \( R \) has infinite transcendence degree as this case is the most computationally complex).

**Lemma 8.2.** Let \( R \) be a computable real closed field, and let \( k \) be a \( \Pi^0_2 \) residue field for \( R \). There is a transcendence basis \( B = \{ r_1, r_2, \ldots \} \) for \( R \) and, for all \( n \), a group \( G_n \) contained in the divisible closure of \( \{ r_1, \ldots, r_n \} \) such that if \( R_n = RC(\langle k, r_1, r_n \rangle) \), the group \( G_n \) is a value group for \( R_n \). Moreover, \( G = \cup_{n \in \omega} G_n \) is a value group for \( R \). The basis \( B \) and the groups \( G_n \) are \( \Delta^0_\omega \).

**Proof.** Let \( \{ a_i \}_{i \in \omega} \) be an effective enumeration of the elements of \( R \). Let \( r_1 \) be the first \( a_i \) such that \( a_i > n \) for all \( n \). Calculating \( r_1 \) is \( \Delta^0_\omega \), as well. Let \( G_1 \) be the divisible closure of \( \{ r_1 \} \). Given \( r_1, \ldots, r_n - 1 \), use \( \Delta^0_\omega \) to check whether there is an \( x \in RC(\langle k, r_1, \ldots, r_n \rangle) \) whose valuation is not in \( G_{n-1} \). If such an \( x \) exists, let \( r_n \) equal the first such element in the enumeration \( \{ a_i \}_{i \in \omega} \) and set \( G_n \) to be the
divisible closure of $G_{n-1}$ and \{r_n\}. Otherwise, set $G_n = G_{n-1}$, and let $r_n$ be the first $a_i$ such that $a_i$ is not in the real closure of $k(r_1, \ldots, r_{n-1})$. In the second case, $r_n$ and $G_n$ are $\Delta^0_\omega$. Thus, this construction is at worst $\Delta^0_4$. \hfill \Box

We prove Theorem 8.1 by showing the how complicated it is to carry out the MR-construction. We show the following.

**Theorem 8.3.** Given a computable real closed field $R$ and a residue field $k$, transcendence basis $B$, and groups $G_n$ as produced in Lemma 8.2, the embedding $\delta : R \rightarrow k\langle\langle G\rangle\rangle$ obtained by the MR-procedure is $\Delta^0_\omega$.

We take a transcendence basis $B = \{r_1, r_2, \ldots\}$ for $R$ and the value group $G_n$ for $R_n = RC(k(r_1, \ldots, r_n))$ as above using only $\Delta^0_4$. Since our later oracle requirements will be much stronger than $\Delta^0_4$, we will assume for simplicity that $k$ and $B$ are in fact computable. We begin by proving the theorem for $R_0$, $R_1$, and $R_2$. We will then prove the theorem in general for $R_n$.

If $R_0 = k$ and $G_0 = \{1\}$, then oracle $\Delta^0_0$ can compute the development of any element of $R_0$ by our assumption that $k$ is computable. The development of $x$ is just $x \cdot 1$, and the integer part is the unique $i \in \mathbb{Z}$ such that $i \leq x < i + 1$. Since $B$ is a transcendence basis and $r_1 \in G_1$ by construction, we set $\delta(r_1) = r_1$. The other elements of $R_1$ are roots of polynomials over $k[r_1]$. Using $\Delta^0_3$, we can find the development of $t \in R_1$, a root of the polynomial $p(x) \in k[r_1][x]$. We use $\Delta^0_3$ to determine the valuation of $t$ in $G_1$, call it $g_0$, and the $a_0 \in k$ such that $w(t - a_0g_0) > g_0$. Our development for $t$ is complete at a finite stage if the difference between $t$ and its purported development is 0. Otherwise, by our bounding results, the development for $t$ will have length $\omega$. Thus, the oracle $\Delta^0_0$ can compute the developments of $R_1$. Note that $\Delta^0_3$ can determine whether or not an element will have a development of finite length or not. Using $\Delta^0_3$, we can determine whether the terms are all negative. If so, then $|t| = t$. If not, then we find the finite initial segment $\sum_{i<k} a_i g_i$ that is negative and we add an appropriate integer $z$ to get an appropriate integer part $[t]$.

