The Degree Spectra of Homogeneous Models

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Abstract

Much previous study has been done on the degree spectra of prime models of a complete atomic decidable theory. Here we study the analogous questions for homogeneous models. We say a countable model $\mathcal{A}$ has a $\mathbf{d}$-*basis* if the types realized in $\mathcal{A}$ are all computable and the Turing degree $\mathbf{d}$ can list Δ^0_0-indices for all types realized in $\mathcal{A}$. We say $\mathcal{A}$ has a $\mathbf{d}$-decidable copy if there exists a model $\mathcal{B} \cong \mathcal{A}$ such that the elementary diagram of $\mathcal{B}$ is $\mathbf{d}$-computable. Goncharov, Millar, and Peretyat'kin independently showed there exists a homogeneous $\mathcal{A}$ with a $\mathbf{0}$-basis but no decidable copy.

We prove that any homogeneous $\mathcal{A}$ with a $\mathbf{0}'$-basis has a low decidable copy. This implies Csima's analogous result for prime models. In the case where all types of the theory T are computable and $\mathcal{A}$ is a homogeneous model with a $\mathbf{0}$-basis, we show $\mathcal{A}$ has copies decidable in every nonzero degree. A degree $\mathbf{d}$ is $\mathbf{0}$-*homogeneous bounding* if any automorphically nontrivial homogenous $\mathcal{A}$ with a $\mathbf{0}$-basis has a $\mathbf{d}$-decidable copy. We show that the nonlow$_2$ Δ^0_2 degrees are $\mathbf{0}$-homogeneous bounding.

1 Introduction

In 1961, Vaught [27] introduced the concepts of prime, saturated, and homogeneous models. These *Vaughtian* models provided a different perspective on model theory that led to many new avenues of research. Inspired by these results, researchers in computability, including Goncharov, Harrington, Peretyat'kin, Morley, and others, began studying the computable content of these and other model theoretic structures and thus needed to effectivize the objects under consideration.

Convention 1.1. *We assume throughout that all theories T are complete and decidable (CD) and all models $\mathcal{A}$ of T are countable.*

In addition, we assume that all models under consideration are *automorphically nontrivial* (see Definition 2.9), since these models are the only interesting ones as we see in §2.4.

Let T be a complete decidable (CD) theory. A model is called ($\mathbf{d}$-)*decidable* if its elementary diagram $D^e(\mathcal{A})$ is ($\mathbf{d}$-)computable. We discuss these and related definitions from computable model theory in more detail in §2.4.

Early researchers showed that a decidable copy of a prime, saturated, or specific homogeneous model of a CD theory does not necessarily exist. Thus, the Turing degree $\mathbf{0}$ is weak in this sense with respect to Vaughtian models. On the other hand, it is easy to see that any Vaughtian model of a theory satisfying reasonable computability conditions has a $\mathbf{0}'$-decidable copy. Given this $\mathbf{0}$ and $\mathbf{0}'$ dichotomy, recent research has focused on studying when an intermediate or other degree decides a copy of a Vaughtian model. In this paper, we will study the homogeneous model case and compare it with the prime model case.

1.1 Recent results on prime models

Let T be a complete atomic decidable (CAD) theory. As mentioned, T has a $\mathbf{0}'$-decidable prime model. Csima [3] greatly improved this result by showing that T always has a prime model decidable in some low degree. A degree $\mathbf{d}$ is *low*$_n$ if $\mathbf{d}^{(n)} = \mathbf{0}^{(n)}$, the lowest possible value. A degree $\mathbf{d} \leq \mathbf{0}'$ is *high*$_n$ if $\mathbf{d}^{(n)} = \mathbf{0}^{(n+1)}$, the highest possible value, and $\mathbf{d}$ is *low* if it is *low*$_1$ and *high* if it is *high*$_1$. Csima, Hirschfeldt, Knight, and Soare [5] studied prime bounding degrees. A degree $\mathbf{d}$ is *prime bounding* if for *any* CAD theory T, $\mathbf{d}$ decides a prime model of T. They showed that the Δ^0_2 prime bounding degrees are exactly the nonlow$_2$ degrees below $\mathbf{0}'$. Csima [3] also studied the case where T was not only a CAD theory but also had only computable types. Hirschfeldt [12] gave a surprising proof generalizing her result to show that in this case *any* noncomputable degree can decide a prime model of T. Since every prime model is homogeneous, it is natural to ask whether these results can be extended to homogeneous models in general.

1.2 Effectivizing Homogeneous Models

One major difference between the prime and homogeneous model cases is that a CAD theory T has a single prime model but a CD theory can have many nonisomorphic homogeneous models. Thus, there are two natural approaches to considering the effectiveness of homogeneous models for a CD theory T. One approach would be to ask whether T has *any* $\mathbf{d}$-decidable homogeneous models for a particular degree $\mathbf{d}$. This question was completely answered by Csima, Harizanov, Hirschfeldt, and Soare in [4]. They showed that a degree $\mathbf{d}$ bounds the elementary diagram of *some* homogeneous model of every CD theory if and only if $\mathbf{d}$ is a PA degree, where $\mathbf{d}$ is a *Peano Arithmetic (PA) degree* if $\mathbf{d}$ is the degree of a complete extension of the effectively axiomatized theory of Peano Arithmetic.

The second approach, begun by Goncharov, Peretyat'kin, and Millar, is to fix a homogeneous model $\mathcal{A}$ of a CD theory T and ask whether $\mathcal{A}$ has a $\mathbf{d}$-decidable copy for various degrees $\mathbf{d}$. If $\mathcal{A}$ has a decidable copy, then there exists a uniformly computable listing of the types realized in $\mathcal{A}$. We call such a listing a $\mathbf{0}$-*basis* of $\mathcal{A}$. Millar [20], Goncharov [7], and Peretyat'kin [23], however, gave examples of homogeneous models with $\mathbf{0}$-bases that have no decidable copy. Moreover, Goncharov and Peretyat'kin described an additional function on the $\mathbf{0}$-basis which exactly characterizes when a decidable copy of the homogeneous $\mathcal{A}$ exists. We discuss their characterization and its relativization in §3.2.

1.3 Homogeneous Low Basis Result

Let T be a complete decidable theory. Let $\mathcal{A}$ be a homogeneous model of T with a $\mathbf{0}'$-basis, *i.e.*, a listing of the types realized in $\mathcal{A}$ where $\mathbf{0}'$ uniformly computes a Δ^0_0-index for each type in the list. We show that there exists a $\mathcal{B} \cong \mathcal{A}$ such that $\mathcal{B}$ is decidable in a low degree.

Theorem 4.1. *Let T be a CD theory and $\mathcal{A}$ a homogeneous model of T with a $\mathbf{0}'$-basis. Then $\mathcal{A}$ has an isomorphic copy $\mathcal{B}$ decidable in a low degree.*

Theorem 4.1 also gives as a corollary Csima's result [3] that any complete atomic decidable theory T has a prime model decidable in a low degree. (See §4 for details). Thus, all prime and homogeneous models satisfying the reasonable computability assumptions have degree theoretically weak isomorphic copies.

1.4 Case where T has types all computable (TAC)

In Theorems 4.1 and 6.2, we only put computability restrictions on the types realized in the model $\mathcal{A}$. Surprisingly the computability of the types *not* realized in $\mathcal{A}$ actually impacts the decidability of $\mathcal{A}$. Let $S(T)$ denote all the types consistent with a theory T. If we also assume that all types in $S(T)$ are computable, not only the types realized in $\mathcal{A}$, we obtain the following strong result.

Theorem 5.2. *Let T be a complete decidable theory with all types in $S(T)$ computable. Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}$-basis. Then $\mathcal{A}$ has an isomorphic copy $\mathcal{B}$ decidable in any nonzero degree.*

1.5 0-basis homogeneous bounding

We next consider the $\mathbf{0}$-basis homogeneous bounding degrees. We say a degree $\mathbf{d}$ is $\mathbf{0}$-*basis homogeneous bounding* if any homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis has a $\mathbf{d}$-decidable isomorphic copy. In §6 we obtain the following.

Theorem 6.2. *Let* $\mathbf{d} \leq \mathbf{0}'$. *If* $\mathbf{d}$ *is nonlow*$_2$*, then* $\mathbf{d}$ *is* $\mathbf{0}$*-homogeneous bounding.*

In a future paper [15], we will show that below $\mathbf{0}'$ the nonlow$_2$ degrees exactly characterize the $\mathbf{0}$-homogeneous bounding degrees. (See §6.3 at the end of this paper.) This full characterization also is analogous to the result about prime bounding degrees by Csima, Hirschfeldt, Knight, and Soare [5].

1.6 Connections between the prime and homogeneous cases

All the theorems presented in this paper on homogeneous models have analogous counterparts for the prime model case. These earlier results on prime models were the original motivation for studying homogeneous models. Since all prime models are homogeneous, we hoped that results on homogeneous models might give the results on prime models as corollaries. Theorem 4.1, which says any homogeneous model with a $\mathbf{0}'$-basis has a low copy, does give the analogous result for prime models as a corollary (see Corollary 4.2). However, it was not clear how Theorems 5.2 and 6.2 on homogeneous models related to their prime counterparts. Recently, we have discovered some underlying connections between the prime and homogeneous cases and are working on fully developing these ideas (see §7.1).

2 Basic Definitions and Techniques

Let $\mathcal{L}$ be a countable language and T be a complete theory on $\mathcal{L}$. Here we fix our notation for various structures under consideration and discuss some basic model theory. See [2] or [18] for an introduction to the model theory found here.

2.1 Types as Paths in the tree $\mathcal{T}_n(T)$ and $\mathbb{T}(\mathcal{A})$

Types, maximal consistent sets of formulas on a finite set of free variables, play a key role in understanding homogeneous models. We define types formally in Definition 2.3. We begin by defining our conventions about formulas in a way that will make types easy to effectivize.

Definition 2.1. (i) Let $F_n(\mathcal{L})$ be the set of the formulas $\theta(x_0, \ldots, x_{n-1})$ of $\mathcal{L}$ with free variables included in $x_0, \ldots, x_{n-1}$. Let $F(\mathcal{L}) = \cup_{n \geq 0} F_n(\mathcal{L})$.

