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To Michael, Mom, and Dad

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CHAPTER 1
INTRODUCTION AND BASIC DEFINITIONS

1.1 Introduction

In 1961, Vaught [33] introduced the concepts of prime, saturated, and homogeneous models. We call such structures *Vaughtian models*. The Vaughtian models provided a different perspective on model theory that led to many new avenues of research. Inspired by these results, researchers in computability, including Goncharov, Harrington, Peretyat'kin, Morley, and others, began studying the computable content of related model theoretic constructions and structures. In order to effectivize these results, they considered complete *decidable* theories and the *countable* models of these theories. In particular, they focused on studying the effectiveness of Vaughtian models.

Convention 1.1.1. *We assume throughout that all theories T are complete and decidable (CD) and all models $\mathcal{A}$ of T are countable.*

In addition, we assume that all models we consider in this paper are *automorphically nontrivial*, defined in Definition 1.2.13, since these models are the only interesting ones as we see in §1.2.8.

1.1.1 Early Effectiveness Results on Vaughtian Models

Let T be a complete decidable (CD) theory. Early researchers wanted to study the effectiveness of various countable models associated with T. A model $\mathcal{A}$ is called $\mathbf{d}$-*computable* if its atomic diagram $D^a(\mathcal{A})$ is $\mathbf{d}$-computable; $\mathcal{A}$ is called $\mathbf{d}$-*decidable* if its elementary diagram $D^e(\mathcal{A})$ is $\mathbf{d}$-computable. We discuss these and related definitions from computable model theory in more detail in §1.2.8. Although researchers knew early on that every CD theory T has *some* decidable model, they constructed CD theories T and countable models $\mathcal{A}$ of T with no decidable or even computable copies. Researchers began looking for added conditions on a countable model $\mathcal{A}$ of a CD theory T that would guarantee the existence of a decidable copy $\mathcal{B}$ of $\mathcal{A}$. Vaughtian models by definition have a precise structure. In particular, a countable theory has exactly one countable prime or saturated model up to isomorphism if these models exist. Although such theories might have many non-isomorphic countable homogeneous models, countable homogeneous models which realize the same types are isomorphic (see Theorem 1.2.9). Since Vaughtian models have these uniqueness properties, researchers asked whether decidable Vaughtian models always exist for CD theories with these models. They proved, however, that decidable Vaughtian models do not always exist. Millar [22] and Goncharov and Nurtazin [9] gave examples of complete atomic decidable (CAD) theories T with no computable prime model (and hence no decidable model). Millar [22] also built a complete decidable theory with all types computable but which has no decidable saturated model. Millar [23], Goncharov [8], and Peretyat'kin [27] provided examples of a homogeneous model with a uniformly computable list of types that has no decidable copy.

These researchers additionally characterized when the Vaughtian structures have decidable copies. Harrington [11] and independently Goncharov and Nurtazin [9] described an exact criterion for when a CAD theory T has a decidable prime model, and Morley [26] and Millar [22] independently showed that the same criterion works in the saturated case. Goncharov [8] and Peretyat'kin [27] independently provided a computability condition on the type spectrum of a homogeneous model which is equivalent to the existence of a decidable copy of the given homogeneous model. The prime and saturated criteria are discussed in §1.3.1, and the homogeneous characterization is examined in §1.3.2.

1.1.2 Degrees between $\mathbf{0}$ and $\mathbf{0}'$

The work of the above researchers showed that a decidable copy of a prime, saturated, or specific homogeneous model does not necessarily exist. Thus, the Turing degree

$\mathbf{0}$ is weak in this sense with respect to Vaughtian models. On the other hand, using the characterizations mentioned above, it is easy to see that any Vaughtian model of a theory satisfying reasonable computability conditions has a $\mathbf{0}'$-decidable copy.

With the solution to Post's Problem [28], attention in computability turned to studying the intermediate degrees between $\mathbf{0}$ and $\mathbf{0}'$ and other classes of degrees. Since then, many of these degree classes and their corresponding properties have played an important role in modern computability theory.

Some examples of such degree classes include the high$_n$, low$_n$, and Peano Arithmetic degrees. A degree $\mathbf{d} \leq \mathbf{0}'$ is *high*$_n$ if $\mathbf{d}^{(n)} = \mathbf{0}^{(n+1)}$, the highest possible value. A degree $\mathbf{d}$ is *low*$_n$ if $\mathbf{d}^{(n)} = \mathbf{0}^{(n)}$, the lowest possible value, and $\mathbf{d}$ is *high* if it is *high*$_1$ and *low* if it is *low*$_1$. A degree $\mathbf{d}$ is a *Peano Arithmetic (PA) degree* if $\mathbf{d}$ is the degree of a complete extension of the effectively axiomatized theory of Peano Arithmetic.

Given the $\mathbf{0}$ and $\mathbf{0}'$ dichotomy seen in early computability results on Vaughtian models, recent research has focused on studying when an intermediate or other degree decides a copy of a Vaughtian model. Others have examined this question in the prime and saturated model cases. Here we will examine which intermediate degrees can decide copies of a given homogeneous model satisfying certain computability conditions.

1.1.3 Recent results on prime models

Let T be a complete atomic decidable (CAD) theory. As mentioned earlier, T has a $\mathbf{0}'$-decidable prime model. Csima [4] greatly improved this result by showing that T always has a prime model decidable in some low degree. Csima, Hirschfeldt, Knight, and Soare [6] studied prime bounding degrees. A degree $\mathbf{d}$ is *prime bounding* if for *any* CAD theory T, $\mathbf{d}$ decides a prime model of T. They showed that the Δ_2^0 prime bounding degrees are exactly the nonlow$_2$ degrees below $\mathbf{0}'$. Csima [4] also studied the case where T was not only a CAD theory but also had only computable types. Hirschfeldt [12] gave a surprising proof generalizing her result to show that in this case *any* noncomputable degree can decide a prime model of T. Since every prime model is homogeneous, it is natural to ask whether these results can be extended to homogeneous models in general.

1.1.4 Effectivizing Homogeneous Models

One major difference between the prime and homogeneous model cases is that a CAD theory T has a single prime model but a CD theory can have many nonisomorphic homogeneous models. Thus, there are two natural approaches to considering the effectiveness of homogeneous models for a CD theory T. One approach would be to ask whether T has *any* $\mathbf{d}$-decidable homogeneous models for a particular degree $\mathbf{d}$. This question was completely answered by Csima, Harizanov, Hirschfeldt, and Soare in [5]. They showed that a degree $\mathbf{d}$ bounds the elementary diagram of *some* homogeneous model of every CD theory if and only if $\mathbf{d}$ is a PA degree. PA degrees can compute completions of consistent computably axiomatizable partial theories. PA degrees then are intuitively homogeneous bounding because the classical construction of a homogeneous extension of a model requires repeatedly finding such completions.

The second approach to studying homogeneous models, begun by Goncharov, Peretyat'kin, and Millar, is to fix a homogeneous model $\mathcal{A}$ of a CD theory T and ask whether $\mathcal{A}$ has a $\mathbf{d}$-decidable copy for various degrees $\mathbf{d}$. Suppose $\mathcal{A}$ has a decidable copy. Then all the types realized in $\mathcal{A}$ must be computable and indeed are uniformly computable. We call such a uniformly computable listing of the types a $\mathbf{0}$-*basis* of $\mathcal{A}$. We define a $\mathbf{0}$-basis and the more general $\mathbf{d}$-basis in §1.3.1. Hence, in order for $\mathcal{A}$ to even possibly have a decidable copy, $\mathcal{A}$ must have a $\mathbf{0}$-basis. Morley knew that the existence alone of a $\mathbf{0}$-basis for the appropriate class of types is sufficient to guarantee the existence of a decidable prime or saturated model. Thus, Morley [26] asked whether the existence of a $\mathbf{0}$-basis for a homogeneous $\mathcal{A}$ guarantees the existence of a decidable copy of $\mathcal{A}$. As mentioned earlier, Millar [23], Goncharov [8], and Peretyat'kin [27] gave counterexamples showing that a $\mathbf{0}$-basis alone is not enough. Moreover, Goncharov and Peretyat'kin described an additional function on the $\mathbf{0}$-basis which exactly characterizes when a decidable copy of the homogeneous $\mathcal{A}$ exists. In addition to a $\mathbf{0}$-basis, the extra requirement is a computable function which takes an index of a type in the basis and an index of a formula consistent with the given type, and approximates monotonically a type in the basis containing the original type and the formula. We call such a function a *monotone extension function*, and we define these functions more formally in Definition 1.3.6. By relativizing Goncharov and Peretyat'kin's result, we see that $\mathcal{A}$ has a $\mathbf{d}$-decidable model exactly if $\mathcal{A}$ has a basis uniformly computable in $\mathbf{d}$ with a $\mathbf{d}$-computable monotone extension function. We discuss their characterization and this relativization more in §1.3.2.

1.1.5 Positive Results on Copies of Homogeneous Models

Homogeneous Low Basis Results

Let T be a complete decidable theory. Let $\mathcal{A}$ be a homogeneous model of T with a $\mathbf{0}$-basis. We show that there exists a $\mathcal{B} \cong \mathcal{A}$ such that $\mathcal{B}$ is decidable in a low degree. We can generalize this result to any homogeneous $\mathcal{A}$ where $\mathcal{A}$ has a $\mathbf{0}'$-*basis*, *i.e.*, a listing of the types realized in $\mathcal{A}$ where $\mathbf{0}'$ uniformly computes a Δ^0_0-index for each type in the list. We prove the following theorem in §2.1.

Theorem 2.1.1. *Let T be a CD theory and $\mathcal{A}$ a homogeneous model of T with a $\mathbf{0}'$-basis. Then $\mathcal{A}$ has an isomorphic copy $\mathcal{B}$ decidable in a low degree.*

Given a $\mathbf{0}'$-basis X for $\mathcal{A}$, we build simultaneously a low degree $\mathbf{d}$ and a basis $Y = X$ for $\mathcal{B}$ with an extension function which is uniformly computable in $\mathbf{d}$. Then there exists a $\mathbf{d}$-decidable $\mathcal{B}$ isomorphic to $\mathcal{A}$ by the relativization of Goncharov and Peretyat'kin's result. We build these objects by forcing the jump using a $\mathbf{0}'$ oracle. At stage s of the construction, we ensure that the first s rows of basis Y are completely filled in and that the first s types in X are placed in Y. Then the forcing must be done respecting these types and the extension function.

Any prime model of a CAD theory has a $\mathbf{0}'$-basis since $\mathbf{0}'$ can uniformly list the formulas which are atoms and hence computably generate a principal type. Thus, Theorem 2.1.1 gives as a corollary Csima's analogous result for prime models.

Corollary 2.1.2. (Csima [4]) *Let T be a complete atomic decidable theory. Then T has a prime model $\mathcal{A}$ decidable in a low degree.*

Although some prime and homogeneous models satisfying the reasonable computability assumptions we discussed do not have decidable copies, these theorems together demonstrate that all such models do have degree theoretically weak isomorphic copies.

Case where T has types all computable (TAC)

In §2.1 and §2.3, we only assume that T is a complete decidable theory and the homogeneous model $\mathcal{A}$ has a $\mathbf{0}$ or $\mathbf{0}'$-basis. We only put computability restrictions on the types realized in the model $\mathcal{A}$. Surprisingly the computability of the types *not* realized in $\mathcal{A}$ actually impacts the decidability of $\mathcal{A}$. Let $S(T)$ denote all the types consistent with a theory T. If we also assume that all types in $S(T)$ are computable, not only the types realized in $\mathcal{A}$, we are in a position to omit the types not realized in $\mathcal{A}$. In §2.2, we prove:

Theorem 2.2.3. *Let T be a complete decidable theory with all types in $S(T)$ computable. Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}$-basis. Then $\mathcal{A}$ has an isomorphic copy $\mathcal{B}$ decidable in any nonzero degree.*

Harris (in a personal communication) noticed that given a type p and a formula consistent with p, there exists a type q principal over p (a notion we formalize in Definition 2.2.5) containing the formula and extending p. Using this idea, we then apply Hirschfeldt's elegant omitting types technique to build a monotone extension function for the $\mathbf{0}$-basis, thereby obtaining Theorem 2.2.3. Hirschfeldt used this technique originally to obtain the analogous result for prime models [12].

$\mathbf{0}$*-basis homogeneous bounding*

We next consider the $\mathbf{0}$-basis homogeneous bounding degrees. We say a degree $\mathbf{d}$ is $\mathbf{0}$*-basis homogeneous bounding* if any homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis has a $\mathbf{d}$-decidable isomorphic copy. In §2.3 we show the following theorem.

Theorem 2.3.2. *Let* $\mathbf{d} \leq \mathbf{0}'$*. If* $\mathbf{d}$ *is nonlow*$_2$ *then* $\mathbf{d}$ *is* $\mathbf{0}$*-homogeneous bounding.*

Given any nonlow$_2$ degree $\mathbf{d} \leq \mathbf{0}'$ and a $\mathbf{0}$-basis for a homogeneous model $\mathcal{A}$, we will construct a $\mathbf{d}$-computable monotone extension function. Thus given a computable index for a type and a formula consistent with that type, $\mathbf{d}$ must enumerate a type realized in $\mathcal{A}$ which contains the given type and formula. Since determining whether a single formula is in a computable type is computable, the problem is using $\mathbf{d}$ to enumerate a type in the $\mathbf{0}$-basis which contains the original type. We will see that Δ^0_2 nonlow$_2$ degrees can approximate this information well enough to compute a monotone extension function.