By construction of our transcendence basis and by the MR-procedure, either $r_2$ is in $G_2$ and has a development of length $1$, or $r_2$ has a development of length $\omega$. Using $\Delta^0_3$, we can determine whether $w(r_2)$ is in $G_1$. If not, $r_2 \in G_2$, and $r_2$ is its own development. In this case, if $r_2$ is infinite, then it is its own integer part, and if $r_2$ is infinitesimal, its integer part is the appropriate integer. If $w(r_2)$ is in $G_1$, the development of length $\omega$ of $r_2$ is computable in $\Delta^0_3$ in the same way the development for an element of $R_1$ is computable. As before, $\Delta^0_3$ computes the integer part of $r_2$.

We now determine how complicated it is to determine the development of $r_n$ (and its integer part) given the developments of the elements of $R_{n-1}$.

**Lemma 8.4.** If $X$ calculates the developments of $R_{n-1}$ for $n \geq 3$, then we have $\Delta^0_{\omega n - 2}(X \oplus \Delta^0_3)$ calculates the development of $r_n$, and $\Delta^0_{\omega n - 2 + 2}(X \oplus \Delta^0_3)$ calculates the integer part of $r_n$.

**Proof.** By the construction of our transcendence basis, the development of $r_n$ has length at most $\omega^{\omega n-2}$ by Theorem 1.1. For any $\delta(r_n) \mid \gamma$ with $\gamma < \omega^{\omega n-2}$, we must check that $\delta(r_n) \mid \gamma$ suffices to be the development of $r_n$, i.e., that there is no element in $R_{n-1}$ with this development.

Following the MR-procedure, we compute the development of $r_n$ as in the case for $r_2$ above. To find the $\omega$-development of $r_n$, we use $\Delta^0_3$, and we use $\Delta^0_3(X \oplus \Delta^0_3)$.
to check whether some element of $R_{n-1}$ already claims this development. To find the $\omega \cdot l$-development of $r_n$, we use $\Delta^0_l$ (and finitely much $\Delta^0(X \oplus \Delta^0)$ information). To see whether $\omega \cdot l$-development of $r_n$ is already claimed, we use $\Delta^0_l (X \oplus \Delta^0)$. In case, we are not done, we use $\Delta^0_l (X \oplus \Delta^0)$ to find the element representing the $\omega \cdot l$-development. To check whether this process ever stops with a development of length $\omega \cdot l$, we use $\Delta^0_l (X \oplus \Delta^0)$. To find the $\omega'$-development, we use $\Delta^0_{l+1} (X \oplus \Delta^0)$, and to check that this development has not already been claimed, we use $\Delta^0_{l+1} (X \oplus \Delta^0)$. To find the $\omega$'-development of $r_n$, we use $\Delta^0_{l+1} (X \oplus \Delta^0)$. We check whether this development is already claimed using $\Delta^0_{l+2} (X \oplus \Delta^0)$ and if so, we find the element representing this development, using $\Delta^0_{l+1} (X \oplus \Delta^0)$. To find the $\omega^2$-development, we use $\Delta^0_{l+2} (X \oplus \Delta^0)$, and we check whether we are done using $\Delta^0_{l+3} (X \oplus \Delta^0)$ and if not, we find the representative, we use $\Delta^0_{l+4} (X \oplus \Delta^0)$. To find the $\omega^2$-development, we use $\Delta^0_{l+3} (X \oplus \Delta^0)$. To check whether we are done, we use $\Delta^0_{l+4} (X \oplus \Delta^0)$, and we find a representative, we use $\Delta^0_{l+3} (X \oplus \Delta^0)$. Since the development of $r_n$ has length at most $\omega^{n-2}$ by Theorem 1.1, $\Delta^0_{\omega^{n-2}} (X)$ computes the development of $r_n$.

Using $\Delta^0_{\omega^{n-2}+2} (X)$, we can determine whether all the terms in the development of $r_n$ are negative. If so, then $|t| = t$. If not, then we find the finite initial segment $\sum_{i<k} a_i g_i$ that is negative and we add an appropriate integer $z$ to get an appropriate integer part $[t]$.