(ii) Let $\{\, \theta_i(\overline{x}) \,\}_{i \in \omega}$ be an effective listing of $F_n(\mathcal{L})$. For every string $\alpha \in 2^{<\omega}$ define

$$\theta_\alpha(\overline{x}) \quad = \quad \bigwedge \{\, \theta_i^{\alpha(i)}(\overline{x}) \ : \ i < |\alpha| \,\}$$

where $\theta^1 = \theta$ and $\theta^0 = \neg\, \theta$.

We view types as paths on certain trees contained in $2^{<\omega}$. We use the terminology and notation for trees developed in §2.1 in [17]. In particular, for $\mathcal{T} \subseteq 2^{<\omega}$ a tree, let $[\mathcal{T}]$ denote the set of infinite paths through $\mathcal{T}$. More information about trees and computability in general can be found in [24] and [25].

Definition 2.2. Let T be a complete $\mathcal{L}$-theory.

(i) A formula $\theta(\overline{x}) \in F_n(\mathcal{L})$ is *consistent with* T if $T \cup (\exists \overline{x})\theta(\overline{x})$ is consistent, *i.e.*, if $(\exists \overline{x})\theta(\overline{x}) \in T$, because T is complete. Let $F_n(T)$ be the subset of $F_n(\mathcal{L})$ consisting of all formulas consistent with T.

(ii) Define the tree of n-ary formulas consistent with T

$$\mathcal{T}_n(T) \ = \ \{\, \theta_\alpha(\overline{x}) \in F_n(\mathcal{L}) \ : \ \alpha \in 2^{<\omega} \quad \& \quad (\exists \overline{x})\theta_\alpha(\overline{x}) \in T \,\}.$$

If $\alpha \subset \beta$, then we say that θ_β *extends* θ_α.

(iii) We regard α as an *index* of θ_α. Define the tree of indices,

$$\widehat{\mathcal{T}}_n(T) = \{\, \alpha \ : \ \theta_\alpha \in \mathcal{T}_n(T) \,\}.$$

The trees $\mathcal{T}_n(T)$ and $\widehat{\mathcal{T}}_n(T)$ are effectively isomorphic but $\widehat{\mathcal{T}}_n(T) \subseteq 2^{<\omega}$ is notationally simpler. Hence, any definitions or results on trees $\widehat{\mathcal{T}} \subset 2^{<\omega}$ carry over to $\mathcal{T}_n(T)$. We eventually simply identify α and $\theta_\alpha(\overline{x})$.

Definition 2.3. [Types] (i) An *n-type of T* is a maximal consistent subset p of formulas of $F_n(T)$. There is a 1-1 correspondence between paths $f \in [\, \widehat{\mathcal{T}}_n(T) \,] \subset 2^\omega$ and the corresponding types $p_f \in [\, \mathcal{T}_n(T) \,]$ where

$$p_f = \{\, \theta_\alpha(\overline{x}) \ : \ \theta_\alpha(\overline{x}) \in \mathcal{T}_n(T) \quad \& \quad \alpha \subset f \,\}.$$

(ii) $S_n(T)$ is the set of all n-types of T, and let $S(T) = \cup_{n \geq 1} S_n(T)$.

(iv) For a complete theory T , we say $p \in S_n(T)$ is a *principal type* if there exists an n-ary formula $\psi \in p$ such that for all n-ary formulas θ, $T \vdash \psi \rightarrow \theta$

or $T \vdash \psi \to \neg\theta$. We call ψ a *generator of* p and say ψ *generates* or *isolates* p. Let $S^P(T) = \{ p \ : \ p \text{ is a principal type of } S(T) \}$.

(v) A complete theory T is *atomic* if every formula $\theta \in F(T)$ is an element of some type $p \in S^P(T)$.

Let T be a theory and $\mathcal{A}$ a model of T.

(vi) An n-tuple $\overline{a} \in A$ *realizes* an n-type $p(\overline{x}) \in S_n(T)$ if $\mathcal{A} \models \theta(\overline{a})$ for all $\theta(\overline{x}) \in p(\overline{x})$. In this case we say that $\overline{a}$ *realizes* p in $\mathcal{A}$.

(vii) Define the *type spectrum* of $\mathcal{A}$

$$\mathbb{T}(\mathcal{A}) = \{ p \ : \ p \in S(T) \quad \& \quad \mathcal{A} \text{ realizes } p \} \quad \text{and}$$

let $\mathbb{T}_n(\mathcal{A}) = \mathbb{T}(\mathcal{A}) \cap S_n(T)$ be the n-types realized in $\mathcal{A}$.

As we will now see, homogeneity can be described in terms of the behavior of types. Moreover, $\mathbb{T}(\mathcal{A})$ plays an important role in understanding the isomorphism class of a given homogeneous model.

2.2 Homogeneous Models and their Uniqueness Theorem

Let $\mathcal{A} \equiv \mathcal{B}$ denote elementary equivalence, $\mathcal{A} \cong \mathcal{B}$ denote isomorphism, and Aut $\mathcal{A}$ denote the group of automorphisms of $\mathcal{A}$.

Definition 2.4. A model $\mathcal{A} \models T$ is *(countably) homogeneous* if for all n-tuples $\overline{a}$ and $\overline{b}$, if $\overline{a}$ and $\overline{b}$ realize the same n-type, *i.e.*, $(\mathcal{A}, \overline{a}) \equiv (\mathcal{A}, \overline{b})$, then

$$(\exists \Phi \in \text{Aut } \mathcal{A}) [\, \Phi(\overline{a}) = \overline{b} \,].$$

Recall that prime and saturated models are necessarily homogeneous.

Definition 2.5. Let T be a complete theory.

(i) A model $\mathcal{A}$ of T is *prime* if $\mathcal{A}$ can be elementarily embedded in any other model $\mathcal{B}$ of T.

(ii) A countable model $\mathcal{A}$ is *(countably) saturated* if $\mathcal{A}$ realizes every type defined over any finite set $F \subseteq A$.

Vaught [27] proved that for any countable theory T, a model $\mathcal{A}$ of T is prime if and only if $\mathcal{A}$ is countable and *atomic*, *i.e.*, realizes only principal types. This is often taken as the defining property of prime models of countable theories. When we write "prime," we shall always mean "countable and atomic."

The next property of homogeneous models demonstrates the usefulness of the notion $\mathbb{T}(\mathcal{A})$. (See Marker [18] Theorem 4.3.23.)

Theorem 2.6. **[Uniqueness Theorem for Homogeneous Models]** *Given a countable complete theory T and homogeneous models $\mathcal{A}$ and $\mathcal{B}$ of T of the same cardinality*

$$\mathbb{T}(\mathcal{A}) \;=\; \mathbb{T}(\mathcal{B}) \quad \Longrightarrow \quad \mathcal{A} \;\cong\; \mathcal{B}. \tag{1}$$

Thus, to construct an isomorphic copy $\mathcal{B}$ of a homogeneous model $\mathcal{A}$, we build $\mathcal{B}$ so that: (1) $\mathcal{B}$ is homogeneous; and (2) the types realized in $\mathcal{B}$ satisfy $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$.

2.3 Presenting Types for a Complete Decidable Theory

From now on we assume that T is a complete decidable (CD) theory in a computable language $\mathcal{L}$. We now extend the noneffective definitions for formulas and types presented in §2.1 for the computable case.

We define an effective enumeration of all formulas consistent with T and show how we describe types using this enumeration. We will use this fixed universal enumeration throughout the remainder of this paper.

Definition 2.7. (i) Given the fixed CD theory T let $\{\theta_i\}_{i\in\omega}$ be an effective numbering of $F(T) = \cup_n F_n(T)$, all the formulas consistent with T.

(ii) For every $n > 0$ and n-type p we may assume p decides every k-ary formula $\theta(\overline{x})$ for every $k < n$ as follows. Define

$$\theta'(x_0, \ldots, x_{n-1}) = \theta(\overline{x}) \wedge (\bigwedge_{j<n} (x_j = x_j)).$$

Add θ to p just if $\theta' \in p$ already. Now associate with p a function $f \in 2^\omega$ such that $f(i) = 1$ if $\theta_i \in p$. Hence, every type corresponds to a function over *all* formulas $\theta_i \in F(T)$ but clearly $f_p(j) = 0$ if θ_j is a k-ary formula for $k > n$.

(iii) Let p be an n-type and q be a k-type for $k < n$. Then p and q are *inconsistent* if there exists a k-ary formula $\theta_i(\overline{x})$ such that $f_p(i) \neq f_q(i)$, and are *consistent* otherwise. If p and q are computable types then their consistency is a Π^0_1 condition.

(iv) For any type $p \in S(T)$ define $p{\restriction}s \;=\; p \cap \{\theta_i\}_{i<s}$. Identify $p{\restriction}s$ with the function $f_p{\restriction}s$ where $f_p(i) = 1$ if $\theta_i \in p$.

2.4 Degree Spectra

Our object of study (and one often studied in computable model theory) will be the collection of Turing degrees of the (elementary) diagrams of isomorphic copies of a model $\mathcal{A}$. See [1] and [9] for wider overviews of computable model theory.

We first define the diagrams associated with a given model. Let $\mathcal{A}$ be a model with universe A. Let $\mathcal{L}_A$ be the language $\mathcal{L} \cup \{c_a : a \in A\}$. Let $\mathcal{A}_A = (\mathcal{A}, a)_{a \in A}$ be the expansion of model $\mathcal{A}$ for language $\mathcal{L}_A$ such that c_a is interpreted by a for every $a \in A$. The *elementary diagram* $D^e(\mathcal{A})$ (atomic diagram $D^a(\mathcal{A})$) of $\mathcal{A}$ is the set of all (atomic) sentences of $\mathcal{L}_A$ which are true in $\mathcal{A}_A$. For any $X \subseteq \omega$, let $deg(X)$ denote the Turing degree of X.