1.1.6 Negative Results on Copies of Homogeneous Models

In Chapter 3, we will show that below $\mathbf{0}'$, the nonlow$_2$ degrees exactly characterize the $\mathbf{0}$-homogeneous bounding degrees.

Theorem 3.1.2 Let $\mathbf{d} \leq \mathbf{0}'$. The degree $\mathbf{d}$ is $\mathbf{0}$-bounding if and only if $\mathbf{d}$ is nonlow$_2$.

We proved the forward direction in Theorem 2.3.2. The remaining direction, which we prove in Theorem 3.5.4, is an extension of the result by Goncharov [8], Peretyat'kin [27], and Millar [23] that there exists a homogeneous model with a $\mathbf{0}$-basis but no decidable copy. Given a degree $\mathbf{d} \leq \mathbf{0}'$ which is low$_2$, we construct a homogeneous $\mathcal{A}$ with a $\mathbf{0}$-basis but no $\mathbf{d}$-monotone extension function. This full characterization also is analogous to the result about prime bounding degrees by Csima, Hirschfeldt, Knight, and Soare [6].

1.1.7 Reverse Mathematics of Homogeneous Models

All the theorems presented in this paper on homogeneous models have analogous counterparts for the prime model case. These earlier results on prime models were the original motivation for studying homogeneous models. Since all prime models are homogeneous, we hoped that results on homogeneous models might give the results on prime models as corollaries. In one case, it was evident that this was the case.

Theorem 2.1.1, which says any homogeneous model with a $\mathbf{0}'$-basis has a low copy, does give the analogous result for prime models as a corollary (See Corollary 2.1.2). However, it was not clear how Theorems 2.2.3 and 2.3.2 on homogeneous models related to their prime counterparts. Recently, we have discovered some underlying connections between the prime and homogeneous cases and are working on fully developing these ideas. We present our current results in Chapter 4.

1.2 Basic Definitions and Techniques

Let $\mathcal{L}$ be a countable language and T be a complete theory on $\mathcal{L}$. Here we fix our notation for various structures under consideration and discuss some basic model theory. See [2] or [20] for an introduction to the model theory found here.

1.2.1 The Lindenbaum Algebra $B_n(T)$ of Formulas

Types, maximal consistent sets of formulas on a finite set of free variables, play a key role in understanding homogeneous models. We define types formally in Definition 1.2.4. We begin by defining our conventions about formulas in a way that will make types easy to effectivize.

Definition 1.2.1. [Lindenbaum Algebra] (i) Let $F_n(\mathcal{L})$ be the set of the formulas $\theta(x_0, \ldots, x_{n-1})$ of $\mathcal{L}$ with free variables included in $x_0, \ldots, x_{n-1}$. Let

$$F(\mathcal{L}) = \cup_{n \geq 0} F_n(\mathcal{L}).$$

(ii) The *equivalence class* of $\theta(\overline{x}) \in F_n(\mathcal{L})$ under the complete theory T is

$$\theta^*(\overline{x}) \quad = \quad \{ \gamma(\overline{x}) \ : \quad (\forall \overline{x}) [\theta(\overline{x}) \ \leftrightarrow \ \gamma(\overline{x})] \in T \},$$

and the *Lindenbaum algebra* $B_n(T)$ consists of these equivalence classes with the induced conjunction, disjunction, and negation operations. We often identify $\theta(\overline{x})$ and its equivalence class $\theta^*(\overline{x})$.

(iii) Let $\{\, \theta_i(\overline{x}) \,\}_{i \in \omega}$ be an effective listing of $F_n(\mathcal{L})$. For every string $\alpha \in 2^{<\omega}$ define

$$\theta_\alpha(\overline{x}) \quad = \quad \bigwedge \{\, \theta_i^{\alpha(i)}(\overline{x}) \ : \ i < |\alpha| \,\}$$

where $\theta^1 = \theta$ and $\theta^0 = \neg\,\theta$.

1.2.2 Trees and Π^0_1 Classes

Our computability theoretic results and the tools they require make considerable use of trees. Types will be viewed as paths on trees, and all the familiar terminology and results about types will be developed first for trees. We begin with basic definitions about trees. We then relate these definitions to various kinds of theories and models. More information about trees and computability in general can be found in [30] and [31].

Definition 1.2.2. [Trees, Part 1] (i) A *tree* $\mathcal{T} \subseteq 2^{<\omega}$ is a subset of $2^{<\omega}$ closed under initial segments, *i.e.*, $\tau \subset \sigma \in \mathcal{T}$ implies $\tau \in \mathcal{T}$. We define the set of (infinite) *paths*,

$$[\,\mathcal{T}\,] \;=\; \{\; f \;:\; f \in 2^{\omega} \quad \& \quad (\forall x)[\; f {\restriction} x \in \mathcal{T}\;]\;\}. \tag{1.1}$$

(ii) *Cantor space* is 2^ω with the following topology. Let $\sigma \subset f$ denote that f extends σ. For every $\sigma \in 2^{<\omega}$ define the *basic open set*,

$$\mathcal{U}_\sigma = \{\; f \;:\; f \in 2^\omega \quad \& \quad \sigma \subset f \;\}$$

(iii) A class $\mathcal{C} \subset 2^\omega$ is *closed* if $2^\omega - \mathcal{C}$ is open or equivalently if $\mathcal{C} = [\,\mathcal{T}\,]$ for some tree as in (1.1).

(iv) If $\mathcal{C} = [\,\mathcal{T}\,]$ for some *computable* tree $\mathcal{T}$, then $\mathcal{C}$ is *effectively closed*, and is called a Π^0_1-*class*.

(v) A tree $\mathcal{T}$ is *extendible* if every node σ can be extended to an infinite path in $\mathcal{T}$, *i.e.*,

$$(\forall \sigma \in \mathcal{T})\,(\exists f \supset \sigma)\,[\; f \in [\,\mathcal{T}\,]\;].$$

A Π^0_1-class $\mathcal{C}$ is *extendible* if $\mathcal{C} = [\,\mathcal{T}\,]$ for some computable extendible tree $\mathcal{T}$, and $\mathcal{C}$ is *nonextendible* otherwise.

1.2.3 The Stone Space $S_n(T)$ as Paths in the Tree $\mathcal{T}_n(T)$

We now formally define types and describe how they can be viewed as paths through a particular tree contained in $2^{<\omega}$.

Definition 1.2.3. Let T be a complete $\mathcal{L}$-theory.

(i) A formula $\theta(\overline{x}) \in F_n(\mathcal{L})$ is *consistent with* T if $T \cup (\exists\overline{x})\theta(\overline{x})$ is consistent, *i.e.*, if $(\exists\overline{x})\theta(\overline{x}) \in T$, because T is complete. Let $F_n(T)$ be the subset of $F_n(\mathcal{L})$ consisting of all formulas consistent with T.

(ii) Define the tree of n-ary formulas consistent with T

$$\mathcal{T}_n(T) \;=\; \{\,\theta_\alpha(\overline{x}) \in F_n(\mathcal{L}) \;:\; \alpha \in 2^{<\omega} \quad \& \quad (\exists\overline{x})\theta_\alpha(\overline{x}) \in T\,\}.$$

If $\alpha \subset \beta$, then we say that β lies *below* α, that θ_β *extends* θ_α and contains more information. (Note that the equivalence classes $\{\,\theta^*_\alpha \;:\; \alpha \in \mathcal{T}_n(T)\}$ generate the Lindenbaum algebra $B_n(T)$ under the Boolean operations.)

(iii) We regard α as an *index* of θ_α. Define the tree of indices,

$$\widehat{\mathcal{T}}_n(T) = \{\,\alpha \;:\; \theta_\alpha \in \mathcal{T}_n(T)\,\}.$$

The trees $\mathcal{T}_n(T)$ and $\widehat{\mathcal{T}}_n(T)$ are effectively isomorphic but $\widehat{\mathcal{T}}_n(T) \subseteq 2^{<\omega}$ and is notationally simpler. Hence, any definitions or results on trees $\widehat{\mathcal{T}} \subset 2^{<\omega}$ automatically carry over to $\mathcal{T}_n(T)$. We eventually suppress explicit mention of $\widehat{\mathcal{T}}_n(T)$ and simply identify α and $\theta_\alpha(\overline{x})$.

Definition 1.2.4. [Types and the Stone Space] (i) An *n-type of* T is a maximal consistent subset p of formulas of $F_n(T)$. There is a 1-1 correspondence between paths $f \in [\,\widehat{\mathcal{T}}_n(T)\,] \subset 2^\omega$ and the corresponding types $p_f \in [\,\mathcal{T}_n(T)\,]$ where

$$p_f = \{\;\theta_\alpha(\overline{x}) \;:\; \theta_\alpha(\overline{x}) \in \mathcal{T}_n(T) \quad \& \quad \alpha \subset f\;\}.$$

(ii) $S_n(T)$ is the set of all n-types of T, with the usual topology as in Definition 1.2.2, and it is also called the *Stone space* (*i.e.*, the dual space of the Boolean algebra $B_n(T)$). The clopen sets of the Stone space are given by $\mathcal{U}_\sigma$ as defined in Definition 1.2.2 (ii).

(iii) Define $S(T) = \cup_{n\geq 1} S_n(T)$.

(We can also regard $S(T)$ as homeomorphic to a subset of 2^ω as follows. Build a tree $\mathcal{T} \subset 2^{<\omega}$ by putting $1^n{}^\frown 0$ on $\mathcal{T}$ and then putting an isomorphic copy of $\widehat{\mathcal{T}}_n(T)$ above it on $\mathcal{T}$.)

(iv) Hence, $S_0(T)$ is the set of complete extensions of T, *i.e.*, 0-types of T. Since we assume T to be a complete theory, $S_0(T)$ consists of a single path.

1.2.4 Atomic Trees and Principal Types

We continue Definition 1.2.2 with properties of trees which will also apply to types. Given $\beta \in 2^n$, let the *length* of β be $|\beta| = n$. We say β converges on k, denoted $\beta(k)\downarrow$, if $k < |\beta|$. If $\beta(k)\downarrow$ and $\beta(k) = i$, we identify $\beta(k)\downarrow$ with the value i.

We begin by defining principal paths and atomic trees. Similar to before, we see how the model theoretic definition of an atomic theory (given below) is connected to the analogous definition for a tree.

Definition 1.2.5. [Trees, Part 2] Let $\mathcal{T} \subseteq 2^{<\omega}$ be an extendible tree.

(i) Nodes β, γ are *incomparable*, written $\beta \,|\, \gamma$, if $(\exists k)[\beta(k)\downarrow \neq \gamma(k)\downarrow]$.

(ii) Nodes $\beta, \gamma \in \mathcal{T}$ *split* node α on $\mathcal{T}$ if $\alpha \subset \beta$, $\alpha \subset \gamma$, and $\beta \,|\, \gamma$.

(iii) Node $\alpha \in \mathcal{T}$ is an *atom* if no extensions split α on $\mathcal{T}$. If α is an atom then the unique extension $f \supset \alpha$ on $\mathcal{T}$ is an *isolated* or *principal path* of $[\mathcal{T}]$, α is a *generator* of f, and we say α *isolates* or *generates* f. Note that f is isolated in the topological sense by the basic open set $\mathcal{U}_\alpha$ because

$$\mathcal{U}_\alpha \ \cap \ [\mathcal{T}] \quad = \quad \{f\}.$$

(iv) For a complete theory T, we call an n-type p *principal* or *isolated* if the corresponding path $f \in [\widehat{\mathcal{T}}_n(T)]$ (where $p_f = p$) is principal. We call any formula φ corresponding to a generator α of f a *generator* of p. Let

$$S^P(T) = \{\, p \;:\; p \text{ is a principal type of } S(T)\,\}.$$

(v) $\mathcal{T}$ is *atomic* if for every $\beta \in \mathcal{T}$ there is an atom $\alpha \supseteq \beta$ with $\alpha \in \mathcal{T}$, or equivalently if the isolated points of $[\mathcal{T}]$ are dense in $[\mathcal{T}]$.

(vi) A complete theory T is *atomic* if the tree $\mathcal{T}_n(T)$ is atomic for every $n \geq 1$.

1.2.5 The Type Spectrum $\mathbb{T}(\mathcal{A})$ of a model $\mathcal{A}$

In §1.2.6 we define what it means for a model to be homogeneous. We will also see that a countable homogeneous model is uniquely determined by the set of types that are satisfied by elements in the model. In this section we define some notation and terminology for such types.

Definition 1.2.6. Let T be a theory and $\mathcal{A}$ a model of T.

(i) An n-tuple $\overline{a} \in A$ *realizes* an n-type $p(\overline{x}) \in S_n(T)$ if $\mathcal{A} \models \theta(\overline{a})$ for all $\theta(\overline{x}) \in p(\overline{x})$. In this case we say that $\mathcal{A}$ *realizes* p via $\overline{a}$.

(ii) Define the *type spectrum* of $\mathcal{A}$

$$\mathbb{T}(\mathcal{A}) = \{\, p \;:\; p \in S(T) \quad \& \quad \mathcal{A} \text{ realizes } p\,\} \quad \text{and}$$

(iii) $\mathbb{T}_n(\mathcal{A}) = \mathbb{T}(\mathcal{A}) \cap S_n(T)$, the n-types realized in $\mathcal{A}$.