We now analyze the complexity of computing the developments of arbitrary elements of $R_n$ for $n \geq 2$. If $t \in R_2$ other than $r_2$, then $t$ is a root of a polynomial $p(x)$ over $R_1[R_2]$. By Theorem 1.1, the development of $t$ has length at most $\omega^4$. We calculate longer and longer initial segments of $t$. At each limit ordinal $\alpha$, we check whether we are done, or whether the initial segment is the development of some other element of $R_2$. Let $\delta(p)(x)$ be the polynomial obtained by taking $p$ and expressing the coefficients from $p$ in terms of their developments. We are done if, when we substitute the development so far for $t$ into $\delta(p)(x)$, expand, and cancel, we are left with 0. We now put an upper bound on the complexity of expanding this substitution.

**Proposition 8.5.** Suppose $X$ computes the developments of the elements of $R_{n-1}$ and $r_n$, and $X$ is also given an initial segment of the development of an element $t$ of $R_n$. Given $p(x) \in R_{n-1}[r_n][x]$, let $\delta(p)(x)$ be the polynomial obtained by substituting the developments of the coefficients of $p$ in for the coefficients themselves. Oracle $\Delta^0_2 (X)$ can find the development that results from plugging the initial segment of the development of $t$ into $\delta(p)(x)$, and oracle $\Delta^0_2 (X)$ can check whether this result is 0.

**Proof.** We begin by showing that $\Delta^0_2(X)$ is capable of adding and multiplying developments.

**Lemma 8.6.** If $X$ computes developments for $s$ and $t$, then $\Delta^0_2(X)$ computes the development of $s + t$.

**Proof.** Using $X$, we can enumerate the group elements $g_i$ and $h_j$, putting them in order and noting (enumerating) equalities. Let $\gamma$ be the order type of this set. Using $\Delta^0_2$, we can compute the set of pairs $(u,c)$, where $u$ is one of the group elements that appears, and $c = a_i$ if $u$ appears only as $a_i$, $c = b_j$ if $u$ appears only as $b_j$, and $c = a_i + b_j$ if $u = g_i = h_j$. 


Lemma 8.7. If $X$ computes developments for $s$ and $t$, then $\Delta_0^0(X)$ computes the development of $s \cdot t$.

Proof. Using $X$, we can enumerate the pairs $g_i \cdot h_j$, putting them in order and noting (enumerating) equalities. Using $\Delta_0^0(X)$, we can compute the set of pairs $(u,c)$, where $u$ is one of the group elements that appears, and $c$ is the sum of the finitely many products $a_ib_j$, where $g_ih_j = u$.

Similarly, if $X$ can compute the developments for $t_1, \ldots, t_n$, then $\Delta_0^0(X)$ can compute the developments for $t_1 + \ldots + t_n$ and $t_1 \cdot \ldots \cdot t_n$. For a finite sum of finite products of elements chosen from among $t_1, \ldots, t_n$, $X$ can enumerate the finite products that occur, putting them in order, and note (enumerate) equalities. Then using $\Delta_0^0(X)$, we can compute the set of pairs $(u,c)$, where $u$ is one of the group elements that appears and $c$ is the coefficient. Oracle $\Delta_0^0(X)$ can check whether the coefficients are all 0.

Now we bound the complexity of computing the development and integer part of an arbitrary element in $R_n$, given the complexity of computing the developments of $R_{n-1}$ and $r_n$.

Lemma 8.8. Suppose $X$ calculates the developments of $r_n$ and $R_{n-1}$ for $n \geq 2$. Oracle $\Delta_0^0_{n-1}(X \oplus \Delta_0^0)$ calculates the development of any element of $R_n$, and $\Delta_0^0_{n-1+2}(X \oplus \Delta_0^0)$ calculates the integer part of any element of $R_n$.