Definition 2.8. Let $dSp^a(\mathcal{A}) = \{deg(D^a(\mathcal{B})) : \mathcal{B} \cong \mathcal{A}\}$. Then $dSp^a(\mathcal{A})$ is called the *atomic degree spectrum of* $\mathcal{A}$ and we similarly define $dSp^e(\mathcal{A})$, the *elementary degree spectrum of* $\mathcal{A}$.

We say $\mathcal{A}$ is *computable* if $D^a(\mathcal{A})$ is computable and $\mathcal{A}$ is *decidable* if $D^e(\mathcal{A})$ is computable, and we similarly define **d**-computable and **d**-decidable for any degree **d**. By Theorem 2.6, we see that

$$dSp^a(\mathcal{A}) = \{deg(D^a(\mathcal{B})) : \mathcal{B} \text{ homogeneous } \& \ \mathbb{T}(\mathcal{B}) = \mathbb{T}(\mathcal{A})\}$$

where $\mathcal{B}$ ranges only over countable models (and similarly $dSp^e(\mathcal{A})$).

Definition 2.9. A structure $\mathcal{A}$ is called *automorphically trivial* if there exists a finite set $F \subset A$ such that any permutation π of A fixing F is an automorphism of $\mathcal{A}$.

The following theorem is a useful fact about degree spectra.

Theorem 2.10. (Knight [14]) *Let $\mathcal{A}$ be a countable structure in a relational language. If $\mathcal{A}$ is not automorphically trivial, then $dSp^a(\mathcal{A})$ is closed upwards. This result also holds for $dSp^e(\mathcal{A})$. Note that $dSp^e(\mathcal{A}) \subseteq dSp^a(\mathcal{A})$.*

Since automorphically trivial models are structurally uninteresting, we only consider the degree spectra of automorphically nontrivial models, which are closed upwards by Theorem 2.10. Thus to show $\mathbf{d} \in dSp^e(\mathcal{A})$, it suffices to show $\mathcal{A}$ has an isomorphic copy decidable below **d**.

Convention 2.11. *We assume all models considered are automorphically nontrivial.*

3 Decidability of Homogeneous Models

In this section, we lay out the terminology required to understand Goncharov and Peretyat'kin's characterization (discussed in §3.2) of when a homogeneous model has a decidable isomorphic copy. For more information on this characterization and how it relates to the analogous characterizations for prime and saturated models, consult [17].

3.1 d-Bases and d-Uniform Bases

Recall if $\mathcal{A}$ has a decidable copy, then there exists a uniformly computable listing of $\mathbb{T}(\mathcal{A})$.

Definition 3.1. We say that a countable model $\mathcal{A}$ has a $\mathbf{0}$-*basis* $X = \{p_j\}_{j\in\omega}$ if X is a uniformly computable listing of $\mathbb{T}(\mathcal{A})$.

We encode a basis X as an infinite two-dimensional matrix of zeros and ones (*i.e.*, an element of $2^{\omega\times\omega}$) where the ith row $X^i \in 2^\omega$ corresponds to the type p_i according to the enumeration of the formulas $F(T)$ we fixed in Definition 2.7. The standard relativization of this idea to degree $\mathbf{d}$ is as follows.

Definition 3.2. We say $\mathcal{A}$ has a $\mathbf{d}$-*uniform basis* $X = \{p_j\}_{j\in\omega}$ if X is a $\mathbf{d}$-uniformly computable listing of $\mathbb{T}(\mathcal{A})$.

Note that a $\mathbf{0}$-basis is a $\mathbf{0}$-uniform basis and vice versa. To obtain stronger results later, we relativize the idea of a $\mathbf{0}$-basis in a nonstandard way. When we relativize the definition of a $\mathbf{0}$-basis to a $\mathbf{d}$-basis, the notions of $\mathbf{d}$-basis and $\mathbf{d}$-uniform basis will differ.

Definition 3.3. A countable model $\mathcal{A}$ has a $\mathbf{d}$-*basis* $X = \{p_j\}_{j\in\omega}$ if

1. the types realized in $\mathcal{A}$ are all computable and
2. there exists a $\mathbf{d}$-computable function g such that $g(j)$ is a Δ^0_0-index for type p_j for all $j \in \omega$.

Note that any $\mathbf{0}$-basis (as defined in Definition 3.1) satisfies this definition and any $\mathbf{d}$-basis can effectively be viewed as a $\mathbf{d}$-uniform basis, but not conversely.

Goncharov, Millar, and Peretyat'kin separately showed that a $\mathbf{0}$-basis alone does not guarantee the existence of a decidable copy of a homogeneous model by building counterexamples [7], [20], [23]. Goncharov and Peretyat'kin, however, exactly characterized when a homogeneous model has a decidable copy. We now discuss their characterization. For more detail on Goncharov's and Peretyat'kin's characterization, see [17].

3.2 Effective and Monotone Extension functions

Although a $\mathbf{0}$-basis for a homogeneous model $\mathcal{A}$ computably tells us what types are realized in $\mathcal{A}$, Goncharov and Peretyat'kin realized that to produce a decidable copy, we need computable information about how these types extend one another.

Definition 3.4. [Effective Extension Function (EEF) and Monotone Extension Function (MEF)]
Let $\mathcal{A}$ be a homogeneous model of a complete decidable (CD) theory T whose type spectrum $\mathbb{T}(\mathcal{A})$ has a $\mathbf{0}$-basis $X = \{p_i\}_{i\in\omega}$.

(i) A function f is an *extension function (EF)* for X if for every n,

- for every n-type $p_i(\overline{x}) \in X \cap S_n(T)$
- and every $(n+1)$-ary $\theta_j(\overline{x}, x_n) \in F_{n+1}(T)$ consistent with $p_i(\overline{x})$

the $(n+1)$-type $p_{f(i,j)} \in X \cap S_{n+1}(T)$ extends both $p_i(\overline{x})$ and $\theta_j(\overline{x}, x_n)$, *i.e.*,

$$p_i(\overline{x}) \cup \{\theta_j(\overline{x}, x_n)\} \subseteq p_{f(i,j)}(\overline{x}, x_n).$$

(ii) If f is also computable then f is an *effective extension function (EEF)*.

(iii) A function f is a *monotone extension function (MEF)* if there exists a computable function $g(i, j, s)$ such that

- $f(i, j) = \lim_s g(i, j, s)$ is an extension function and
- $p_{g(i,j,s)}\restriction s \subseteq p_{g(i,j,s+1)}\restriction s$.

An effective extension function is a computable function which given any n-type p_i and any consistent $(n+1)$-ary formula θ_j outputs an index k such that p_k is an $(n+1)$ type which amalgamates p_i and θ_j. A monotone extension function given the same data monotonically approximates the index of an amalgamating $(n+1)$-type. Specifically, the approximate amalgamator $p_{g(i,j,s)}(\overline{x}, x_n)$ at stage s agrees with the true amalgamator $p_{f(i,j)}(\overline{x}, x_n)$ on the first s formulas of $F(T)$.

The next result is our main tool for obtaining new results.

Theorem 3.5. (A relativization of Goncharov [7], Peretyat'kin [23]) *Let T be a complete decidable theory and $\mathcal{A}$ of T be homogeneous. Then the following are equivalent:*

1. *$\mathcal{A}$ has a* **d***-decidable isomorphic copy.*

2. *Every* **d***-uniform basis for $\mathcal{A}$ has a* **d***-monotone extension function.*

3. *Some* **d***-uniform basis for $\mathcal{A}$ has a* **d***-monotone extension function.*

4. *Some* **d***-uniform basis for $\mathcal{A}$ has a* **d***-effective extension function.*

Suppose a homogeneous model $\mathcal{A}$ has a **0**-basis X. To show $\mathcal{A}$ has an isomorphic copy of degree **d**, we can build a new **d**-uniform basis Y with an **d**-effective extension function. Alternatively, since a **0**-basis can be effectively viewed as a **d**-uniform basis, we can build a **d**-monotone extension function on the original **0**-basis X. In either case, by Theorem 3.5, $\mathcal{A}$ has a **d**-decidable isomorphic copy $\mathcal{B}$.

3.3 Notation for Bases

Recall we encode a basis X as an infinite two-dimensional matrix with elements from $\{0, 1\}$ where the ith row $X^i \in 2^\omega$ corresponds to the type p_i according to the enumeration of the formulas $F(T)$ in Definition 2.7.

Since we build bases in stages during our constructions, we use upper case letters such as X, Y, M to denote finite or partially defined matrices in $2^{\omega\times\omega}$. Similarly, we let X^i denote the partial string which is the ith row of X. Let $ht(X)$ denote the greatest i such that $(\exists x)X^i(x)\downarrow$. For some formula ψ, let $\exists\psi$ denote the sentence given by ψ with all its free variables quantified out by existentials.

4 Homogeneous Low Basis Theorem

Let $\mathcal{A}$ be an automorphically nontrivial homogeneous model of a complete decidable theory T. We now study the elementary degree spectrum of this fixed model. Goncharov, Millar, and Peretyat'kin showed that an arbitrary homogeneous $\mathcal{A}$ with a **0**-basis does not necessarily have a decidable isomorphic copy [7], [20], [23], and Tusupov [26] showed that any such $\mathcal{A}$ always has a $\mathbf{0}'$-decidable copy.

The negative results above require delicately building counterexamples. Tusupov's result, however, follows easily from the relativized EEF/MEF Theorem 3.5. Here we examine the degrees in between **0** and $\mathbf{0}'$.

4.1 The Basic Result

We prove that $dSp^e(\mathcal{A})$ always contains a low degree for any homogeneous $\mathcal{A}$ with a $\mathbf{0}'$-basis. Using a $\mathbf{0}'$ oracle, we will build a $\mathbf{d}$-uniform basis for $\mathcal{A}$ with an effective extension function for a low degree $\mathbf{d}$. The relativized EEF/MEF Theorem 3.5 then gives that there exists a low $\mathcal{B} \cong \mathcal{A}$. We show that this result implies Csima's analogous result for prime models [3]. Moreover, we show that the proof can be strengthened to obtain results on cone avoidance and minimal pairs in $dSp^e(\mathcal{A})$.