In early papers some authors in computable model theory had used $S(\mathcal{A})$ in place of $\mathbb{T}(\mathcal{A})$. However, this conflicts with the standard usage in ordinary model theory where Marker [20] defines $S_n^{\mathcal{A}}(Y)$ to be the set of n-types in the theory $\mathrm{Th}_Y(\mathcal{A})$ of $\mathcal{A}$ for some $Y \subseteq |\mathcal{A}|$ (p. 115). The use of $\mathbb{T}_n(\mathcal{A})$ is different from Marker's $S_n^{\mathcal{A}}(Y)$ because: (1) we consider only *pure* types in the original language $\mathcal{L} = \mathcal{L}(T)$ and do not allow any extra names $Y \subseteq |\mathcal{A}|$ to be added; (2) we consider only those types actually *realized* in $\mathcal{A}$ not merely those consistent with $\mathrm{Th}_Y(\mathcal{A})$. Marker has no notation for our $\mathbb{T}_n(\mathcal{A})$.

As we will now see, homogeneity can be described in terms of the behavior of types. Moreover, $\mathbb{T}(\mathcal{A})$ plays an important role in understanding the isomorphism class of a given homogeneous model.

1.2.6 Homogeneous Models and their Uniqueness Theorem

Let $\mathcal{A} \equiv \mathcal{B}$ denote elementary equivalence, $\mathcal{A} \cong \mathcal{B}$ denote isomorphism, and Aut $\mathcal{A}$ denote the group of automorphisms of $\mathcal{A}$.

Definition 1.2.7. A model $\mathcal{A} \vDash T$ is *homogeneous* if for all n-tuples $\overline{a}$ and $\overline{b}$, if $\overline{a}$ and $\overline{b}$ realize the same n-type, *i.e.*, $(\mathcal{A}, \overline{a}) \equiv (\mathcal{A}, \overline{b})$, then

$$(\exists \Phi \in \text{Aut } \mathcal{A})\,[\, \Phi(\overline{a}) = \overline{b}\,].$$

Recall that prime and saturated models are necessarily homogeneous.

Definition 1.2.8. Let T be a complete theory.

(i) A model $\mathcal{A}$ of T is *prime* if $\mathcal{A}$ can be elementarily embedded in any other model $\mathcal{B}$ of T.

(ii) $\mathcal{A}$ is *weakly saturated* if $\mathbb{T}(\mathcal{A}) = S(T)$, and a countable model $\mathcal{A}$ is (countably) *saturated* if $\mathcal{A}$ realizes every type defined over any finite set $F \subseteq A$.

Vaught [33] proved that for any countable theory T, a model $\mathcal{A}$ of T is prime if and only if $\mathcal{A}$ is countable and *atomic*, *i.e.*, realizes only principal types. This is often taken as the defining property of prime models of countable theories. When we write "prime" we shall always mean "countable and atomic."

One of the most pleasant properties of all homogeneous models is the following result which demonstrates the usefulness of the notion $\mathbb{T}(\mathcal{A})$. (See Marker [20] Theorem 4.3.23.)

Theorem 1.2.9. [Uniqueness Theorem for Homogeneous Models] *Given a countable complete theory T and homogeneous models $\mathcal{A}$ and $\mathcal{B}$ of T of the same cardinality*

$$\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B}) \implies \mathcal{A} \cong \mathcal{B}. \tag{1.2}$$

Hence, for an arbitrary homogeneous model $\mathcal{A}$ the isomorphism type of $\mathcal{A}$ is completely determined by the types realized in the model. This will be very useful for passing from a homogeneous model $\mathcal{A}$ to an isomorphic copy $\mathcal{B}$ which satisfies certain computability properties. We simply construct a $\mathbf{d}$-decidable model $\mathcal{B}$ with the desired properties such that: (1) $\mathcal{B}$ is homogeneous; and (2) the types realized in $\mathcal{B}$ satisfy $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$.

1.2.7 Presenting Types for a Complete Decidable Theory

From now on we assume that T is a complete decidable (CD) theory in a computable language $\mathcal{L}$. We now extend and sharpen the noneffective definitions for formulas and types presented in §1.2.1 for the computable case.

First we observe that determining whether a formula is consistent with a complete decidable theory or a computable type is computable.

Proposition 1.2.10. *(i) If T is a complete decidable (CD) theory then $F_n(T)$, and more precisely the tree $\mathcal{T}_n(T)$ are both decidable.*

(ii) If T is a CD theory, $n > 1$, and p is a computable $(n-1)$-type then $F_n(T)/p$ and the tree $\mathcal{T}_n(T)/p$ are computable where these are defined to be the formulas of $F_n(T)$ that are consistent with p.

Proof. (i) Since T is complete a formula $\theta(\overline{x}) \in F_n(\mathcal{L})$ is consistent with T iff $(\exists \overline{x})\, \theta(\overline{x}) \in T$. Therefore, since T is decidable $F_n(T)$ is also decidable.

(ii) We can do the same as in (i) but now we use the computability of both T and p to test the projections for consistency with a given formula $\theta(\overline{x}) \in F_n(T)$. □

We define an effective enumeration of all formulas consistent with T and show how we describe types using this enumeration. We will use this fixed universal enumeration throughout the remainder of this paper.

Definition 1.2.11. (i) Given the fixed CD theory T let $\{\theta_i\}_{i\in\omega}$ be an effective numbering of $F(T) = \cup_n F_n(T)$, all the formulas consistent with T.

(ii) For every $n > 0$ and n-type p we may assume p decides every k-ary formula $\theta(\overline{x})$ for every $k < n$ as follows. Define

$$\theta'(x_0, \ldots, x_{n-1}) = \theta(\overline{x}) \wedge (\bigwedge\!\!\!\bigwedge_{j<n} (x_j = x_j)).$$

Add θ to p just if $\theta' \in p$ already. Now associate with p a function $f \in 2^\omega$ such that $f(i) = 1$ if $\theta_i \in p$. Hence, every type corresponds to a function over *all* formulas $\theta_i \in F(T)$ but clearly $f_p(j) = 0$ if θ_j is a k-ary formula for $k > n$.

(iii) Let p be an n-type and q be a k-type for $k < n$. Then p and q are *inconsistent* if there exists a k-ary formula $\theta_i(\overline{x})$ such that $f_p(i) \neq f_q(i)$, and are *consistent* otherwise. If p and q are computable types then their consistency is a Π^0_1 condition.

(iv) For any type $p \in S(T)$ define $p\restriction s = p \cap \{\theta_i\}_{i<s}$. Identify $p\restriction s$ with the function $f_p\restriction s$ where $f_p(i) = 1$ if $\theta_i \in p$.

Suppose p is identified with a computable function f_p. By the above definition, we can compute the arity of the type p by checking whether $f_p(i_n) = 1$ where $\theta_{i_n} = \bigwedge\!\!\!\bigwedge_{j<n}(x_j = x_j)$. The arity of p is the n such that $f_p(i_n) = 1$ and $f_p(i_{n+1}) = 0$

1.2.8 Degree Spectra

Our object of study will be the collection of Turing degrees of the (elementary) diagrams of isomorphic copies of a model $\mathcal{A}$. This collection, known as the *degree spectrum* of a model and defined formally below, is often studied in computable model theory. See [1] and [10] for wider overviews of the subject.

We first define the diagrams associated with a given model. Let $\mathcal{A}$ be a model with universe A. Let $\mathcal{L}_A$ be the language $\mathcal{L} \cup \{c_a : a \in A\}$, $\mathcal{L}$ expanded to include a new constant c_a for each $a \in A$. Let $\mathcal{A}_A = (\mathcal{A}, a)_{a \in A}$ be the expansion of model $\mathcal{A}$ for language $\mathcal{L}_A$ such that the constant c_a is interpreted by a for every $a \in A$. The *atomic diagram* of $\mathcal{A}$ is the set of all atomic sentences of $\mathcal{L}_A$ which hold in $\mathcal{A}_A$. The *elementary diagram* of $\mathcal{A}$ is the set of all sentences of $\mathcal{L}_A$ which are true in $\mathcal{A}_A$. Let $D^a(\mathcal{A})$ denote the atomic diagram and $D^e(\mathcal{A})$ denote the elementary diagram of $\mathcal{A}$. For any $X \subseteq \omega$, let $deg(X)$ denote the Turing degree of X.

Definition 1.2.12. Let $dSp^a(\mathcal{A}) = \{deg(D^a(\mathcal{B})) : \mathcal{B} \cong \mathcal{A}\}$. Then $dSp^a(\mathcal{A})$ is called the *atomic degree spectrum of* $\mathcal{A}$ and we similarly define $dSp^e(\mathcal{A})$, the *elementary degree spectrum of* $\mathcal{A}$.

We say $\mathcal{A}$ is *computable* if $D^a(\mathcal{A})$ is computable and $\mathcal{A}$ is *decidable* if $D^e(\mathcal{A})$ is computable, and we similarly define **d**-computable and **d**-decidable for any degree **d**. We say that $\mathcal{A}$ has a *computable presentation* if there exists a computable $\mathcal{B}$ with $\mathcal{B} \cong \mathcal{A}$, *i.e.*, $\mathbf{0} \in dSp^a(\mathcal{A})$ and a *decidable presentation* if $\mathbf{0} \in dSp^e(\mathcal{A})$.

One often proves results about the atomic degree spectrum of a model. In the following work however we prove results about the elementary degree spectrum. In the positive direction, these results are slightly stronger than the same results about the atomic degree spectrum since $D^a(\mathcal{A}) \leq_T D^e(\mathcal{A})$.

By Theorem 1.2.9 (the uniqueness theorem for homogeneous models), we see that

$$dSp^a(\mathcal{A}) = \{deg(D^a(\mathcal{B})) : \mathcal{B} \text{ homogeneous } \& \ \mathbb{T}(\mathcal{B}) = \mathbb{T}(\mathcal{A})\}$$

where $\mathcal{B}$ ranges only over countable models. We can similarly describe $dSp^e(\mathcal{A})$.

Definition 1.2.13. A structure $\mathcal{A}$ is called *automorphically trivial* if there exists a finite set $F \subset A$ such that any permutation π of A fixing F pointwise is an automorphism of $\mathcal{A}$.

The following theorem is a useful fact about degree spectra.

Theorem 1.2.14. (Knight [15]) *Let $\mathcal{A}$ be a countable structure in a relational language. If $\mathcal{A}$ is not automorphically trivial, then $dSp^a(\mathcal{A})$ is closed upwards. This result also holds for $dSp^e(\mathcal{A})$. Note that $dSp^e(\mathcal{A}) \subseteq dSp^a(\mathcal{A})$.*

Since automorphically trivial models are structurally uninteresting, we only consider the degree spectra of automorphically nontrivial models, which are closed upwards by Theorem 1.2.14. Thus to show $\mathbf{d} \in dSp^e(\mathcal{A})$, it suffices to show $\mathcal{A}$ has an isomorphic copy decidable below $\mathbf{d}$.

Convention 1.2.15. *For the remainder of the paper, all models considered are automorphically nontrivial.*

1.3 Decidability of Homogeneous Models

In this section, we lay out the terminology required to understand Goncharov and Peretyat'kin's characterization (discussed in §1.3.2) of when a homogeneous model has a decidable isomorphic copy. For more information on this characterization and how it relates to the analogous characterizations for prime and saturated models, consult [18].

1.3.1 $\mathbf{0}$-Bases and Morley's Question

Goncharov, Peretyat'kin, and others previously studied what conditions suffice to show that a homogeneous model $\mathcal{A}$ has a decidable isomorphic copy $\mathcal{B}$. If $\mathcal{A}$ has a decidable copy, then there exists a uniformly computable listing of $\mathbb{T}(\mathcal{A})$.

Definition 1.3.1. We say that a countable model $\mathcal{A}$ has a $\mathbf{0}$-*basis* $X = \{p_j\}_{j\in\omega}$ if X is a uniformly computable listing of $\mathbb{T}(\mathcal{A})$.

The standard relativization of this idea to degree $\mathbf{d}$ is as follows.

Definition 1.3.2. We say $\mathcal{A}$ has a $\mathbf{d}$-*uniform basis* $X = \{p_j\}_{j\in\omega}$ if X is a $\mathbf{d}$-uniformly computable listing of $\mathbb{T}(\mathcal{A})$.

Note that a $\mathbf{0}$-basis is a $\mathbf{0}$-uniform basis and vice versa. To obtain stronger results later, we relativize the idea of a $\mathbf{0}$-basis in a nonstandard way. When we relativize the definition of a $\mathbf{0}$-basis to a $\mathbf{d}$-basis, the notions of $\mathbf{d}$-basis and $\mathbf{d}$-uniform basis will differ.

We encode a basis X as an infinite two-dimensional matrix of zeros and ones (*i.e.*, an element of $2^{\omega\times\omega}$) where the i^{th} row $X^i \in 2^\omega$ corresponds to the type p_i according to the enumeration of the formulas $F(T)$ we fixed in Definition 1.2.11. Note that if $\mathcal{A}$ has a $\mathbf{0}$-basis X then:

1. the types realized in $\mathcal{A}$ are all computable and
2. there exists a computable function g such that $g(j)$ is a Δ^0_0-index for type p_j for all $j \in \omega$.

Thus, we define a $\mathbf{d}$-basis as follows.