Proof. Let $t \in R_n$ such that $t$ is a root of $p(x) \in R_{n-1}[r_n][x]$. Calculating $\delta(t) \upharpoonright \omega$ needs only $\Delta_2^0$. By Proposition 8.5, to test whether $\delta(t) \upharpoonright \omega$ suffices as a development for $t$ can be done using $\Delta_0^0(X \oplus \Delta_0^0)$. If so, then we are done. If not, then the development of $t$ continues. We need the element represented by the $\omega$-development. To test whether a given $t'$ serves, we check whether $t'$ has the same $\omega$-development as $t$. For this, we use $\Delta_0^0(X \oplus \Delta_0^0)$. We check whether this is the development for $t'$. We look for the polynomial $q(x)$ for which $t'$ (and some other elements) are roots. We can find $q(x)$ computably. When we plug the $\omega$-development for $t'$ into $\delta(q)(x)$, and simplify the expression, we see whether the coefficients are all 0. For this, we need $\Delta_0^0(X \oplus \Delta_0^0)$ as before. (Note that we are using the fact the MR-procedure produces a truncation closed embedding.) If the answer is positive, then $t'$ is represented by its $\omega$-development, and we can continue with the development of $t$. Since there is some $t'$ that serves, we can find it using $\Delta_0^0(X \oplus \Delta_0^0)$. Given the finitely many $t'_i$ calculated before in determining $\delta(t) \upharpoonright \omega \cdot (n - 1)$, oracle $\Delta_0^0$ can calculate the $\omega \cdot n$ development of $t$. To see if $\delta(t) \upharpoonright \omega \cdot n$ suffices, we use $\Delta_0^0(X \oplus \Delta_0^0)$. If $t \upharpoonright \omega \cdot n$ never suffices for any $n$, initial segment $t \upharpoonright \omega^2$ can be calculated using $\Delta_0^0(X \oplus \Delta_0^0)$ (but determining whether $t \upharpoonright \omega \cdot n$ is the development of $t$ requires $\Delta_0^0_{3+2}(X \oplus \Delta_0^0)$). Similarly, $\Delta_0^0(X \oplus \Delta_0^0)$ (using finitely much $\Delta_0^0(X \oplus \Delta_0^0)$ information) computes $t \upharpoonright \omega^2 \cdot n$, and $\Delta_0^0(X \oplus \Delta_0^0)$ checks whether $t \upharpoonright \omega^2 \cdot n$ suffices as a development of $t$ and finding an appropriate $t'$ if it does not. If the development of $t$ continues, we find that $\Delta_0^0_{2n-1}(X \oplus \Delta_0^0)$ computes $\delta(t) \upharpoonright \omega_n$. Oracle $\Delta_0^0_{2n+1}(X \oplus \Delta_0^0)$ determines whether $\delta(t) \upharpoonright \omega^n$ suffices as a development of $t$ and finds a $t'$ with that development otherwise. Thus, $\Delta_0^0(X \oplus \Delta_0^0)$ computes $\delta(t) \upharpoonright \omega^n$, and $\Delta_0^0_{n+2}(X \oplus \Delta_0^0)$ computes whether another element $t'$ with development $\delta(t) \upharpoonright \omega^n$ exists. To find the
\( \omega \cdot l \)-development, we use \( \Delta^0_{\omega \cdot l} (X \oplus \Delta^0_2) \), and we check whether we are done using 
\( \Delta^0_{\omega \cdot l + 1} (X \oplus \Delta^0_3) \) and if not, we find the representative, we use 
\( \Delta^0_{\omega \cdot l + 2} (X \oplus \Delta^0_3) \). To find the \( \omega \cdot l \)-development, we use \( \Delta^0_{\omega \cdot l} (X \oplus \Delta^0_2) \). To check whether we are done, we use 
\( \Delta^0_{\omega \cdot l + 2} (X \oplus \Delta^0_3) \), and we find a representative, we use 
\( \Delta^0_{\omega \cdot l + 1} (X \oplus \Delta^0_3) \). Since 
the development of \( t \) has length at most \( \omega^{n-1} \) by Theorem 1.1, 
\( \Delta^0_{\omega^n - 1} (X \oplus \Delta^0_2) \) computes the development of \( t \), and 
\( \Delta^0_{\omega^n - 1 + 2} (X \oplus \Delta^0_2) \) computes the integer part 
of \( t \) in the same way as before.

Since \( X = \Delta^0_2 \) computes the developments of \( R_1 \) and \( r_2 \) and the group \( G_2 \), oracle 
\( \Delta^0_\omega \) computes the developments of elements in \( R_2 \), and \( \Delta^0_{\omega + 2} \) computes the integer parts of elements in \( R_2 \). By induction on \( n \), we obtain Theorem 8.3.

**References**


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