Theorem 4.1. *Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}'$-basis. There is a $\mathbf{d}$-decidable $\mathcal{B} \cong \mathcal{A}$ for a low degree $\mathbf{d}$.*

Proof. Let $X = \{p_i\}_{i\in\omega}$ be the $\mathbf{0}'$-basis of types of $\mathcal{A}$. We build a $\mathbf{d}$-uniform basis $Y = \{q_i\}_{i\in\omega}$ for $\mathcal{A}$ for $\mathbf{d}$ low. Let $f(n,m)$ be a computable injective function such that $\mathrm{ran}(f) = \{2n : n \in \omega\}$, and if $f(i,j) = k$ then $i < k$. The range condition simply ensures that there are infinitely many rows not in the range of f. This f will be the effective extension function for the $\mathbf{d}$-uniform basis Y that we build. Since Y is $\mathbf{d}$-uniform and f is computable, by the relativized EEF/MEF Theorem 3.5, $\mathcal{A}$ will have a $\mathbf{d}$-decidable copy $\mathcal{B}$ where $\mathbf{d}$ is low. We will meet the following requirements for all e, i, k:

$\mathbf{P}_e$**: (Lowness)** $\{e\}^Y(e)$ is decided by stage $e+1$ of the construction.

$\mathbf{Q}_i$**: ($Y \subseteq \mathbb{T}(\mathcal{A})$)** $q_i = p_j$ for some $j \in \omega$

$\mathbf{R}_i$**: ($\mathbb{T}(\mathcal{A}) \subseteq Y$)** $p_i = q_j$ for some $j \in \omega$

$\mathbf{S}_k$**: (EEF)** If $k = f(i,j)$, q_i is an n-type, and θ_j a $(n+1)$-formula consistent with q_i, then q_k is an $(n+1)$-type extending q_i and θ_j.

Construction.

We build $Y \in 2^{\omega\times\omega}$ in stages using a $\mathbf{0}'$-oracle so that $Y = \cup_{s\in\omega} Y_s$ where Y_s is a partial infinite matrix of zeroes and ones defined at stage s. $Y^i \in 2^\omega$, the ith row of Y, corresponds to a type $q_i \in S(T)$. We satisfy requirements $\mathbf{Q}_i$ and $\mathbf{R}_i$ to ensure that $Y = \{q_i\}_{i\in\omega} = \mathbb{T}(\mathcal{A})$.

Stage $s = 0$**.**

Define Y_0^0 so that $q_0 = p_0$. Leave the remainder of Y_0 undefined.

Stage $s + 1 = e + 1$**.**

Let $h = ht(Y_s)$ where $ht(X)$ for a partial matrix X denotes the greatest i such that $(\exists x) X^i(x)\downarrow$. We are given Y_s which by induction satisfies:

- $h \geq s$ [At least s many rows are partially filled.]
- $(\forall i \leq h)(\forall x)[Y_s^i(x)\downarrow \ \& \ (\exists j)\ q_i = p_j]$

  [For each $i \leq h$, Y_s^i is total, and Y_s^i corresponds to a type in $\mathbb{T}(\mathcal{A})$.]
- $(\forall i \leq s)(\exists j \leq h)[p_i = q_j]$

  [For all $i \leq s$ there exists $j \leq h$ so that type p_i corresponds to Y_s^j.]
- $(\forall k = f(i,j) \leq h)$

  $\theta_j \in F_{n+1}(T)\ \&\ \exists\theta_j \in q_i \in S_n(T) \implies (q_i \cup \{\theta_j\}) \subset q_k \in S_{n+1}(T)$

  [f is an effective extension function through stage s.]

Using a $\mathbf{0}'$-oracle we test whether there exist a t and a finite partial matrix M such that M has the following computable properties:

1. M respects Y_s

   $(\forall x)[M(x)\downarrow \ \&\ Y_s(x)\downarrow \quad \implies \quad M(x) = Y_s(x)]$

   [M is agrees with Y_s where both are defined.]
2. M respects T

   $(\forall i)(\exists! n)[h < i \leq ht(M) \quad \implies \quad (\exists l)(\exists p \in S_n(T))[M^i = p\restriction l\ ]$

   [M^i can be extended to an n-type of $S(T)$]
3. M respects the EEF f

   For all k where $h < i \leq ht(M)$, $k = f(i,j)$, and $\theta_j \in F_{n+1}(T)$ either:

   (a) M^i (or Y_s^i if $i \leq h$) is not in $S_n(T)$ or is inconsistent with θ_j.
   [M decides q_i is not an n-type or q_i and θ_j are inconsistent.]

   (b) M^i (or Y^i if $i \leq h$) determines q_i is an n-type and $M^i(\exists\theta_j)\downarrow= 1$. Then $M^i(\exists[\theta_j \wedge \theta_{M^k}])\downarrow= 1$ and M^k determines q_k is an $(n+1)$-type.
   [M proves q_i is an n-type consistent with θ_j. Then M proves q_k can be extended to an $(n+1)$-type containing the partial type q_i and θ_j.]
4. M forces the jump

   $\{e\}_t^M(e)\downarrow$

If there exists some such M and t, let M' be the least such M and set $Y'_{s+1} = M' \cup Y_s$. Otherwise let $Y'_{s+1} = Y_s$.

We extend Y'_{s+1} so that for all $k \leq ht(Y'_{s+1})$, M_i is total. If $k \leq ht(Y'_{s+1})$ and $k \notin \mathrm{ran}(f)$, $\mathbf{0}'$ can find a type in $X = \{p_i\}_{i\in\omega}$ that extends the kth row of Y'_{s+1} (since each row of Y'_{s+1} is consistent with T). Similarly, we fill in q_k when $k = f(i,j)$ but q_i and θ_j either have incorrect arities or are inconsistent. Hence we can fill in these finitely many q_k for $i \leq ht(Y'_{s+1})$ while respecting Y'_{s+1}.

Otherwise if $k = f(i,j)$, we use $\mathbf{0}'$ to find an $(n+1)$-amalgamator $p_l \in X$ for q_i and θ_j that extends Y'_{s+1}. We know (since inductively $q_i \in \mathbb{T}(\mathcal{A})$) some such amalgamator exists in X. To find it, we iteratively test each p_l to determine if it extends q_i, θ_j and the kth row of Y'_{s+1} (which is consistent with q_i and θ_j). Since X is a $\mathbf{0}'$-basis, $\mathbf{0}'$ can uniformly compute a Δ^0_0-index for each p_l. Thus, $\mathbf{0}'$ can uniformly determine if p_l satisfies these requirements since consistency between two computable types and a formula is a Π^0_1 statement. We extend row k of Y'_{s+1} to correspond to the first p_l with the desired properties.

Now let Y_{s+1} be Y'_{s+1} with the above extensions (so all q_k are decided for $k \leq ht(Y'_{s+1})$) and with p_{s+1} placed on the next empty row which is not in the range of f.

End Construction.

Verification.

By the conditions put on M and the way in which q_i's are decided, requirements $\mathbf{Q}_i$, $\mathbf{R}_i$, and $\mathbf{S}_i$ are satisfied for all i. Specifically, we completely fill each q_i (*i.e.*, the row Y^i) by copying some $p_j \in X$ which agrees with the finitely many bits already determined. Hence every row in Y is some p_j. Similarly, we know by the last line of the construction that every p_j is some row in Y. Finally, by construction, f is an effective extension function for Y.

At the end of stage $e+1$, $\{e\}^Y(e)$ is decided, satisfying requirement $\mathbf{P}_e$. If $\{e\}^{Y_{e+1}}(e)\downarrow$, the use principle guarantees that $\{e\}^Y(e)\downarrow = \{e\}^{Y_{e+1}}(e)\downarrow$ since $Y \supseteq Y_{e+1}$. Otherwise by construction, no compatible extension σ of Y_{e+1} can cause $\{e\}^\sigma(e)$ to converge, so $\{e\}^Y(e)\uparrow$. Thus Y is low. By applying the relativized EEF/MEF Theorem 3.5 to Y with effective extension function f, there exists a model $\mathcal{B} \cong \mathcal{A}$ of low degree.

□

This result easily provides the following corollary.

Corollary 4.2. (Csima [3]) *Let T be a complete atomic decidable theory. Then there exists a prime model $\mathcal{B}$ of T decidable in a low degree.*

Proof. Let $\mathcal{A}$ be a prime model of T. Then,

$$\mathbb{T}(\mathcal{A})=S^P(T)=\{p(\bar{x}) : p(\bar{x}) \text{ is a principal type in } T\}.$$

By the above result, it suffices to show that $\mathcal{A}$ has a $\mathbf{0}'$-basis. Given a formula $\theta(\bar{x}) \in F^n(T)$, $\mathbf{0}'$ can decide the following Σ^0_1 question:

$$(\ \exists\gamma\ \in\ F^n(T)\ )\ [\ (\exists\bar{x})[\theta\wedge\gamma]\ \in\ T\ \&\ (\exists\bar{x})[\theta\wedge\neg\gamma]\ \in\ T\ ],$$

i.e., whether θ generates a principal type. Hence, $\mathbf{0}'$ can decide whether a formula $\theta(\bar{x})$ is a generator for a principal type. Since T is decidable, the index for a generator in an effective listing of formulas gives rise to a Δ^0_0-index for its corresponding principal type.

We use $\mathbf{0}'$ to determine all formulas which are generators for principal types in T. We then use the indices for these formulas to give a $\mathbf{0}'$-basis for $\mathbb{T}(\mathcal{A})$.

□

4.2 Avoiding Cones and Minimal Pairs

We can strengthen the results of the last section by combining additional requirements with the basic approach. These results tell us two important properties of the degree spectrum of any homogeneous model $\mathcal{A}$ with a $\mathbf{0}'$-basis. The first result shows that $dSp^e(\mathcal{A})$ contains a scattered selection of degrees between $\mathbf{0}$ and $\mathbf{0}'$. The second shows $dSp^e(\mathcal{A})$ always contains a minimal pair, in other words, that noncomputable information cannot be coded into the degree spectrum of any such model.