Definition 1.3.3. A countable model $\mathcal{A}$ has a $\mathbf{d}$-*basis* $X = \{p_j\}_{j\in\omega}$ if

1. the types realized in $\mathcal{A}$ are all computable and
2. there exists a $\mathbf{d}$-computable function g such that $g(j)$ is a Δ^0_0-index for type p_j for all $j \in \omega$.

Note that any $\mathbf{d}$-basis can effectively be viewed as a $\mathbf{d}$-uniform basis, but not conversely. The difference is that in a $\mathbf{d}$-basis $g(j)$ gives a Δ^0_0-index for p_j but in a $\mathbf{d}$-uniform basis $g(j)$ gives only a Δ^D_0-index for $D \in \mathbf{d}$. To understand the importance and utility of bases, we will focus on $\mathbf{0}$-bases and not $\mathbf{d}$-bases for the moment.

Suppose $\mathcal{A}$ has a decidable copy. Then $\mathcal{A}$ has a $\mathbf{0}$-basis X for $T(\mathcal{A})$, *i.e.*, there is a uniformly computable listing of the types realized in $\mathcal{A}$. Moreover, the theory T of $\mathcal{A}$ must be decidable. The converse holds for prime and saturated models.

Theorem 1.3.4. *If $\mathcal{A}$ has a $\mathbf{0}$-basis and is prime or saturated, then $\mathcal{A}$ has a decidable presentation.*

The prime case was proved by Goncharov-Nurtazin [9] and Harrington [11], and the saturated case was proved by Morley [26] and Millar [22]. Morley naturally asked the analogous question for homogeneous models.

Question 1.3.5. (Morley [26]) Let $\mathcal{A}$ be homogeneous. If $\mathcal{A}$ has a $\mathbf{0}$-basis, does $\mathcal{A}$ necessarily have a decidable presentation?

Goncharov, Millar, and Peretyat'kin separately negatively answered this question in [8], [23], [27]. They constructed examples of homogeneous models with **0**-bases but no decidable copies. Thus, a **0**-basis alone does not guarantee the existence of a decidable copy of a homogeneous model. Goncharov and Peretyat'kin, however, exactly characterized when a homogeneous model has a decidable copy. They showed that a **0**-basis together with a certain computable function on indices of types and formulas are necessary and sufficient to guarantee that a homogeneous model has a decidable copy. We now precisely describe these needed functions.

1.3.2 Effective and Monotone Extension Functions

Although a **0**-basis for a homogeneous model $\mathcal{A}$ computably tells us what types are realized in $\mathcal{A}$, Goncharov and Peteyat'kin realized that in order to guarantee $\mathcal{A}$ has a decidable presentation we need computable information about how these types extend one another. Below we define functions that contain such information.

Definition 1.3.6. [Effective Extension Function (EEF) and Monotone Extension Function (MEF)]
Let $\mathcal{A}$ be a homogeneous model of a complete decidable (CD) theory T whose type spectrum $\mathbb{T}(\mathcal{A})$ has a **0**-basis $X = \{p_i\}_{i\in\omega}$.

(i) A function f is an *extension function (EF)* for X if for every n,

- for every n-type $p_i(\overline{x}) \in X \cap S_n(T)$
- and every $(n+1)$-ary $\theta_j(\overline{x}, x_n) \in F_{n+1}(T)$ consistent with $p_i(\overline{x})$

the $(n+1)$-type $p_{f(i,j)} \in X \cap S_{n+1}(T)$ extends both $p_i(\overline{x})$ and $\theta_j(\overline{x}, x_n)$, *i.e.*,

$$p_i(\overline{x}) \cup \{\theta_j(\overline{x}, x_n)\} \subseteq p_{f(i,j)}(\overline{x}, x_n).$$

(ii) If f is also computable then f is an *effective extension function (EEF).*

(iii) A function f is a *monotone extension function (MEF)* if there exists a computable function $g(i, j, s)$ such that

- $f(i,j) = \lim_s g(i,j,s)$ is an extension function and
- $p_{g(i,j,s)} \upharpoonright s \subseteq p_{g(i,j,s+1)} \upharpoonright s$.

An effective extension function is a computable function that given any n-type p_i and any consistent $(n+1)$-ary formula θ_j outputs an index k such that p_k is an $(n+1)$-type that amalgamates p_i and θ_j. A monotone extension function given the same data monotonically approximates the index of an amalgamating $(n+1)$-type. Specifically, the approximate amalgamator $p_{g(i,j,s)}(\overline{x}, x_n)$ at stage s agrees with the true amalgamator $p_{f(i,j)}(\overline{x}, x_n)$ on the first s formulas of $F(T)$. Hence an effective extension function immediately tells us the address in the **0**-basis of the type extending the data, whereas a monotone extension function only describes this type formula by formula. (In the limit, however, a monotone extension function will also describe the correct address.)

If a **0**-basis X for $\mathcal{A}$ has an effective extension function f, then it is easy to see X has a monotone extension function. The converse is not true. Both effective and monotone extension functions are locally basis dependent in that they are defined on a particular **0**-basis X. Effective extension functions are also globally basis dependent in the sense that if $\mathcal{A}$ has two **0**-bases X and Y and X has an effective extension function, Y might not have one. Monotone extension functions are more robust in that they are globally basis independent, *i.e.*, if some **0**-basis for $\mathcal{A}$ has a monotone extension function, every **0**-basis for $\mathcal{A}$ has one.

The next result is our main tool for obtaining new results.

Theorem 1.3.7. (Goncharov [8], Peretyat'kin [27]) *Let T be a complete decidable theory and $\mathcal{A}$ of T be homogeneous. Then the following are equivalent:*

1. *$\mathcal{A}$ has a decidable presentation.*
2. *Every **0**-basis for $\mathcal{A}$ has a monotone extension function.*
3. *Some **0**-basis for $\mathcal{A}$ has a monotone extension function.*
4. *Some **0**-basis for $\mathcal{A}$ has an effective extension function.*

For a given homogeneous model $\mathcal{A}$ and degree $\mathbf{d}$ we want to know whether there exists an isomorphic copy $\mathcal{B}$ of $\mathcal{A}$ of degree $\mathbf{d}$. To achieve this we relativize the previous result to an arbitrary degree $\mathbf{d}$ by replacing every $\mathbf{0}$-concept above by the corresponding $\mathbf{d}$-concept.

Theorem 1.3.8. *Let T be a complete decidable theory and $\mathcal{A}$ of T be homogeneous. Then the following are equivalent:*

1. *$\mathcal{A}$ has a $\mathbf{d}$-decidable presentation.*
2. *Every $\mathbf{d}$-uniform basis for $\mathcal{A}$ has a $\mathbf{d}$-monotone extension function.*
3. *Some $\mathbf{d}$-uniform basis for $\mathcal{A}$ has a $\mathbf{d}$-monotone extension function.*
4. *Some $\mathbf{d}$-uniform basis for $\mathcal{A}$ has a $\mathbf{d}$-effective extension function.*

Suppose a homogeneous model $\mathcal{A}$ has a $\mathbf{0}$-basis X. To show $\mathcal{A}$ has an isomorphic copy of degree $\mathbf{d}$, we can build a new $\mathbf{d}$-uniform basis Y with an $\mathbf{d}$-effective extension function. Alternatively, since a $\mathbf{0}$-basis can be effectively viewed as a $\mathbf{d}$-uniform basis, we can build a $\mathbf{d}$-monotone extension function on the original $\mathbf{0}$-basis X. In either case, by Theorem 1.3.8, $\mathcal{A}$ has a $\mathbf{d}$-decidable isomorphic copy $\mathcal{B}$.

1.3.3 Notation for Bases

Recall we encode a basis X as an infinite two-dimensional matrix with elements from $\{0, 1\}$ where the i^{th} row $X^i \in 2^\omega$ corresponds to the type p_i according to the enumeration of the formulas $F(T)$ in Definition 1.2.11.

We refer to partial functions from $\omega \times \omega$ to 2 as *partial infinite matrices* and partial functions from ω to 2 as *partial strings*. During our constructions of bases, we use upper case letters such as X, Y, M to denote partial infinite matrices. Similarly, we let X^i denote the partial string which is the i^{th} row of X. Let $ht(X)$ denote the greatest i such that $(\exists x)X^i(x)\downarrow$. For some formula ψ, let $\exists\psi$ denote the sentence given by ψ with all its free variables quantified out by existentials, and let $(\exists\bar{x})[\psi]$ denote the formula given by ψ with the free variables $\bar{x}$ quantified out by existentials.

CHAPTER 2
POSITIVE RESULTS ON HOMOGENEOUS MODELS

2.1 Homogeneous Low Basis Theorem

Let $\mathcal{A}$ be an automorphically nontrivial homogeneous model of a complete decidable theory T. We now study the elementary degree spectrum of this fixed model. Goncharov, Millar, and Peretyat'kin showed that an arbitrary homogeneous $\mathcal{A}$ with a $\mathbf{0}$-basis does not necessarily have a decidable isomorphic copy [8], [23], [27], and Tusupov [32] showed that any such $\mathcal{A}$ always has a $\mathbf{0}'$-decidable copy.

The negative results above require delicately building counterexamples. Tusupov's result, however, follows directly from the relativized EEF/MEF Theorem 1.3.8. The theorem says that $\mathcal{A}$ has a $\mathbf{0}'$-decidable copy if $\mathcal{A}$'s $\mathbf{0}$-basis has a $\mathbf{0}'$-effective extension function. Suppose we are given an n-type p_i and an $(n+1)$-formula θ_j consistent with p_i. Determining whether an $(n+1)$-type p_k extends p_i and θ_j is a Π^0_1 property. Thus $\mathbf{0}'$ can compute the extension function. Here we examine the degrees in between $\mathbf{0}$ and $\mathbf{0}'$.

2.1.1 The Basic Result

We prove that $dSp^e(\mathcal{A})$ always contains a low degree for any homogeneous $\mathcal{A}$ with a $\mathbf{0}'$-basis. Using a $\mathbf{0}'$ oracle, we will build a $\mathbf{d}$-uniform basis for $\mathcal{A}$ with an effective extension function for a low degree $\mathbf{d}$. The relativized EEF/MEF Theorem 1.3.8 then gives that there exists a low $\mathcal{B} \cong \mathcal{A}$. We show that this result implies Csima's analogous result for prime models [4]. Moreover, we show that the proof can be strengthened to obtain results on cone avoidance and minimal pairs in $dSp^e(\mathcal{A})$.

Theorem 2.1.1. *Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}'$-basis. There exists a $\mathbf{d}$-decidable $\mathcal{B} \cong \mathcal{A}$ for a low degree $\mathbf{d}$.*

Proof. Let $X = \{p_i\}_{i\in\omega}$ be the $\mathbf{0}'$-basis of types of $\mathcal{A}$. We build a **d**-uniform basis $Y = \{q_i\}_{i\in\omega}$ for $\mathcal{A}$ for **d** low. Let $f(n,m)$ be a computable injective function such that $ran(f) = \{2n : n \in \omega\}$, and if $f(i,j) = k$ then $i < k$. The range condition simply ensures that there are infinitely many rows not in the range of f. This f will be the effective extension function for the **d**-uniform basis Y that we build. Since Y is **d**-uniform and f is computable, by the relativized EEF/MEF Theorem 1.3.8, $\mathcal{A}$ will have a **d**-decidable copy $\mathcal{B}$ where **d** is low. We will meet the following requirements for all e, i, k:

$\mathbf{P}_e$**: (Lowness)** $\Phi_e^Y(e)$ is decided by stage $e+1$ of the construction.

$\mathbf{Q}_i$**:** $(Y \subseteq \mathbb{T}(\mathcal{A}))$ $q_i = p_j$ for some $j \in \omega$

$\mathbf{R}_i$**:** $(\mathbb{T}(\mathcal{A}) \subseteq Y)$ $p_i = q_j$ for some $j \in \omega$

$\mathbf{S}_k$**: (EEF)** If $k = f(i,j)$, q_i is an n-type, and θ_j an $(n+1)$-formula consistent with q_i, then q_k is an $(n+1)$-type extending q_i and θ_j.

Construction.

We build $Y \in 2^{\omega\times\omega}$ in stages using a $\mathbf{0}'$-oracle so that $Y = \cup_{s\in\omega} Y_s$ where Y_s is a partial infinite matrix defined at stage s. $Y^i \in 2^\omega$, the i^{th} row of Y, corresponds to a type $q_i \in S(T)$. We satisfy requirements $\mathbf{Q}_i$ and $\mathbf{R}_i$ to ensure that $Y = \{q_i\}_{i\in\omega} = \mathbb{T}(\mathcal{A})$. At each stage s, Y_s consists of at least s many completely filled in rows. We have a Δ^0_0-index for each of these rows, and they all correspond to types in $\mathbb{T}(\mathcal{A})$. Moreover, for each $i \leq s$, there is a row that corresponds to p_i. Finally, the rows in Y_s and there corresponding types respect the EEF f so far.

Stage $s = 0$**.**

Define Y_0^0 so that $q_0 = p_0$. In other words, place the type p_0 on row 0 of Y_0. Leave the remainder of Y_0 undefined.

Stage $s + 1 = e + 1$**.**

Let $h = ht(Y_s)$ where $ht(X)$ for a partial matrix X denotes the greatest i such that $(\exists x)X^i(x)\downarrow$.

Using a $\mathbf{0}'$-oracle we test whether there exist a t and a finite partial matrix M such that M has the following computable properties:

1. <u>M forces the jump</u>

   $\Phi^M_{e,t}(e)\downarrow$

2. <u>M respects Y_s</u>

   $(\forall x)[M(x)\downarrow \ \& \ Y_s(x)\downarrow \quad \Longrightarrow \quad M(x) = Y_s(x)]$

   [M agrees with Y_s where both are defined.]