4.2.1 Avoiding Cones of Degrees

Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}'$-basis, and let C be a noncomputable low set. We show there exists a low copy $\mathcal{B}$ of $\mathcal{A}$ which avoids the upper and lower cones generated by C. Thus, $dSp^e(\mathcal{A})$ must be scattered in the sense that no low degree $\mathbf{c}$ is below every degree in $dSp^e(\mathcal{A})$. (This result could be generalized further to show a list of low degrees whose jumps are uniformly computable in $\mathbf{0}'$ could not bound from below all of $dSp^e(\mathcal{A})$.)

Theorem 4.3. *Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}'$-basis X. Let C be a noncomputable low set. Then there is a low copy $\mathcal{B}$ of $\mathcal{A}$ such that $C \not\leq_{\mathrm{T}} D^e(\mathcal{B})$ and $D^e(\mathcal{B}) \not\leq_{\mathrm{T}} C$.*

Proof. If $\mathcal{A}$ has a decidable copy, then the theorem follows by upward closure of the degree spectrum Theorem 2.10. Thus we may assume that $\mathcal{A}$ has no decidable copy.

In addition to the requirements in the basic result, Theorem 4.1, we add requirements:

$\mathbf{N}_e$: $C \neq \{e\}^Y$

$\mathbf{N}'_e$: $Y \neq \{e\}^C$

If we satisfy these additional requirements, there exists a $\mathcal{B} \cong \mathcal{A}$ where $D^e(\mathcal{B}) \equiv_{\mathrm{T}} Y$ by Theorem 3.5. Thus $\mathcal{B}$ is low and avoids the upper and lower cones of C. We modify the construction in Theorem 4.1.

Construction.

We alternate between building the low uniform basis Y with effective extension function f for $\mathcal{A}$ as before and satisfying our additional requirements.

Stage $s+1 = 3e+1$. **(Lowness)**

Act as in stage $s+1$ in the last construction to ensure the lowness of Y.

Stage $s+1 = 3e+2$. **(Satisfy $\mathbf{N}_e$)**

We meet $\mathbf{N}_e$. Test using a $\mathbf{0}' = C'$ oracle whether there exist t, y, and a finite partial matrix M such that:

- M respects Y_s, T, and the effective extension function f (as in the construction for Theorem 4.1).
- $C(y) \neq \{e\}_t^M(y)$

If yes, take the least such extension M' that satisfies the above and set $Y'_{s+1} = M' \cup Y_s$. Use $\mathbf{0}'$ and the $\mathbf{0}'$-basis X to completely fill in each partial row in Y'_{s+1} to obtain Y_{s+1}.

Stage $s+1 = 3e+3$. **(Satsify $\mathbf{N}'_e$)**

Use the $\mathbf{0}' = C'$ oracle to ask whether there exist t, y and a finite partial matrix M such that:

- M respects Y_s, T, and the effective extension function f.
- $Y(y) \neq \{e\}_t^C(y)\downarrow$

If so, let M' be the least such M. Then set $Y'_{s+1} = M' \cup Y_s$. Extend Y'_{s+1} to Y_{s+1} as above.

End Construction.

Verification.

We built a low uniform basis Y with effective extension function f as in Theorem 4.1. Let $\mathbf{d}$ be the degree of Y. By Theorem 3.5, we obtain a $\mathbf{d}$-decidable (*i.e.*, low) $\mathcal{B}$ isomorphic to $\mathcal{A}$. It remains to show that $\mathbf{N}_e$ and $\mathbf{N}'_e$ are satisfied for all e.

Suppose $C = \{e\}^Y$. Then we can compute $C(y)$ for any y effectively from the finitely many computable indices for the rows in Y_{3e+2}. To compute $C(y)$, compute $\{e\}_s^M(y)$ for some finite partial matrix M which respects Y_{3e+2}, T, and f where M is defined (these questions are computable since M is finite). Such an M exists since C is total. Then $\{e\}_s^M(y) = C(y) = \{e\}^Y(y)$. If $\{e\}_s^M(y) \neq \{e\}^Y(y)$, M would have been selected to extend Y_{3e+2}, satisfying $\mathbf{N}_e$. Since C was assumed to be noncomputable, $\mathbf{N}_e$ holds for all e. Hence $C \not\leq_{\mathrm{T}} D^e(\mathcal{B})$ since $Y \equiv_{\mathrm{T}} D^e(\mathcal{B})$.

Suppose $Y = \{e\}^C$. Then we can compute $Y(y)$ for any y uniformly from the finitely many computable indices corresponding to the rows of Y_{3e+3}. Find the first such finite partial M that respects Y_{3e+3}, T, and f so that the length of M is greater than y (thinking of M now as a string in $2^{<\omega}$). Such an M exists since Y is total and has these properties. Then $M(y) = Y(y) = \{e\}^C(y)$. (Otherwise we would have extended Y_{3e+3} using M, satisfying $\mathbf{N}'_e$.)

Hence Y is computable. But then $\mathcal{B}$ would be a decidable copy of $\mathcal{A}$, a contradiction. Thus, $\mathbf{N}'_e$ is satisfied for all e and $D^e(\mathcal{B}) \not\leq_{\mathrm{T}} C$.

□

4.2.2 Minimal Pairs

If $\mathcal{A}$ is a homogeneous model with a $\mathbf{0}'$-basis, Theorem 4.3 shows that $dSp^e(\mathcal{A})$ contains a scattering of degrees below $\mathbf{0}'$. Now we show $dSp^e(\mathcal{A})$ contains minimal pairs. Therefore it is impossible to code noncomputable information into the isomorphism class of a homogeneous model with a $\mathbf{0}'$-basis.

Theorem 4.4. *Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}'$-basis X. Then there are low decidable copies $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ such that if $Z \leq_{\mathrm{T}} D^e(\mathcal{B})$ and $Z \leq_{\mathrm{T}} D^e(\mathcal{C})$, then Z is computable.*

Proof. If $\mathcal{A}$ has a decidable copy, then the theorem follows by upward closure of the degree spectrum Theorem 2.10. Thus we may assume that $\mathcal{A}$ has no decidable copy.

We build a minimal pair of low models $\mathcal{B}$ and $\mathcal{C}$ isomorphic to $\mathcal{A}$ using a $\mathbf{0}'$-oracle argument. For $\mathcal{B}$ and $\mathcal{C}$, we have each of the requirements in the basic result Theorem 4.1. Let $Y^{\mathcal{B}}$ and $Y^{\mathcal{C}}$ denote the low uniform bases we are constructing with effective extension functions $f^{\mathcal{B}}$ and $f^{\mathcal{C}}$ for $\mathcal{B}$ and $\mathcal{C}$. We add the following requirement to ensure $\mathcal{B}$ and $\mathcal{C}$ form a minimal pair.

$\mathbf{N}_e$: (Minimal Pair)

$$\{e\}^{Y^{\mathcal{B}}} = \{e\}^{Y^{\mathcal{C}}} = g \text{ total} \quad \Longrightarrow \quad g \text{ computable.}$$

Construction. We build $Y^{\mathcal{B}}$ and $Y^{\mathcal{C}}$ as in Theorem 4.1 with the following additions.

Stage 0**:**

Let the first rows in $Y^{\mathcal{B}}$ and $Y^{\mathcal{C}}$ be two distinct one types in $\mathcal{A}$.

Stage $s+1 = 2e+1$**. (Lowness)**

Act as in stage $s+1$ in the basic result to meet the lowness requirements for $\mathcal{B}$ and $\mathcal{C}$.

Stage $s+1 = 2e+2$**. (Satisfy $\mathbf{N}_e$)**

Use a $\mathbf{0}'$-oracle to test whether there exist t, y and finite partial matrices M and M' such that:

- M respects $Y^{\mathcal{B}}_s$, T, and the effective extension function $f^{\mathcal{B}}$ (as in Theorem 4.1) and similarly M' respects these properties for $Y^{\mathcal{C}}_s$ and $f^{\mathcal{C}}$.
- $\{e\}^M_t(y)\downarrow \neq \{e\}^{M'}_t(y)\downarrow$

If so, take the least such extensions $\tilde{M}$ and $\tilde{M}'$ that satisfy the above and set $\tilde{Y}^{\mathcal{B}}_{s+1} = \tilde{M} \cup Y^{\mathcal{B}}_s$ and $\tilde{Y}^{\mathcal{C}}_{s+1} = \tilde{M}' \cup Y^{\mathcal{C}}_s$. Then let $Y^{\mathcal{B}}_{s+1}$ equal $\tilde{Y}^{\mathcal{B}}_{s+1}$ extended using $\mathbf{0}'$ to completely fill in each partial row in $\tilde{Y}^{\mathcal{B}}_{s+1}$ as in Theorem 4.1, and similarly define $Y^{\mathcal{C}}_{s+1}$.

End Construction.

Verification.

The models $\mathcal{B}$ and $\mathcal{C}$ are low copies of $\mathcal{A}$ exactly as in Theorem 4.1. It remains to show that $\mathbf{N}_e$ is satisfied for all e.

Suppose $g = \{e\}^{Y^{\mathcal{B}}} = \{e\}^{Y^{\mathcal{C}}}$ is total. We can compute g from the finitely many computable indices of the rows in $Y^{\mathcal{B}}_{2e+2}$. To compute $g(y)$, computably find a finite partial M which respects $Y^{\mathcal{B}}_{2e+2}$, T, and $f^{\mathcal{B}}$ such that $\{e\}^M(y)\downarrow$. Some such M must exist since g is total, and then $\{e\}^M_s(y) = g(y)$. Hence g is computable.

Since $\mathcal{A}$ has no decidable copy, $\mathcal{B}$ and $\mathcal{C}$ are not decidable and form a minimal pair.

□

In the next section, we will see how stronger conditions on the theory T can tell us more about the elementary degree spectrum of homogeneous models.