3. <u>M respects T</u>

   For all i such that $h < i \leq ht(M)$, the finite string M^i demonstrates that M^i can only be extended to a type of arity n. Furthermore, the conjunction of formulas θ_{M^i} corresponding to M^i is consistent with T.

   [M^i can be extended to an n-type of $S(T)$]

4. <u>M respects the EEF f</u>

   For all k where $h < i \leq ht(M)$, $k = f(i, j)$, and $\theta_j \in F_{n+1}(T)$ either:

   (a) M^i (or Y^i_s if $i \leq h$) can only be extended to an n-type or is inconsistent with θ_j.

   [The type q_i will be of the wrong arity or will be inconsistent with θ_j.]

   (b) M^i (or Y^i if $i \leq h$) determines q_i is an n-type and $M^i(\ (\exists x_n)[\theta_j]\ )\downarrow= 1$. Then $M^i(\ (\exists x_n)[\ \theta_j \wedge \theta_{M^k}\ ]\ )\downarrow= 1$ and M^k determines q_k will be an $(n+1)$-type.

   [M proves M^i can only be extended to an n-type consistent with θ_j. Then M proves M^k can be extended to an $(n+1)$-type containing the partial type generated by M^i and θ_j.]

If there exists some such M and t, let M' be the least such M and set the matrix $Y'_{s+1} = M' \cup Y_s$. Otherwise let $Y'_{s+1} = Y_s$.

We extend Y'_{s+1} so that for all $k \leq ht(Y'_{s+1})$, row k of Y'_{s+1} is total. If we have $k \leq ht(Y'_{s+1})$ and $k \notin ran(f)$, $\mathbf{0}'$ can find a Δ^0_0-index for a type in $X = \{p_i\}_{i\in\omega}$ that extends the k^{th} row of Y'_{s+1} (since each row of Y'_{s+1} is consistent with T). Similarly, we fill in q_k when $k = f(i,j)$ but q_i and θ_j either have incorrect arities or are inconsistent. Hence we can fill in these finitely many q_k for $i \leq ht(Y'_{s+1})$ while respecting Y'_{s+1}.

Otherwise if $k = f(i,j)$, we use $\mathbf{0}'$ to find an $(n+1)$-amalgamator $p_l \in X$ for q_i and θ_j that extends Y'_{s+1}. We know (since inductively $q_i \in \mathbb{T}(\mathcal{A})$) some such amalgamator exists in X. To find it, we iteratively test each p_l to determine if it extends q_i, θ_j and the k^{th} row of Y'_{s+1} (which is consistent with q_i and θ_j). Since X is a $\mathbf{0}'$-basis, $\mathbf{0}'$ can uniformly compute a Δ^0_0-index for each p_l. Thus, $\mathbf{0}'$ can uniformly determine if p_l satisfies these requirements since consistency between two computable types and a formula is a Π^0_1 statement. We extend row k of Y'_{s+1} to correspond to the first p_l with the desired properties.

Now let Y_{s+1} be Y'_{s+1} with the above extensions and with p_{s+1} placed on the next empty row which is not in the range of f. (Therefore, all q_k are decided for $k \leq ht(Y'_{s+1})$.)

End Construction.

Verification.

By the conditions put on M and the way in which q_i's are decided, requirements $\mathbf{Q}_i$, $\mathbf{R}_i$, and $\mathbf{S}_i$ are satisfied for all i. Specifically, we completely fill each q_i (*i.e.*, the row Y^i) by copying some $p_j \in X$ which agrees with the finitely many bits already determined. Hence every row in Y is some p_j. Similarly, we know by the last line of the construction that every p_j is some row in Y. Finally, by construction, f is an effective extension function for Y.

At the end of stage $e+1$, $\Phi^Y_e(e)$ is decided, satisfying requirement $\mathbf{P}_e$. If $\Phi^{Y_{e+1}}_e(e)\downarrow$, the use principle guarantees that $\Phi^Y_e(e)\downarrow = \Phi^{Y_{e+1}}_e(e)\downarrow$ since $Y \supseteq Y_{e+1}$. Otherwise by construction, no compatible extension σ of Y_{e+1} can cause $\Phi^\sigma_e(e)$ to converge, so $\Phi_e^Y(e)\uparrow$. Thus Y is low. By applying the relativized EEF/MEF Theorem 1.3.8 to Y with effective extension function f, there exists a model $\mathcal{B} \cong \mathcal{A}$ of low degree.

□

Thus any homogeneous model with a $\mathbf{0}'$-basis has a low presentation. Even though not every homogeneous model with a $\mathbf{0}'$ or $\mathbf{0}$-basis has a decidable presentation, each such model does have a presentation of low information content. This result easily provides the following corollary.

Corollary 2.1.2. (Csima [4]) *Let T be a complete atomic decidable theory. Then there exists a prime model $\mathcal{B}$ of T decidable in a low degree.*

Proof. Let $\mathcal{A}$ be a prime model of T. Then,

$$\mathbb{T}(\mathcal{A}) = S^P(T) = \{p(\bar{x}) : p(\bar{x}) \text{ is a principal type in } T\}.$$

By the above result, it suffices to show that $\mathcal{A}$ has a $\mathbf{0}'$-basis. Given a formula $\theta(\bar{x}) \in F^n(T)$, $\mathbf{0}'$ can decide the following Σ_1^0 question:

$$(\ \exists\gamma \ \in \ F^n(T)\ )\ [\ (\exists\bar{x})[\theta \wedge \gamma] \ \in \ T \ \&\ (\exists\bar{x})[\theta \wedge \neg\gamma] \ \in \ T\ ],$$

i.e., whether θ generates a principal type. Hence $\mathbf{0}'$ can decide whether a formula $\theta(\bar{x})$ is a generator for a principal type. Since T is decidable, the index for a generator in an effective listing of formulas gives rise to a Δ_0^0-index for its corresponding principal type.

We use $\mathbf{0}'$ to determine all formulas which are generators for principal types in T. We then use the indices for these formulas to give a $\mathbf{0}'$-basis for $\mathbb{T}(\mathcal{A})$.

□

2.1.2 Avoiding Cones and Minimal Pairs

We can strengthen the results of the last section by combining additional requirements with the basic approach. These results tell us two important properties of the degree spectrum of any homogeneous model $\mathcal{A}$ with a $\mathbf{0}'$-basis. The first result shows that $dSp^e(\mathcal{A})$ contains a scattered selection of degrees between $\mathbf{0}$ and $\mathbf{0}'$. The second shows $dSp^e(\mathcal{A})$ always contains a minimal pair, in other words, that noncomputable information cannot be coded into the degree spectrum of any such model.

Avoiding Cones of Degrees

Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}'$-basis, and let C be a noncomputable low set. We show there exists a low copy $\mathcal{B}$ of $\mathcal{A}$ which avoids the upper and lower cones generated by C. Thus, $dSp^e(\mathcal{A})$ must be scattered in the sense that no low degree $\mathbf{c}$ is below every degree in $dSp^e(\mathcal{A})$. (This result could be generalized further to show a list of low degrees whose jumps are uniformly computable in $\mathbf{0}'$ could not bound from below all of $dSp^e(\mathcal{A})$.)

Theorem 2.1.3. *Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}'$-basis X. Let C be a noncomputable low set. Then there is a low copy $\mathcal{B}$ of $\mathcal{A}$ such that $C \not\leq_{\mathrm{T}} D^e(\mathcal{B})$ and $D^e(\mathcal{B}) \not\leq_{\mathrm{T}} C$.*

Proof. If $\mathcal{A}$ has a decidable copy, then the theorem follows by upward closure of the degree spectrum Theorem 1.2.14. Thus we may assume that $\mathcal{A}$ has no decidable copy.

In addition to the requirements in the basic result, Theorem 2.1.1, we add requirements:

$\mathbf{N}_e$: $\quad C \neq \Phi_e^Y$

$\mathbf{N}'_e$: $\quad Y \neq \Phi_e^C$

If we satisfy these additional requirements, there exists a $\mathcal{B} \cong \mathcal{A}$ where $D^e(\mathcal{B}) \equiv_{\mathrm{T}} Y$ by Theorem 1.3.8. Thus $\mathcal{B}$ is low and avoids the upper and lower cones of C. We modify the construction in Theorem 2.1.1.

Construction.

We alternate between building the low uniform basis Y with effective extension function f for $\mathcal{A}$ as before and satisfying our additional requirements.

Stage $s+1 = 3e+1$**.** **(Lowness)**

Act as in stage $s+1$ in the previous construction to ensure the lowness of Y.

Stage $s+1 = 3e+2$**.** **(Satisfy $\mathbf{N}_e$)**

We meet $\mathbf{N}_e$. Test using a $\mathbf{0}' = C'$ oracle whether there exist t, y, and a finite partial matrix M such that:

- M respects Y_s, T, and the effective extension function f (as in the construction for Theorem 2.1.1).
- $C(y) \neq \Phi^M_{e,t}(y)$

If yes, take the least such extension M' that satisfies the above and set $Y'_{s+1} = M' \cup Y_s$. Use $\mathbf{0}'$ and the $\mathbf{0}'$-basis X to completely fill in each partial row in Y'_{s+1} to obtain Y_{s+1}.

Stage $s+1 = 3e+3$**.** **(Satsify $\mathbf{N}'_e$)**

Use the $\mathbf{0}' = C'$ oracle to ask whether there exist t, y and a finite partial matrix M such that:

- M respects Y_s, T, and the effective extension function f.
- $Y(y) \neq \Phi^C_{e,t}(y)\downarrow$

If so, let M' be the least such M. Then set $Y'_{s+1} = M' \cup Y_s$. Extend Y'_{s+1} to Y_{s+1} as above.

End Construction.

Verification.

We built a low uniform basis Y with effective extension function f as in Theorem 2.1.1. Let $\mathbf{d}$ be the degree of Y. By Theorem 1.3.8, we obtain a $\mathbf{d}$-decidable (*i.e.*, low) $\mathcal{B}$ isomorphic to $\mathcal{A}$. It remains to show that $\mathbf{N}_e$ and $\mathbf{N}'_e$ are satisfied for all e.

Suppose $C = \Phi_e^Y$. Then we can compute $C(y)$ for any y effectively from the finitely many computable indices for the rows in Y_{3e+2}. To compute $C(y)$, compute $\Phi_{e,s}^M(y)$ for some finite partial matrix M that respects Y_{3e+2}, T, and f where M is defined (these questions are computable since M is finite). Such an M exists since C is total. Then $\Phi_{e,s}^M(y) = C(y) = \Phi_e^Y(y)$. If $\Phi_{e,s}^M(y) \neq \Phi_e^Y(y)$, M would have been selected to extend Y_{3e+2}, satisfying $\mathbf{N}_e$. Since C was assumed to be noncomputable, $\mathbf{N}_e$ holds for all e. Hence $C \not\leq_{\mathrm{T}} D^e(\mathcal{B})$ since $Y \equiv_{\mathrm{T}} D^e(\mathcal{B})$.

Suppose $Y = \Phi_e^C$. Then we can compute $Y(y)$ for any y uniformly from the finitely many computable indices corresponding to the rows of Y_{3e+3}. Find the first such finite partial M that respects Y_{3e+3}, T, and f so that the length of M is greater than y (thinking of M now as a string in $2^{<\omega}$). Such an M exists since Y is total and has these properties. Then $M(y) = Y(y) = \Phi_e^C(y)$. (Otherwise we would have extended Y_{3e+3} using M, satisfying $\mathbf{N}'_e$.)

Hence Y is computable. But then $\mathcal{B}$ would be a decidable copy of $\mathcal{A}$, a contradiction. Thus, $\mathbf{N}'_e$ is satisfied for all e and $D^e(\mathcal{B}) \not\leq_{\mathrm{T}} C$.

□

Minimal Pairs

If $\mathcal{A}$ is a homogeneous model with a $\mathbf{0}'$-basis, Theorem 2.1.3 shows that $dSp^e(\mathcal{A})$ contains a scattering of degrees below $\mathbf{0}'$. Now we show $dSp^e(\mathcal{A})$ contains minimal pairs. Therefore it is impossible to code noncomputable information into the isomorphism class of a homogeneous model with a $\mathbf{0}'$-basis.

Theorem 2.1.4. *Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}'$-basis X. Then there are low decidable copies $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ such that if $Z \leq_{\mathrm{T}} D^e(\mathcal{B})$ and $Z \leq_{\mathrm{T}} D^e(\mathcal{C})$, then Z is computable.*

Proof. If $\mathcal{A}$ has a decidable copy, then the theorem follows by upward closure of the degree spectrum Theorem 1.2.14. Thus we may assume that $\mathcal{A}$ has no decidable copy.

We build a minimal pair of low models $\mathcal{B}$ and $\mathcal{C}$ isomorphic to $\mathcal{A}$ using a $\mathbf{0}'$-oracle argument. For $\mathcal{B}$ and $\mathcal{C}$, we have each of the requirements in the basic result Theorem 2.1.1. Let $Y^{\mathcal{B}}$ and $Y^{\mathcal{C}}$ denote the low uniform bases we are constructing with effective extension functions $f^{\mathcal{B}}$ and $f^{\mathcal{C}}$ for $\mathcal{B}$ and $\mathcal{C}$. We add the following requirement to ensure $\mathcal{B}$ and $\mathcal{C}$ form a minimal pair.