5 Full Basis Theorem for Homogeneous Models

As mentioned in §3.1, early researchers first asked when a homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis has a decidable copy. In §4, to generalize this work, we studied the degree spectrum of a homogeneous model $\mathcal{A}$ with a $\mathbf{0}'$-basis. We saw that the level of computability of the types in $\mathbb{T}(\mathcal{A})$ directly impacts how decidable copies of $\mathcal{A}$ can be. Thus restricting our study to any homogeneous model $\mathcal{A}$ with a $\mathbf{d}$-basis for some degree $\mathbf{d}$ is a natural requirement to obtain useful results on $dSp^e(\mathcal{A})$. This basis condition implies that all of the types *realized in* $\mathcal{A}$, *i.e.*, in $\mathbb{T}(\mathcal{A})$, are computable. In this section, we assume that not only are the types in $\mathbb{T}(\mathcal{A})$ computable but also that all the types in $S(T)$ are computable. It is surprising that the computability of these types outside of $\mathbb{T}(\mathcal{A})$ greatly affects $dSp^e(\mathcal{A})$.

If $S(T)$ or even $S^c(T)$, the set of computable types in T, is uniformly computable and $\mathcal{A}$ has a $\mathbf{0}$-basis, we obtain the strongest possible result.

Theorem 5.1. (Goncharov [7], Millar [21]) *Let T be a complete decidable theory with $S^c(T)$ uniformly computable. If $\mathcal{A}$ is a homogeneous model with a $\mathbf{0}$-basis, then $\mathcal{A}$ has a decidable copy.*

We explore the case where all the types in $S(T)$ are computable but not uniformly computable. We obtain almost as strong of a result.

Theorem 5.2. *Let T be a complete decidable theory with $S(T)$ consisting of computable types. Let $\mathcal{A}$ of T be a homogeneous model with a $\mathbf{0}$-basis. Then for every nonzero degree $\mathbf{d}$ there is a copy $\mathcal{B}$ of $\mathcal{A}$ of degree $\mathbf{d}$.*

This theorem is the strongest possible. Goncharov built a homogeneous model $\mathcal{A}$ of T with a $\mathbf{0}$-basis but no decidable copy where $S(T)$ consists only of computable types.

Theorem 5.3. (Goncharov [7]) *There exists a complete decidable theory T with $S(T)$ consisting of computable types and a homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis but no decidable copy.*

Definition 5.4. Let $p(\bar{x})$ be an n-type and $\varphi(\bar{x}, y)$ be an $(n+1)$-formula consistent with p. We say that an $(n+1)$-type $q(\bar{x}, y)$ is *principal over* p if there exists some formula $\psi(\bar{x}, y) \in q(\bar{x}, y)$ (called the *generator* of q over p) such that

$$q = \{\zeta(\bar{x}, y) : (\forall y)(\psi(\bar{x}, y) \to \zeta(\bar{x}, y)) \in p\}. \tag{2}$$

Harris (personal communication) noticed that one can effectively find an $(n+1)$-type that extends p and φ which is principal over p if $S(T)$ is uniformly computable. We generalize this idea to the case where all the types in $S(T)$ are simply computable. In Lemma 5.5 we show that any noncomputable degree $\mathbf{d}$ can uniformly compute an $(n+1)$-type q principal over p which contains φ from a Δ^0_0-index for p and φ.

To compute q, we use $\mathbf{d}$ to omit any nonprincipal types over p that extend the type we are building. Suppose we must decide whether an $(n+1)$-formula $\zeta(\bar{x}, y)$ is in our $(n+1)$-type $q(\bar{x}, y)$. If one of ζ or $\neg\zeta$ is inconsistent with $p(\bar{x})$, we are forced to put the consistent one in q. Otherwise, we let $\mathbf{d}$ decide which formula should be included in q by consulting the next unexamined bit of a fixed $C \in \mathbf{d}$. Since the $(n+1)$-type q we build is in $S(T)$, q is computable by assumption. If q was nonprincipal over p, we would have consulted C infinitely often in building q, thus inadvertently coding the degree $\mathbf{d}$ into q. Since q is computable, we could show $\mathbf{d}$ was computable, contradicting our assumption that $\mathbf{d} > \mathbf{0}$. Hence, q must be principal over p. Hirschfeldt first applied this slick technique to prove the analogous result for prime models [12].

Now suppose $p \in \mathbb{T}(\mathcal{A})$ for some homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis. Given a computable index for p and a formula φ, $\mathbf{d}$ can uniformly compute a type q principal over p and containing φ by Lemma 5.5 below. Since q is principal over p, there exists a generating formula ψ as in (2) which is consistent with p. Since $p \in \mathbb{T}(\mathcal{A})$, $q \in \mathbb{T}(\mathcal{A})$. Thus, we can use $\mathbf{d}$ to build a $\mathbf{d}$-monotonic extension function for the $\mathbf{0}$-basis of $\mathcal{A}$ (See *Proof of Theorem 5.2* at the end of this section). By Theorem 3.5, $\mathcal{A}$ has a $\mathbf{d}$-decidable copy, proving Theorem 5.2. We first prove the omitting types lemma.

Lemma 5.5. *Let T be a complete decidable theory where $S(T)$ consists only of computable types. Let $\mathbf{d} > \mathbf{0}$. Given $p(\bar{x}) \in S(T)$ and $\varphi(\bar{x}, y)$ a formula consistent with p, there exists a type $q(\bar{x}, y) \in S(T)$ such that*

- $\varphi \in q$
- $p \subset q$

- *q is principal over p*

Moreover, q is uniformly computable in $\mathbf{d}$ given a Δ^0_0-index for p and φ.

Proof. Let $C \in \mathbf{d}$. We build q so that it is a type of T, and for all e, if φ_e is nonprincipal over p and $\varphi \in \varphi_e$, then $\varphi_e \neq q$.

Construction.

We define $q = \cup_{s\in\omega} q_s$ in stages using C as an oracle. As usual, defining q means determining if θ_i or $\neg\theta_i$ is an element of q for all $i \in \omega$. At stage s we will have defined q up to length s. We use v_s as an indicator of how much of C we have used in the construction at stage s.

Stage 0: Let $q_0 = \emptyset$. Let $v_0 = 0$.

Stage s+1: We assume q_s has length s and is consistent with p and φ. We define q_{s+1} to have length $s+1$. Let $q_{s+1}\restriction s = q_s \restriction s$. If θ_s is not a formula in $x_0, ..., x_n$, define $q_{s+1}(s) = 0$ since θ_s is not an $(n+1)$-formula.

Otherwise, we can effectively test whether the $(n+1)$-formulas

$$\theta_s \wedge \varphi \wedge \theta_{q_{s+1}\restriction s} \text{ and } \neg\theta_s \wedge \varphi \wedge \theta_{q_{s+1}\restriction s}$$

are consistent with p. Since q_s is consistent with p and φ, one of these two formulas must be consistent with p. If only the first is consistent, let $q_{s+1}(s) = 1$ (*i.e.*, q_{s+1} contains θ_s), and if only the second is consistent, let $q_{s+1}(s) = 0$ (*i.e.*, q_{s+1} contains $\neg\theta_s$). Set $v_{s+1} = v_s$.

If both the formulas are consistent, let $q_{s+1}(s) = C(v_s)$ and define $v_{s+1} = v_s + 1$. In other words, $\theta_s \in q_{s+1}$ if and only if $v_s \in C$.

End Construction.

Verification.

By construction q contains p and φ, and q is uniformly computable in $\mathbf{d}$ from a computable index for p and φ. Moreover $q \in S(T)$ and hence is computable.

Suppose for a contradiction that φ_e is a nonprincipal type over p consistent with φ and $q = \varphi_e$. Hence $\varphi_e = q$ is total and computable. Since $\varphi_e = q$ is nonprincipal over p, there were infinitely many stages in the above construction where C decided which formula to place in q. Note that the construction is computable except for when C is consulted. We show by induction that C is computable. Suppose we have computed $C\restriction n$. To decide whether $n \in C$, we can follow the construction computably to the stage s where the nth digit of C is consulted (*i.e.*, $n = v_{s-1}$). Then $n \in C$ if and only if $\theta_{s-1} \in q$. Since q is computable, C is computable, a contradiction. Thus q is a principal type over p containing φ. □

Proof of Theorem 5.2. Using Lemma 5.5, we can build a monotone extension function for a given **0**-basis X in any nonzero degree. If p is an n-type realized in $\mathcal{A}$ and φ is an $(n+1)$-ary formula consistent with p, then the amalgamating $(n+1)$-type q constructed in Lemma 5.5 must also be realized in $\mathcal{A}$ and hence be listed in X. This follows since $(\exists y)\psi \in p$ where ψ is the generator for q over p and since any tuple which realizes ψ realizes all of q. To build the monotone extension function g in some nonzero degree $\mathbf{d}$, let p_i be an n-type in X consistent with the $(n+1)$-formula θ_j. By Lemma 5.5 and the comment above, there exists a least indexed $(n+1)$-type p_l in X that is uniformly computable in $\mathbf{d}$ and amalgamates p_i and θ_j. Define $g(i,j,s)$ to be the least k such that the type p_k in X satisfies $p_k \restriction s = p_l \restriction s$. Clearly $\lim_{s\in\omega} g(i,j,s) = l$, and g is a $\mathbf{d}$-monotone extension function for X. By Theorem 3.5, this guarantees that $dSp^e(\mathcal{A})$ contains all nonzero degrees. $\square$

We now return to assuming only that T is complete and decidable. For the remainder of the paper we make no assumptions on what types can be in $S(T)$.

6 Bounding Results

In §4 and §5, we fixed a homogeneous model $\mathcal{A}$ with a $\mathbf{d}$-basis and studied its degree spectrum. Now we will consider which degrees must be in the degree spectrum of any automorphically nontrivial homogeneous model with a **0**-basis. In other words, we will attempt to find degrees $\mathbf{d}$ that are strong enough to compute a $\mathbf{d}$-decidable copy of *any* homogeneous model with a **0**-basis. This idea gives rise to a definition.

Definition 6.1. A degree $\mathbf{d}$ is **0**-*basis homogeneous bounding* or simply **0**-*bounding* if for any automorphically nontrivial homogeneous model $\mathcal{A}$ with a **0**-basis, there exists a $\mathcal{B} \cong \mathcal{A}$ such that $\mathcal{B}$ is $\mathbf{d}$-decidable, *i.e.*, $\mathbf{d} \in dSp^e(\mathcal{A})$.