N$_e$: (Minimal Pair)

$$\Phi_e^{Y^{\mathcal{B}}} = \Phi_e^{Y^{\mathcal{C}}} = g \text{ total} \quad \Longrightarrow \quad g \text{ computable.}$$

Construction.

We build $Y^{\mathcal{B}}$ and $Y^{\mathcal{C}}$ as in Theorem 2.1.1 with the following additions.

Stage 0:

Let the first rows in $Y^{\mathcal{B}}$ and $Y^{\mathcal{C}}$ be two distinct 1-types in $\mathcal{A}$.

Stage $s+1 = 2e+1$. **(Lowness)**

Act as in stage $s+1$ in the basic result to meet the lowness requirements for $\mathcal{B}$ and $\mathcal{C}$.

Stage $s+1 = 2e+2$. **(Satisfy N$_e$)**

Use a $\mathbf{0}'$-oracle to test whether there exist t, y and finite partial matrices M and M' such that:

- M respects $Y_s^{\mathcal{B}}$, T, and the effective extension function $f^{\mathcal{B}}$ (as in Theorem 2.1.1) and similarly M' respects these properties for $Y_s^{\mathcal{C}}$ and $f^{\mathcal{C}}$.
- $\Phi_{e,t}^{M}(y)\downarrow \neq \Phi_{e,t}^{M'}(y)\downarrow$

If so, take the least such extensions $\tilde{M}$ and $\tilde{M}'$ that satisfy the above and set $\tilde{Y}_{s+1}^{\mathcal{B}} = \tilde{M} \cup Y_s^{\mathcal{B}}$ and $\tilde{Y}_{s+1}^{\mathcal{C}} = \tilde{M}' \cup Y_s^{\mathcal{C}}$. Then let $Y_{s+1}^{\mathcal{B}}$ equal $\tilde{Y}_{s+1}^{\mathcal{B}}$ extended using $\mathbf{0}'$ to completely fill in each partial row in $\tilde{Y}_{s+1}^{\mathcal{B}}$ as in Theorem 2.1.1, and similarly define $Y_{s+1}^{\mathcal{C}}$.

End Construction.

Verification.

The models $\mathcal{B}$ and $\mathcal{C}$ are low copies of $\mathcal{A}$ exactly as in Theorem 2.1.1. It remains to show that $\mathbf{N}_e$ is satisfied for all e.

Suppose $g = \Phi_e^{Y^{\mathcal{B}}} = \Phi_e^{Y^{\mathcal{C}}}$ is total. We can compute g from the finitely many computable indices of the rows in $Y^{\mathcal{B}}_{2e+2}$. To compute $g(y)$, computably find a finite partial M which respects $Y^{\mathcal{B}}_{2e+2}$, T, and $f^{\mathcal{B}}$ such that $\Phi_e^M(y)\downarrow$. Some such M must exist since g is total, and then $\Phi_{e,s}^M(y) = g(y)$. Hence g is computable. Since $\mathcal{A}$ has no decidable copy, $\mathcal{B}$ and $\mathcal{C}$ are not decidable and form a minimal pair.

□

Although these previous two theorems are simple extensions of the basic result, they provide useful information about the degree spectrum of a homogeneous model with a $\mathbf{0}'$-basis. The avoiding cones result tells us that the degree spectrum of such a model is scattered between $\mathbf{0}$ and $\mathbf{0}'$. The minimal pair result tells us that noncomputable information cannot be coded into an isomorphism class of such a model.

In the next section, we will see how stronger conditions on the theory T can tell us more about the elementary degree spectrum of homogeneous models.

2.2 Full Basis Theorem for Homogeneous Models

As mentioned in §1.3.1, early researchers first asked when a homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis has a decidable copy. In §2.1, to generalize this work, we studied the degree spectrum of a homogeneous model $\mathcal{A}$ with a $\mathbf{0}'$-basis. We saw that the level of computability of the types in $\mathbb{T}(\mathcal{A})$ directly impacts how decidable copies of $\mathcal{A}$ can be. Thus restricting our study to any homogeneous model $\mathcal{A}$ with a $\mathbf{d}$-basis for some degree $\mathbf{d}$ is a natural requirement to obtain useful results on $dSp^e(\mathcal{A})$. This basis condition implies that all of the types *realized in* $\mathcal{A}$, *i.e.*, in $\mathbb{T}(\mathcal{A})$, are computable. In this section, we assume that not only are the types in $\mathbb{T}(\mathcal{A})$ computable but also that all the types in $S(T)$ are computable. It is surprising that the computability of these types outside of $\mathbb{T}(\mathcal{A})$ greatly affects $dSp^e(\mathcal{A})$.

Definition 2.2.1. The notation $S^c(T)$ denotes the set of computable types in T.

If $S(T)$ or even $S^c(T)$ is uniformly computable and $\mathcal{A}$ has a $\mathbf{0}$-basis, we obtain the strongest possible result.

Theorem 2.2.2. (Goncharov [8], Millar [24]) *If T is a complete decidable theory with $S^c(T)$ uniformly computable and $\mathcal{A}$ is a homogeneous model with a $\mathbf{0}$-basis, then $\mathcal{A}$ has a decidable copy.*

We explore the case where all the types in $S(T)$ are computable but not uniformly computable. We obtain almost as strong a result.

Theorem 2.2.3. *Let T be a complete decidable theory with $S(T)$ consisting of computable types. Let $\mathcal{A}$ of T be a homogeneous model with a $\mathbf{0}$-basis. Then for every nonzero degree $\mathbf{d}$ there is a copy $\mathcal{B}$ of $\mathcal{A}$ of degree $\mathbf{d}$.*

This theorem is the strongest possible. Goncharov built a homogeneous model $\mathcal{A}$ of T with a $\mathbf{0}$-basis but no decidable copy where $S(T)$ consists only of computable types.

Theorem 2.2.4. (Goncharov [8]) *There exists a complete decidable theory T with $S(T)$ consisting of computable types and a homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis but no decidable copy.*

Definition 2.2.5. Let $p(\bar{x})$ be an n-type. We say that an $(n+1)$-type $q(\bar{x}, y)$ is *principal over* p if there exists some formula $\psi(\bar{x}, y) \in q(\bar{x}, y)$ (called the *generator* of q over p) such that

$$q = \{\zeta(\bar{x}, y) : (\forall y)(\psi(\bar{x}, y) \rightarrow \zeta(\bar{x}, y)) \in p\}. \tag{2.1}$$

Harris (personal communication) noticed that one can find an $(n+1)$-type that extends p and φ which is principal over p effectively if $S(T)$ is uniformly computable. We generalize this idea to the case where all the types in $S(T)$ are simply computable. In Lemma 2.2.6 we show that any noncomputable degree $\mathbf{d}$ can uniformly compute an $(n+1)$-type q principal over p which contains φ from a Δ^0_0-index for p and φ.

To compute q, we use $\mathbf{d}$ to omit any nonprincipal types over p that extend the type we are building. Suppose we must decide whether an $(n+1)$-formula $\zeta(\bar{x}, y)$ is in our $(n+1)$-type $q(\bar{x}, y)$. If one of ζ or $\neg\zeta$ is inconsistent with $p(\bar{x})$, we are forced to put the consistent one in q. Otherwise, we let $\mathbf{d}$ decide which formula should be included in q by consulting the next unexamined bit of a fixed $C \in \mathbf{d}$. Since the $(n+1)$-type q we build is in $S(T)$, q is computable by assumption. If q was nonprincipal over p, we would have consulted C infinitely often in building q, thus inadvertently coding the degree $\mathbf{d}$ into q. Since q is computable, we could show $\mathbf{d}$ was computable, contradicting our assumption that $\mathbf{d} > \mathbf{0}$. Hence, q must be principal over p. Hirschfeldt first applied this slick technique to prove the analogous result for prime models [12].

Now suppose $p \in \mathbb{T}(\mathcal{A})$ for some homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis. Given a computable index for p and a formula φ, $\mathbf{d}$ can uniformly compute a type q principal over p and containing φ by Lemma 2.2.6 below. Since q is principal over p, there exists a generating formula ψ as in (2.1) which is consistent with p. Since $p \in \mathbb{T}(\mathcal{A})$, $q \in \mathbb{T}(\mathcal{A})$. Thus, we can use $\mathbf{d}$ to build a $\mathbf{d}$-monotonic extension function for the $\mathbf{0}$-basis of $\mathcal{A}$ (See *Proof of Theorem 2.2.3* at the end of this section). By Theorem 1.3.8, $\mathcal{A}$ has a $\mathbf{d}$-decidable copy, proving Theorem 2.2.3.

We first prove the omitting types lemma.

Lemma 2.2.6. *Let T be a complete decidable theory where $S(T)$ consists only of computable types. Let $\mathbf{d} > \mathbf{0}$. Given $p(\bar{x}) \in S(T)$ and $\varphi(\bar{x}, y)$ a formula consistent with p, there exists a type $q(\bar{x}, y) \in S(T)$ such that*

- $\varphi \in q$
- $p \subset q$
- *q is principal over p*

Moreover, q is uniformly computable in $\mathbf{d}$ given a Δ^0_0-index for p and φ.

Proof. Let $C \in \mathbf{d}$. We build q so that it is a type of T, and for all e, if Φ_e is nonprincipal over p and $\varphi \in \Phi_e$, then $\Phi_e \neq q$.

Construction.

We define $q = \bigcup_{s\in\omega} q_s$ in stages using C as an oracle. As usual, defining q means determining whether θ_i or $\neg\theta_i$ is an element of q for all $i \in \omega$. At stage s we will have defined q up to length s. We use v_s as an indicator of how much of C we have used in the construction at stage s.

Stage 0: Let $q_0 = \emptyset$. Let $v_0 = 0$.

Stage s+1: We assume q_s has length s and is consistent with p and φ. We define q_{s+1} to have length $s+1$. Let $q_{s+1}\restriction s = q_s\restriction s$. If θ_s is not a formula in $x_0, \ldots, x_n$, define $q_{s+1}(s) = 0$ since θ_s is not an $(n+1)$-formula.

Otherwise, we can effectively test whether the $(n+1)$-formulas

$$\theta_s \wedge \varphi \wedge \theta_{q_{s+1}\restriction s} \quad \text{and} \quad \neg\theta_s \wedge \varphi \wedge \theta_{q_{s+1}\restriction s}$$

are consistent with p. Since q_s is consistent with p and φ, one of these two formulas must be consistent with p. If only the first is consistent, let $q_{s+1}(s) = 1$ (*i.e.*, q_{s+1} contains θ_s), and if only the second is consistent, let $q_{s+1}(s) = 0$ (*i.e.*, q_{s+1} contains $\neg\theta_s$). Set $v_{s+1} = v_s$.

If both the formulas are consistent, let $q_{s+1}(s) = C(v_s)$ and define $v_{s+1} = v_s + 1$. In other words, $\theta_s \in q_{s+1}$ if and only if $v_s \in C$.

End Construction.

Verification.

By construction q contains p and φ, and q is uniformly computable in $\mathbf{d}$ from a computable index for p and φ. Moreover $q \in S(T)$ and hence is computable.

Suppose for a contradiction that Φ_e is a nonprincipal type over p consistent with φ and $q = \Phi_e$. Hence $\Phi_e = q$ is total and computable. Since $\Phi_e = q$ is nonprincipal over p, there were infinitely many stages in the above construction where C decided which formula to place in q. Note that the construction is computable except for when C is consulted. We show by induction that C is computable. Suppose we have computed $C\restriction n$. To decide whether $n \in C$, we can follow the construction computably to the stage s where the n^{th} digit of C is consulted (*i.e.*, $n = v_{s-1}$). Then $n \in C$ if and only if $\theta_{s-1} \in q$. Since q is computable, C is computable, a contradiction. Thus q is a principal type over p containing φ. □

Proof of Theorem 2.2.3. Using Lemma 2.2.6, we can build a monotone extension function for a given $\mathbf{0}$-basis X in any nonzero degree. If p is an n-type realized in $\mathcal{A}$ and φ is an $(n+1)$-ary formula consistent with p, then the amalgamating $(n+1)$-type q constructed in Lemma 2.2.6 must also be realized in $\mathcal{A}$ and hence be listed in X. This follows since $(\exists y)\psi \in p$ where ψ is the generator for q over p and since any tuple which realizes ψ realizes all of q. To build the monotone extension function g in some nonzero degree $\mathbf{d}$, let p_i be an n-type in X consistent with the $(n+1)$-formula θ_j, and let q be the type constructed in Lemma 2.2.6. By the above comment, there exists a least indexed $(n+1)$-type p_l in X equal to q. Define $g(i, j, s)$ to be the least k such that the type p_k in X satisfies $p_k \restriction s = g \restriction s$. Clearly $\lim_{s\in\omega} g(i, j, s) = l$, and g is a $\mathbf{d}$-monotone extension function for X. By Theorem 1.3.8, this guarantees that $dSp^e(\mathcal{A})$ contains all nonzero degrees. □

We now return to assuming only that T is complete and decidable. From now on we make no assumptions on the computational complexity of the types in $S(T)$.