The definition above is distinct from the idea of homogeneous bounding in work by Csima, Harizanov, Hirschfeldt, and Soare [4]. In that work, a degree $\mathbf{d}$ is *homogeneous bounding* if for any complete decidable theory T, there exists *some* homogeneous model $\mathcal{A}$ of T which is $\mathbf{d}$-decidable. They exactly characterized the homogeneous bounding degrees as the degrees of Peano arithmetic. The definition of **0**-bounding requires that $\mathbf{d}$ be able to decide a copy of any homogeneous model with a **0**-basis.

We show the following result.

Theorem 6.2. *Let* $\mathbf{d}$ *be a* Δ^0_2 *degree. If* $\mathbf{d}$ *is nonlow*$_2$ *(*i.e., $\mathbf{d}'' > \mathbf{0}''$*) then* $\mathbf{d}$ *is* **0**-*basis homogeneous bounding.*

This result uses the characterization of nonlow$_2$ Δ^0_2 degrees used to prove the analogous result for prime models proved by Csima, Hirschfeldt, Knight, and Soare [5].

6.1 Nonlow$_2$ Δ^0_2 Characterization

Like in the previous theorems, the difficulty in these constructions is determining if an $(n+1)$-type amalgamates an n-type and an $(n+1)$-formula. Deciding this question comes down to determining the consistency of two infinite types. Determining consistency of a finite formula with a computable n-type is computable. Determining if an $(n+1)$-type extends an n-type and an $(n+1)$-formula, however, requires asking a Π^0_1 and hence $\mathbf{0}'$ question.

We use the following equivalence for nonlow$_2$ degrees below $\mathbf{0}'$ to obtain an approximation to the answer to these Π^0_1 questions. This equivalence is a relativization of Martin's theorem.

Theorem 6.3. (See [5]) *Assume $\mathbf{d} \in \Delta^0_2$. Then $\mathbf{d}$ is nonlow$_2$ if and only if $\mathbf{d}$ satisfies*

$$(\forall\ g \leq_{\mathrm{T}} \mathbf{0}')\ (\exists\ f \leq \mathbf{d})\ (\exists^\infty x)\ [\ g(x) \leq f(x)\ ]. \tag{3}$$

In other words, $\mathbf{d} \leq \mathbf{0}'$ is nonlow$_2$ exactly if given a $\mathbf{0}'$-computable function g, there exists a $\mathbf{d}$-computable function f such that infinitely often $f(x)$ is at least as big as $g(x)$. This theorem leads to the following definitions.

Definition 6.4. We call condition (3) the *escape property*. We say that f *escapes* g at x if $f(x) \geq g(x)$.

6.2 Proof of the 0-Bounding Theorem

Suppose we are given a $\mathbf{0}$-basis X for some homogeneous model $\mathcal{A}$. By Theorem 3.5, to show $\mathcal{A}$ has a $\mathbf{d}$-decidable copy for a nonlow$_2$ $\mathbf{d} \leq \mathbf{0}'$, we must construct a $\mathbf{d}$-monotone extension function for X. If we can uniformly compute an amalgamator from computable indices for an n-type $p \in \mathbb{T}(\mathcal{A})$ and an $(n+1)$-formula φ, we can build a monotone extension function as we did in the proof of Theorem 5.2. We will carefully define a $\mathbf{0}'$-computable function that outputs a stage by which we will witness an inconsistency between q and p or φ if $q \in \mathbb{T}(\mathcal{A})$ is not an amalgamator. By Theorem 6.3, any nonlow$_2$ degree $\mathbf{d} \leq \mathbf{0}'$ computes a function that infinitely often escapes (*i.e.*, is greater than) this $\mathbf{0}'$-computable function. We will then use this $\mathbf{d}$-computable escape function to compute a $\mathbf{d}$-monotone extension function for X. The challenge will be to define the $\mathbf{0}'$-computable function in a robust enough way to ensure that escaping it only infinitely often ensures that we will settle on a correct amalgamator.

Proof of Theorem 6.2. Let $\mathbf{d} \leq \mathbf{0}'$ be a nonlow$_2$ degree. Let $\mathcal{A}$ be a nontrivial homogeneous model with a $\mathbf{0}$-basis X of a complete decidable theory T. We show $\mathcal{A}$ has a $\mathbf{d}$-decidable copy $\mathcal{B}$. By Theorem 3.5, it suffices to show there exists a $\mathbf{d}$-monotone extension function for that X.

Let $X = \{p_i\}_{i\in\omega}$ be the computable enumeration of $\mathbb{T}(A)$. We build a $\mathbf{d}$-monotone extension function for X. We first define a set of triples that encodes which types amalgamate others. Then we use an approximation of this set to define our best guess for amalgamators at a given stage. Finally we use the escape property of nonlow$_2$ Δ^0_2 degrees to infinitely often find stages where these guess amalgamators are true amalgamators.

Let $S = \{\langle i, \alpha, j\rangle |\ (n+1)\text{-type } p_j \text{ extends } (n+1)\text{-ary } \theta_\alpha \text{ and } n\text{-type } p_i\}$.

Since $X = \{p_i\}_{i\in\omega}$ is uniformly computable, S is a Π^0_1 set. Hence there exists a computable sequence $\{S_s\}_{s\in\omega}$ such that for all $x \in \omega$, $S(x) = \lim_s S_s(x)$. We may assume for every $\alpha \in 2^{<\omega}$ and $i, s \in \omega$ where θ_α and p_i are consistent, S_s contains an element $\langle i, \alpha, j\rangle$ for some $j \in \omega$ (*i.e.*, S_s provides some guess amalgamator for such p_i and θ_α).

For all $i \in \omega$ and $\alpha \in 2^{<\omega}$ where p_i is an n-type, θ_α is an $(n+1)$- formula, and p_i is consistent with θ_α define:

- the *true amalgamator target* $y_{\langle i,\alpha\rangle} = (\mu\langle i, \alpha, j\rangle)[\langle i, \alpha, j\rangle \in S]$ and
- the *approximate amalgamator target* $y^s_{\langle i,\alpha\rangle} = (\mu\langle i, \alpha, j\rangle)[\langle i, \alpha, j\rangle \in S_s]$.

Let

(4) $\quad h(n) = (\mu s)(\forall i, |\alpha| \leq n)(\forall w \leq y^s_{\langle i,\alpha\rangle})(\forall t \geq s)[S_t(w) = S_s(w) = S(w)].$

In other words, $h(n)$ is the least stage s by which for all $i, |\alpha| \leq n$, $S_s(w)$ has settled forever for all $w \leq y^s_{\langle i,\alpha\rangle}$.

Since $\mathbf{d} \leq \mathbf{0}'$ is nonlow$_2$ and h is a Π^0_1 function, by Theorem 6.3 (the escape characterization),

$$(\exists f \leq \mathbf{d})(\exists^\infty x)[h(x) \leq f(x)].$$

We may assume that f is increasing. Let $T = \{x \in \omega | h(x) \leq f(x)\}$. By above, T is an infinite set. We speed up our computable approximation to S

by setting $\hat{S}_s = S_{f(s)}$. We now use the computable approximation $\{\hat{S}_s\}_{s\in\omega}$. We define $\hat{y}^s_{\langle i,\alpha\rangle} = y^{f(s)}_{\langle i,\alpha\rangle}$. By (4), any apparent target $\hat{y}^t_{\langle i,\alpha\rangle} = y^{f(t)}_{\langle i,\alpha\rangle}$ at a true stage $t \in T$ is the true target $y_{\langle i,\alpha\rangle}$ if $i, |\alpha| \leq t$. Since an apparent target is a true target for $t \in T$ only if $i, |\alpha| \leq t$, we will be careful to ensure that we only lay down at most s many formulas at stage s. We call T the set of *true stages*.

To build a **d**-monotone extension function for X it suffices to show that given an index i for an n-type p_i and an $(n+1)$-formula θ_β consistent with p_i, we can **d**-uniformly compute for all s an index j_s such that $j = \lim_s j_s$ exists, $p_j \upharpoonright s = p_{j_s} \upharpoonright s$, and p_j amalgamates p_i and θ_β. (To then find the amalgamator for p_i and θ_k, we first computably find a θ_β such that $|\beta| = k+1$, $\beta(k) = 1$ and θ_β is consistent with p_i.) This is equivalent to **d**-uniformly building an $(n+1)$-type q that amalgamates p_i and θ_β in stages such that q_s has length s and $q = p_j$ for some j.

Let p_i be an n-type and θ_β an $(n+1)$-formula consistent with p_i.

Construction.

We construct q in stages so that $q = \cup_{s\in\omega} q_s$ and $|q_s| = s$ for all s. At each stage we also have a guide $(n+1)$-formula ψ_s that determines which target $\hat{y}^s_\alpha = \hat{y}^s_{<i,\alpha>}$ we should rely on to determine q. We ensure that ψ_s always has the form θ_α for some $\alpha \in \{0,1\}^{<\omega}$ for all $s \in \omega$.

Stage 0**.** Let $q_0 = \emptyset$. Let $\psi_0 = \theta_\beta$.

Stage $s+1$**.**

Assume we are given q_s such that $|q_s| = s$ and q_s is consistent with p_i and θ_β. Let ψ_s have the form θ_γ and be consistent with p_i and q_s.

If $|\gamma| \geq s+1$, set $q_{s+1} = \gamma \upharpoonright (s+1)$ and $\psi_{s+1} = \psi_s$, *i.e.*, follow θ_γ. (Since θ_γ is consistent with q_s and $|q_s| = s$, $q_s \subseteq q_{s+1}$.)