2.3 Bounding Results

In §2.1 and §2.2, we fixed a homogeneous model $\mathcal{A}$ with a $\mathbf{d}$-basis and studied its degree spectrum. Now we will consider which degrees must be in the degree spectrum of any automorphically nontrivial homogeneous model with a $\mathbf{0}$-basis. In other words, we will attempt to find degrees $\mathbf{d}$ that are strong enough to compute a $\mathbf{d}$-decidable copy of *any* homogeneous model with a $\mathbf{0}$-basis. This idea gives rise to a definition.

Definition 2.3.1. A degree $\mathbf{d}$ is $\mathbf{0}$-*basis homogeneous bounding* or simply $\mathbf{0}$-*bounding* if for any automorphically nontrivial homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis, there exists a $\mathcal{B} \cong \mathcal{A}$ such that $\mathcal{B}$ is $\mathbf{d}$-decidable, *i.e.*, $\mathbf{d} \in dSp^e(\mathcal{A})$.

The definition above is distinct from the idea of homogeneous bounding in work by Csima, Harizanov, Hirschfeldt, and Soare [5]. In that work, a degree $\mathbf{d}$ is *homogeneous bounding* if for any complete decidable theory T, there exists *some* homogeneous model $\mathcal{A}$ of T which is $\mathbf{d}$-decidable. They exactly characterized the homogeneous bounding degrees as the degrees of Peano arithmetic. The definition of $\mathbf{0}$-bounding requires that $\mathbf{d}$ be able to decide a copy of any homogeneous model with a $\mathbf{0}$-basis.

We show the following result.

Theorem 2.3.2. *Let* $\mathbf{d}$ *be a* Δ^0_2 *degree. If* $\mathbf{d}$ *is nonlow*$_2$ *(*i.e.*,* $\mathbf{d}'' > \mathbf{0}''$*) then* $\mathbf{d}$ *is* $\mathbf{0}$*-basis homogeneous bounding.*

This result uses the characterization of nonlow$_2$ Δ^0_2 degrees used to prove the analogous result for prime models proved by Csima, Hirschfeldt, Knight, and Soare [6].

2.3.1 Nonlow$_2$ Δ^0_2 Characterization

Like in the previous theorems, the difficulty in these constructions is determining whether an $(n+1)$-type amalgamates an n-type and an $(n+1)$-formula. Deciding this question comes down to determining the consistency of two infinite types. Determining consistency of a finite formula with a computable n-type is computable. Determining whether an $(n+1)$-type extends an n-type and an $(n+1)$-formula, however, requires asking a Π^0_1 and hence $\mathbf{0}'$ question.

We use the following equivalence for nonlow$_2$ degrees below $\mathbf{0}'$ to obtain an approximation to the answer to these Π^0_1 questions. This equivalence is a relativization of Martin's theorem.

Theorem 2.3.3. (Martin [21], see also [6]) *Assume* $\mathbf{d} \in \Delta^0_2$*. Then* $\mathbf{d}$ *is nonlow*$_2$ *if and only if* $\mathbf{d}$ *satisfies*

$$(\forall\ g \leq_{\mathrm{T}} \mathbf{0}')\ (\exists\ f \leq \mathbf{d})\ (\exists^{\infty} x)\ [\ g(x) \leq f(x)\ ]. \tag{2.2}$$

In other words, $\mathbf{d} \leq \mathbf{0}'$ is nonlow$_2$ exactly if given a $\mathbf{0}'$-computable function g, there exists a $\mathbf{d}$-computable function f such that infinitely often $f(x)$ is at least as big as $g(x)$. This theorem leads to the following definitions.

Definition 2.3.4. We call condition (2.2) the *escape property*. We say that f *escapes* g at x if $f(x) \geq g(x)$.

2.3.2 Proof of the $\mathbf{0}$-Bounding Theorem

Suppose we are given a $\mathbf{0}$-basis X for some homogeneous model $\mathcal{A}$. By Theorem 1.3.8, to show $\mathcal{A}$ has a $\mathbf{d}$-decidable copy for a nonlow$_2$ $\mathbf{d} \leq \mathbf{0}'$, we must construct a $\mathbf{d}$-monotone extension function for X. If we can uniformly compute an amalgamator from computable indices for an n-type $p \in \mathbb{T}(\mathcal{A})$ and an $(n+1)$-formula φ, we can build a monotone extension function as we did in the proof of Theorem 2.2.3. We will carefully define a $\mathbf{0}'$-computable function that outputs a stage by which we will witness an inconsistency between q and p or φ if $q \in \mathbb{T}(\mathcal{A})$ is not an amalgamator. By Theorem 2.3.3, any nonlow$_2$ degree $\mathbf{d} \leq \mathbf{0}'$ computes a function that infinitely often escapes (*i.e.*, is greater than) this $\mathbf{0}'$-computable function. We will then use this $\mathbf{d}$-computable escape function to compute a $\mathbf{d}$-monotone extension function for X. The challenge will be to define the $\mathbf{0}'$-computable function in a robust enough way to ensure that escaping it only infinitely often ensures that we will settle on a correct amalgamator.

Proof of Theorem 2.3.2. Let $\mathbf{d} \leq \mathbf{0}'$ be a nonlow$_2$ degree. Let $\mathcal{A}$ be a nontrivial homogeneous model with a $\mathbf{0}$-basis X of a complete decidable theory T. We show $\mathcal{A}$ has a $\mathbf{d}$-decidable copy $\mathcal{B}$. By Theorem 1.3.8, it suffices to show there exists a $\mathbf{d}$-monotone extension function for that X.

Let $X = \{p_i\}_{i\in\omega}$ be the computable enumeration of $\mathbb{T}(A)$. We build a $\mathbf{d}$-monotone extension function for X. We first define a set of triples that encodes which types amalgamate others. Then we use an approximation of this set to define our best guess for amalgamators at a given stage. Finally we use the escape property of nonlow$_2$

Δ^0_2 degrees to infinitely often find stages where these guess amalgamators are true amalgamators.

Let $S = \{\langle i, \alpha, j\rangle |\ (n+1)\text{-type } p_j \text{ extends } (n+1)\text{-ary } \theta_\alpha \text{ and } n\text{-type } p_i\}$.

Since $X = \{p_i\}_{i\in\omega}$ is uniformly computable, S is a Π^0_1 set. Hence there exists a computable sequence $\{S_s\}_{s\in\omega}$ such that for all $x \in \omega$, $S(x) = \lim_s S_s(x)$. We may assume for every $\alpha \in 2^{<\omega}$ and $i, s \in \omega$ where θ_α and p_i are consistent, S_s contains an element $\langle i, \alpha, j\rangle$ for some $j \in \omega$ (*i.e.*, S_s provides some guess amalgamator for such p_i and θ_α).

For all $i \in \omega$ and $\alpha \in 2^{<\omega}$ where p_i is an n-type, θ_α is an $(n+1)$- formula, and p_i is consistent with θ_α define:

- the *true amalgamator target* $y_{\langle i,\alpha\rangle} = (\mu\langle i, \alpha, j\rangle)[\langle i, \alpha, j\rangle \in S]$ and
- the *approximate amalgamator target* $y^s_{\langle i,\alpha\rangle} = (\mu\langle i, \alpha, j\rangle)[\langle i, \alpha, j\rangle \in S_s]$.

Let

$$h(n) = (\mu\, s)\,(\forall\, i, |\alpha| \,\le\, n)\,(\forall\, w \,\le\, y^s_{\langle i,\alpha\rangle})\ ,(\forall\, t \,\ge\, s)\,[\,S_t(w) = S_s(w) = S(w)\,].$$

In other words, $h(n)$ is the least stage s by which for all $i, |\alpha| \le n$, $S_s(w)$ has settled forever for all $w \le y^s_{\langle i,\alpha\rangle}$.

Since $\mathbf{d} \le \mathbf{0'}$ is nonlow$_2$ and h is a Π^0_1 function, by Theorem 2.3.3 (the escape characterization),

$$(\exists f \le \mathbf{d})(\exists^\infty x)[h(x) \le f(x)].$$

We may assume that f is increasing. Let $T = \{x \in \omega | h(x) \le f(x)\}$. By above, T is an infinite set. We speed up our computable approximation to S by setting $\hat{S}_s = S_{f(s)}$. We now use the computable approximation $\{\hat{S}_s\}_{s\in\omega}$. We define $\hat{y}^s_{\langle i,\alpha\rangle} = y^{f(s)}_{\langle i,\alpha\rangle}$. By definition of h, any apparent target $\hat{y}^t_{\langle i,\alpha\rangle} = y^{f(t)}_{\langle i,\alpha\rangle}$ at a true stage $t \in T$ is the true target $y_{\langle i,\alpha\rangle}$ if $i, |\alpha| \le t$. Since an apparent target is a true target for $t \in T$ only if $i, |\alpha| \leq t$, we will be careful to ensure that we only lay down at most s many formulas at stage s. We call T the set of *true stages*.

To build a **d**-monotone extension function for X it suffices to show that given an index i for an n-type p_i and an $(n+1)$-formula θ_β consistent with p_i, we can **d**-uniformly compute for all s an index j_s such that $j = \lim_s j_s$ exists, $p_j \restriction s = p_{j_s} \restriction s$, and p_j amalgamates p_i and θ_β. (To then find the amalgamator for p_i and θ_k, we first computably find a θ_β such that $|\beta| = k+1$, $\beta(k) = 1$ and θ_β is consistent with p_i.) This is equivalent to **d**-uniformly building an $(n+1)$-type q that amalgamates p_i and θ_β in stages such that q_s has length s and $q = p_j$ for some j.

Let p_i be an n-type and θ_β an $(n+1)$-formula consistent with p_i.

Construction.

We construct q in stages so that $q = \cup_{s \in \omega} q_s$ and $|q_s| = s$ for all s. At each stage we also have a guide $(n+1)$-formula ψ_s that determines which target $\hat{y}^s_\alpha = \hat{y}^s_{<i,\alpha>}$ we should rely on to determine q. We ensure that ψ_s always has the form θ_α for some $\alpha \in \{0,1\}^{<\omega}$ for all $s \in \omega$.

Stage 0**.** Let $q_0 = \emptyset$. Let $\psi_0 = \theta_\beta$.

Stage $s+1$**.**

Assume we are given q_s such that $|q_s| = s$ and q_s is consistent with p_i and θ_β. Let ψ_s have the form θ_γ and be consistent with p_i and q_s.

If $|\gamma| \geq s+1$, set $q_{s+1} = \gamma \restriction (s+1)$ and $\psi_{s+1} = \psi_s$, *i.e.*, follow θ_γ. (Since θ_γ is consistent with q_s and $|q_s| = s$, $q_s \subseteq q_{s+1}$.)

Otherwise, if $|\gamma| < s+1$, let $m = \langle \hat{y}^{s+1}_\gamma \rangle_3$ where $\langle\rangle_3$ denotes the computable function where $\langle (\langle a, b, c \rangle) \rangle_3 = c$. (In plain language, p_m is thought to be an amalgamator for p_i and θ_γ.) Then consider $p_m \restriction (s+1)$. Check whether $p_m \restriction (s+1)$ is consistent with p_i and $\psi_s = \theta_\gamma$ and q_s and whether p_m is an $(n+1)$-type. If it is consistent and an $(n+1)$-type, set $q_{s+1} = p_m \restriction (s+1)$ and set $\psi_{s+1} = \psi_s = \theta_\gamma$. *We trust the guess amalgamator.*

If it is not consistent or not an $(n+1)$-type, since q_s is consistent with p_i and ψ_s and $|q_s| = s$, we can extend q_s by one digit to q_{s+1} while maintaining consistency with p_i and ψ_s and q_s. Since our target amalgamator was incorrect, we update it while respecting the choices made in q_{s+1}. Let $\psi_{s+1} = \theta_\delta$ where θ_δ is the $(n+1)$-formula θ_α with $|\alpha|$ minimal and θ_α proves the conjunction of θ_β and the formulas in q_{s+1} and is consistent with p_i. Note that $|\delta| \leq \max(|\beta|, |q_s|)$. *We have been shown that the guess amalgamator is invalid, and hence, we must find a new guide formula ψ_{s+1} based on what we have already laid down in q_{s+1}.*

End Construction.

Verification.

We show $q = \cup_{s\in\omega} q_s$ is an $(n+1)$-type in $\mathbb{T}(\mathcal{A})$ that extends p_i and θ_β as desired. First note that for all s, we have $q_s \subseteq q_{s+1}$, and q_{s+1} is consistent with p_i and θ_β. Hence q is consistent with p_i and θ_β. Since $|q_s| = s$ for all s, the type q is total. We must show that $q \in \mathbb{T}(\mathcal{A})$ (so then $q = p_j$ for some j).

Let $|\psi_s|$ by definition be $|\gamma|$ where $\psi_s = \theta_\gamma$. At stage s, $|\psi_s| \leq \max(|\beta|, |q_s|)$. Since $|q_s| = s$ for all s, $|\psi_s| \leq s$ for all stages $s \geq |\beta|$. Since the set of true stages T is infinite, we may choose a true stage $t' \geq \max(|\beta|, i)$. Suppose $\psi_{t'} = \theta_\gamma$. Since $|\psi_{t'}|, i \leq t'$ and t' is a true stage, $\hat{y}_\gamma^{t'}$ equals y_γ a true target. Hence, $p = p_{\langle \hat{y}_\gamma^{t'} \rangle_3}$ truly amalgamates p_i and $\psi_{t'}$ and thus p_i and θ_β. Moreover, by definition, p is an $(n+1)$-type realized in $\mathcal{A}$. By construction, (since p is actually a true amalgamator for p_i and $\psi_{t'}$) $\psi_s = \psi_{t'}$ and $q_s = p\restriction s$ for all $s \geq t'$. Hence q equals p, an $(n+1)$-type realized in $\mathcal{A}$, and q extends p_i and θ_β as desired. In other words, once we have unwittingly stumbled onto a true amalgamator at such a true stage picked above, we will never abandon this target.