Otherwise, if $|\gamma| < s+1$, let $m = \langle \hat{y}^{s+1}_\gamma \rangle_3$ where $\langle\rangle_3$ denotes the computable function where $\langle(\langle a,b,c\rangle)\rangle_3 = c$. (In plain language, p_m is thought to be an amalgamator for p_i and θ_γ.) Then consider $p_m \upharpoonright (s+1)$. Check if $p_m \upharpoonright (s+1)$ is consistent with p_i and $\psi_s = \theta_\gamma$ and q_s and if p_m is an $(n+1)$-type. If it is consistent and an $(n+1)$-type, set $q_{s+1} = p_m \upharpoonright (s+1)$ and set $\psi_{s+1} = \psi_s = \theta_\gamma$. *We trust the guess amalgamator*

If it is not consistent or not an $(n+1)$-type, since q_s is consistent with p_i and ψ_s and $|q_s| = s$, we can extend q_s by one digit to q_{s+1} while maintaining consistency with p_i and ψ_s and q_s. Since our target amalgamator was incorrect, we update it while respecting the choices made in q_{s+1}. Let $\psi_{s+1} = \theta_\delta$ where θ_δ is the $(n+1)$-formula θ_α with $|\alpha|$ minimal and θ_α proves the conjunction of θ_β and the formulas in q_{s+1} and is consistent with p_i.

Note that $|\delta| \leq \max(|\beta|, |q_s|)$. *We have been shown that the guess amalgamator is invalid, and hence, we must find a new guide formula ψ_{s+1} based on what we have already laid down in q_{s+1}.*

End Construction.

Verification.

We show $q = \cup_{s\in\omega} q_s$ is an $(n+1)$-type in $\mathbb{T}(\mathcal{A})$ that extends p_i and θ_β as desired. First note that for all s, $q_s \subseteq q_{s+1}$, and q_{s+1} is consistent with p_i and θ_β. Hence q is consistent with p_i and θ_β. Since $|q_s| = s$ for all s, q is total. We must show that $q \in \mathbb{T}(\mathcal{A})$ (so then $q = p_j$ for some j).

Let $|\psi_s|$ by definition be $|\gamma|$ where $\psi_s = \theta_\gamma$. At stage s, $|\psi_s| \leq \max(|\beta|, |q_s|)$. Since $|q_s| = s$ for all s, $|\psi_s| \leq s$ for all stages $s \geq |\beta|$. Since the set of true stages T is infinite, we may choose a true stage $t' \geq \max(|\beta|, i)$. Suppose $\psi_{t'} = \theta_\gamma$. Since $|\psi_{t'}|, i \leq t'$ and t' is a true stage, $\hat{y}_\gamma^{t'}$ equals y_γ a true target. Hence, $p = p_{\langle \hat{y}_\gamma^{t'} \rangle_3}$ truly amalgamates p_i and $\psi_{t'}$ and thus p_i and θ_β. Moreover, by definition, p is an $(n+1)$-type realized in $\mathcal{A}$. By construction, (since p is actually a true amalgamator for p_i and $\psi_{t'}$) $\psi_s = \psi_{t'}$ and $q_s = p \restriction s$ for all $s \geq t'$. Hence q equals p, an $(n+1)$-type realized in $\mathcal{A}$, and q extends p_i and θ_β as desired. In other words, once we have unwittingly stumbled onto a true amalgamator at such a true stage picked above, we will never abandon this target.

The construction is clearly $\mathbf{d}$-effective, and hence q is $\mathbf{d}$-uniformly computable from a computable index for p_i. By this fact, we can build the desired $\mathbf{d}$-monotone extension function for X as we did in the proof of Theorem 5.2. □

Thus if $\mathbf{d} \leq \mathbf{0}'$ and $\mathbf{d}$ is nonlow$_2$, there exists a copy $\mathcal{B}$ of $\mathcal{A}$ of degree $\mathbf{d}$, *i.e.*, $\mathbf{d} \in dSp^e(\mathcal{A})$, for any automorphically nontrivial homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis.

6.3 Negative Results on 0-Bounding Degrees

In a later paper [15], we will show that the nonlow$_2$ degrees exactly characterize the $\mathbf{0}$-bounding degrees within the Δ_2^0 degrees.

Theorem 6.5. (Lange [15]) *Let $\mathbf{d} \leq \mathbf{0}'$. The degree $\mathbf{d}$ is $\mathbf{0}$-bounding if and only if $\mathbf{d}$ is nonlow$_2$.*

Theorem 6.2 gives one direction of this result. The other direction is an extension of the result by Goncharov [7], Peretyat'kin [23], and Millar [20] that there exists a homogenous model with a $\mathbf{0}$-basis but no decidable copy. Specifically, given a degree $\mathbf{d} \leq \mathbf{0}'$ which is low_2, we construct a homogeneous $\mathcal{A}$ with a $\mathbf{0}$-basis but no $\mathbf{d}$-monotone extension function. By Theorem 3.5, $\mathcal{A}$ has no $\mathbf{d}$-decidable copy $\mathcal{B}$. Therefore $\mathbf{d}$ is not a $\mathbf{0}$-bounding degree.

The characterization in Theorem 6.5 is analogous to the following result about prime models.

Theorem 6.6. (Csima, Hirschfeldt, Knight, Soare [5]) *Let* $\mathbf{d} \in \Delta^0_2$. *The degree* $\mathbf{d}$ *is nonlow*$_2$ *if and only if* $\mathbf{d}$ *is prime bounding.*

7 Conclusion

Many others are exploring related results on the degree spectra of Vaughtian (prime, saturated, and homogeneous) models beyond the ones mentioned in this paper (See [6], [11]). Simultaneously, researchers, including the author, are also studying the reverse mathematics of Vaughtian models.

7.1 Reverse Mathematics of Vaughtian Models

We [16] are exploring similar reverse mathematics questions for homogeneous models. For example, we have shown that the classical statement "A complete theory has a homogeneous model" is equivalent over RCA_0 to WKL_0 by carefully examining the related computability result by Csima, Harizanov, Hirschfeldt, and Soare [4]. As we have seen throughout this paper and noted in §1.6, many of the degree theoretic results are exactly the same for the prime and homogeneous model cases. Recently we have found some reverse mathematical connections between the prime and homogeneous cases that explain some of this degree theoretic evidence (the prime case is studied by Hirschfeldt, Shore, and Slaman in [13]). We are currently exploring these connections more deeply, and we will present them in [16].

References

[1] C.J. Ash and J.F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy*, 1st edn., Stud. Logic Found. Math., vol. 144, Amsterdam, 2000.

[2] C.C. Chang and H.J. Keisler, *Model Theory*, 3rd edn., Stud. Logic Found. Math., vol. 73, North-Holland, Amsterdam, 1990 [1st edn. 1973, 2nd edn. 1977].

[3] B. F. Csima, Degree spectra of prime models, J. Symbolic Logic, vol. 69 (2004), 430-442.

[4] B. F. Csima, V. S. Harizanov, D. R. Hirschfeldt, and R. I. Soare, Bounding homogeneous models, in preparation.

[5] B. F. Csima, D. R. Hirschfeldt, J. F. Knight, and R. I. Soare, Bounding prime models, J. Symbolic Logic, vol. 69 (2004), pp. 1117-1142.

[6] R. Epstein, Computably enumerable degrees of Vaught's models, submitted.

[7] S.S. Goncharov, Strong constructivizability of homogeneous models (Russian), Algebra i Logika, **17** (1978) 363-388, 490; [translated in: Algebra and Logic, **17** (1978) 247-263].

[8] S. S. Goncharov and A. T. Nurtazin, Constructive models of complete decidable theories, Algebra i Logika, **12** (1973), pp. 125-42, 243; [translated in: Algebra and Logic, **12** (1973), 67-77].

[9] V. S. Harizanov, Pure computable model theory, in *Handbook of Recursive Mathematics* (Yu. L. Ershov, S. S. Goncharov, A. Nerode, J. B. Remmel, eds.), Stud. Logic Found. Math., vol. 138–139, Elsevier Science, Amsterdam, 1998, pp. 3–114.

[10] L. Harrington, Recursively presentable prime models, The Journal of Symbolic Logic, vol. 39 (1974), pp. 305-9.

[11] K. Harris, Bounding saturated models, in preparation.

[12] D. R. Hirshfeldt, Computable trees, prime models, and relative decidability, Proceedings of the Amer. Math. Soc., **134** (2006), 1495-1498.

[13] D. R. Hirschfeldt, R. A. Shore, and T. A. Slaman, The atomic model theorem, in preparation.

[14] J. F. Knight, Degrees coded in jumps of orderings, J. Symbolic Logic, vol. 51 (1986), pp. 1034-1042.

[15] K. M. Lange, A Characterization of the **0**-basis Homogeneous Bounding Degrees, in preparation.

[16] K. M. Lange, Reverse mathematics of Homogeneous models, in preparation.

[17] K. M. Lange and R. I. Soare, Computability of homogeneous models, Notre Dame J. of Formal Logic, **48** no. 1 (2007), 143-170.

[18] D. Marker, *Model theory: an introduction*, Grad. texts in math., 277, New York, 2002.

[19] T. S. Millar, Foundations of recursive model theory, Ann. Math. Logic, vol. 13 (1978), 45–72.

[20] T. S. Millar, Homogeneous models and decidability, Pacific J. Math., **91** (1980), 407-418.

[21] Type structure complexity and decidability, *Trans. Amer. Math. S.*, **271** (1982), 73–81.

[22] M. Morley, Decidable models, Israel J. Math., **25** (1976), 233-240.

[23] M. G. Peretyat'kin, A criterion for strong constructivizability of a homogeneous model (Russian), Algebra i Logika, **17** (1978) 436-454, 491; [translated in: Algebra and Logic, **19** (1980) 202-229].

[24] [Soare, cta]
R. I. Soare, *Computability Theory and Applications* Springer-Verlag, Heidelberg, in preparation.

[25] R. I. Soare, *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets*, Springer-Verlag, Heidelberg, 1987.

[26] D. A. Tusupov. Numerations of homogeneous models of decidable complete theories with a computable family of types, in *Theory of Algorithms and Its Applications* (V.N. Remeslennikov ed.), Computable Systems, vol. 129.

[27] R. L. Vaught, Denumerable models of complete theories, pp. 301-21 in Infinitistic Methods (Proceedings of Symposium on Foundations of Mathematics, Warsaw, 1959), Pergamon Press, 1961.