The construction is clearly $\mathbf{d}$-effective, and hence q is $\mathbf{d}$-uniformly computable from a computable index for p_i. By this fact, we can build the desired $\mathbf{d}$-monotone extension function for X as we did in the proof of Theorem 2.2.3. □

Thus if $\mathbf{d} \leq \mathbf{0}'$ and $\mathbf{d}$ is nonlow$_2$, there exists a copy $\mathcal{B}$ of $\mathcal{A}$ of degree $\mathbf{d}$, in other words $\mathbf{d} \in dSp^e(\mathcal{A})$, for any automorphically nontrivial homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis. Furthermore, note that any degree $\mathbf{d} \notin \mathbf{GL_2}$, *i.e.*, $(\mathbf{d} \cup \mathbf{0}')' < \mathbf{d}''$, is also $\mathbf{0}$-basis homogeneous bounding by the same construction in Theorem 2.3.2 since $\mathbf{d}$ satisfies the escape property (see §6. in [3]).

In Chapter 3, we will show that the nonlow$_2$ degrees exactly characterize the $\mathbf{0}$-bounding degrees within the Δ^0_2 degrees.

CHAPTER 3
NEGATIVE RESULTS ON HOMOGENEOUS MODELS

3.1 Non 0-Basis Homogeneous Bounding Degrees

Recall the following definition.

Definition 3.1.1. A degree $\mathbf{d}$ is $\mathbf{0}$-*basis homogeneous bounding* if for any nontrivial homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis, there exists a $\mathcal{B} \cong \mathcal{A}$ such that $\mathcal{B}$ is $\mathbf{d}$-decidable, *i.e.*, $\mathbf{d} \in dSp^e(\mathcal{A})$.

We showed in Theorem 2.3.2 that any nonlow$_2$ $\mathbf{d} \leq \mathbf{0}'$ ($\mathbf{d}'' > \mathbf{0}''$) is $\mathbf{0}$-basis homogeneous bounding. We now show that the $\mathbf{0}$-basis homogeneous bounding degrees exactly characterize the nonlow$_2$ degrees below $\mathbf{0}'$.

Theorem 3.1.2. *Let* $\mathbf{d} \leq \mathbf{0}'$. *The degree* $\mathbf{d}$ *is* $\mathbf{0}$-*basis homogeneous bounding if and only if* $\mathbf{d}$ *is nonlow*$_2$.

For brevity, we substitute the shorter term $\mathbf{0}$-*bounding* for $\mathbf{0}$-*basis homogeneous bounding*. The remaining direction, which we prove in Theorem 3.5.4, is an extension of the result by Goncharov [8], Peretyat'kin [27], and Millar [23] that there exists a homogeneous model with a $\mathbf{0}$-basis but no decidable copy. Given a low$_2$ degree $\mathbf{d} \leq \mathbf{0}'$, we construct a homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis but no $\mathbf{d}$-decidable copy $\mathcal{B}$. Therefore $\mathbf{d}$ is not a $\mathbf{0}$-bounding degree. The characterization in Theorem 3.1.2 is analogous to the following result about prime models. A degree $\mathbf{d}$ is *prime bounding* if any complete atomic decidable theory has a prime model decidable in degree $\mathbf{d}$.

Theorem 3.1.3. (Csima, Hirschfeldt, Knight, Soare [6]) *Let* $\mathbf{d} \in \Delta^0_2$. *The degree* $\mathbf{d}$ *is nonlow*$_2$ *if and only if* $\mathbf{d}$ *is prime bounding.*

We fix a low$_2$ degree $\mathbf{d} \leq \mathbf{0}'$ and construct a nontrivial homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis but no $\mathbf{d}$-decidable copy. Remember that $\mathcal{A}$ has a $\mathbf{d}$-decidable copy if and only if every $\mathbf{d}$-uniform basis for $\mathcal{A}$ has a $\mathbf{d}$-monotone extension function by Theorem 1.3.8, the relativized Monotone Extension Function Theorem. Notice that a $\mathbf{0}$-basis can be effectively made into a $\mathbf{d}$-uniform basis for all degrees $\mathbf{d}$. Thus we build a homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis such that the $\mathbf{0}$-basis has no $\mathbf{d}$-monotone extension function. Then $\mathcal{A}$ has no $\mathbf{d}$-decidable copy.

To build such a counterexample, we satisfy two general requirements that will be described in more detail later. First, we ensure that we are building a *homogeneous* model $\mathcal{A}$ with a $\mathbf{0}$-basis X.

P: $\mathcal{A}$ is a homogeneous model with a $\mathbf{0}$-basis of types X.

Second, we require that $\mathcal{A}$ has no $\mathbf{d}$-decidable copies. In other words,

N: The $\mathbf{0}$-basis X for $\mathcal{A}$ has no $\mathbf{d}$-monotone extension function.

3.2 The Positive Requirements

First, we explore how to satisfy $\mathbf{P}$. The next section describes how to build a basis of types that is realized by some homogeneous model.

3.2.1 Building Homogeneous Models

Our counterexample will model a complete decidable theory T that we describe later. We will construct a $\mathbf{0}$-basis of types of T that satisfies the closure properties described in Theorem 3.2.1. Then, this $\mathbf{0}$-basis will equal $\mathbb{T}(\mathcal{A})$ for some homogeneous model $\mathcal{A}$ of T.

Theorem 3.2.1 (Goncharov [8], Peretyat'kin [27]). *Let T be a complete theory, and suppose X is a countable set contained in $S(T)$. Then there exists a homogeneous model realizing exactly the types in X if and only if*

1. $T \in X$

2. *X is closed under permutations of variables*

3. *X is closed under taking subtypes*

4. **Extension Property (EP)**
If $p(\bar{x}) \in X$ and $\varphi(\bar{x}, y)$ are consistent, then there exists a type $q(\bar{x}, y) \in X$ such that $p \cup \{\varphi\} \subseteq q$.

5. **Type Amalgamation Property (TAP)**
For any pair of types $p_1(\bar{x}, y), p_2(\bar{x}, z) \in X$ such that $p_1 \restriction \bar{x} = p_2 \restriction \bar{x}$, there exists a type $q(\bar{x}, y, z) \in X$ containing p_1 and p_2.

3.2.2 Refining the Positive Requirements

The **0**-basis $X = \{p_i\}_{i\in\omega}$ is encoded as a function from $\omega \times \omega$ to 2, where the restriction of this function to the domain $\{i\} \times \omega$ encodes the type p_i. We build X computably in stages, and view X as a uniformly computable infinite matrix where the i^{th} row corresponds to a type p_i in $\mathbb{T}(\mathcal{A})$.

To ensure that a given homogeneity closure condition with respect to a row or pair of rows of X is satisfied, we place down a marker H on an empty row of X. Then we ensure that H moves to a new row finitely often during the construction and that the row on which it settles satisfies the given homogeneity closure condition. Hence **P** can be restated as:

P: All homogeneity markers settle, and the rows that they settle on satisfy the required homogeneity closure condition.

More specifically, we must satisfy for all i and j the following requirements.

Q$_j$**:** Let π_k denote the k^{th} permutation of the free variables in the type on row j. We assign markers $Perm_{j,k}$ to **Q**$_j$. Then $Perm_{j,k}$ settles on a row corresponding to the type generated by the permutation π_k of the free variables applied to p_j.

S$_j$**:** Let V_k denote the k^{th} distinct subset of the free variables in p_j. For each V_k, marker $Sub_{j,k}$ will settle on a row corresponding to the subtype of p_j generated by the free variables in V_k.

$\mathbf{R}_{i,j}$: If an $(n+1)$-ary θ_j is consistent with the n-type p_i, then marker $Tr_{i,j}$ will settle on a row k such that p_k extends p_i and $\{\theta_j\}$.

$\mathbf{T}_{i,j}$: If rows i and j correspond to types $p_i(\bar{x}, y), p_j(\bar{x}, z)$ and $p_i \restriction \bar{x} = p_j \restriction \bar{x}$, then marker $Tap_{i,j}$ settles on a row k of type $q(\bar{x}, y, z)$ containing p_i and p_j. (We allow the possibility that $\bar{x}$ is empty).

In the construction, we place the theory T, a 0-type, on row 0 of X. Note that the closure conditions for permutations of variables, subtypes, and type amalgamation are trivially satisfied for row 0 (together with any other row in the case of type amalgamation). To satisfy $\mathbf{R}_{0,j}$, we will ensure that for every 1-ary formula $\theta_j(x)$ consistent with T, there is some row that corresponds to a 1-type containing θ_j. Hence we only use homogeneity markers to satisfy the above requirements for rows that correspond to n-types for $n \geq 1$. We will not use homogeneity markers to satisfy positive requirements involving row 0, *i.e.*, the 0-type T.

3.3 The Negative Requirements

Let $\mathbf{d} \leq \mathbf{0}'$ be a low$_2$ degree. Our goal is to build a $\mathbf{0}$-basis of types $X = \{p_i\}_{i \in \omega}$ for a homogeneous model $\mathcal{A}$ such that X has no $\mathbf{d}$-monotone extension function. For X to be a $\mathbf{0}$-basis of a homogeneous model, X must satisfy the homogeneity closure conditions described above in Theorem 3.2.1.

3.3.1 A Characterization of Low$_2$ Δ^0_2 Degrees

We use the following characterization of the low$_2$ Δ^0_2 degrees to enumerate all the $\mathbf{d}$-computable functions. We use this enumeration to ensure that no $\mathbf{d}$-computable function can be a $\mathbf{d}$-monotone extension function for the $\mathbf{0}$-basis X that we are building.

Theorem 3.3.1 (Derived from Jockusch [14], see [6] p. 1125)**.** *A degree $\mathbf{d} \leq \mathbf{0}'$ is low$_2$ if and only if the $\mathbf{d}$-computable functions are $\mathbf{0}'$-uniform. In other words, there exists a function $g \leq \mathbf{0}'$ such that if $g_e(x) = g(e,x)$, then $\{g_e\}_{e \in \omega} = \{f : f \leq \mathbf{d}\}$.*

Applying the limit lemma to the above, we obtain:

Corollary 3.3.2. *If* $\mathbf{d} \leq \mathbf{0}'$ *is low*$_2$*, there exists a computable function* $g(e, x, s)$ *such that* $g_e(x) = \lim_s g(e, x, s)$ *exists for all* e *and* x *and* $\{g_e : e \in \omega\} = \{f : f \leq \mathbf{d}\}$.

Let g_e denote the e^{th} **d**-computable function. Let $g_{e,s}(x) = g(e, x, s)$ be the computable approximation to g_e at stage s. At each stage s we have a computable approximation $\{g_{e,s}\}_{e\in\omega}$ to the list $\{g_e\}_{e\in\omega}$ and hence to a list of the **d**-computable functions.

3.3.2 Refining the Negative Requirements

Given this listing of **d**-computable functions, **N** can be restated as, for all e,

N$_e$**:** The function g_e is not an MEF for the **0**-basis X.

To show that g_e is not an MEF, it suffices to show that g_e does not behave like an MEF on a particular p_i and θ_j. Fix i and j. Let $\Lambda_s(t) = g_{e,s}(\langle i, j, t\rangle)$ and $\Lambda(t) = \lim_{s\to\infty} \Lambda(t) = g_e(\langle i, j, t\rangle)$.

Definition 3.3.3. Let $X = \{p_i\}_{i\in\omega}$ be a basis. Let p_i be an n-type and θ_j be an $(n+1)$-ary formula consistent with p_i. Let $\Lambda(t)$ be a function from ω to ω.

1. We say Λ *rests on row* k *at level* t if $\Lambda(t) = k$ and Λ *settles* on row k if $\lim_t \Lambda(t) = k$ exists.

2. We say that Λ *traces out an amalgamator through level* t for an n-type p_i and $(n+1)$-ary formula θ_j if:

   (a) $\Lambda(t) = k$

   (Λ rests on row k at level t),

   (b) $p_{\Lambda(t')}\restriction t' = p_{\Lambda(t'+1)}\restriction t'$ for all $t' < t$

   (Λ respects the formula monotonicity property of MEFs), and

   (c) p_k is an $(n+1)$-type containing θ_j if $j < t$ and the formulas in the partial type $p_i\restriction t$

(Through level t, the type p_k extends p_i and $\{\theta_j\}$).

(d) $\Lambda(t') \leq \Lambda(t'+1)$ for all $t' < t$.

(Λ is monotonically increasing.)

(e) $p_{\Lambda(t')}\restriction t' \neq p_j\restriction t'$ for all $j < \Lambda(t')$ for all $t' \leq t$.

(Λ rests on the least possible row in X that satisfies the above conditions at each level t.)

3. We say Λ *traces out an amalgamator* for p_i and θ_j if Λ does so through all levels.

Let p_i be any n-type and θ_j be any $(n+1)$-ary formula consistent with p_i. If g_e is an MEF, Λ will both trace out an amalgamator for p_i and θ_j and settle on some row (which corresponds to the amalgamator being traced out). To ensure that g_e is not an MEF, we fix a 1-type $p_i(x)$ and a 2-ary formula $\theta_j(x,y)$ consistent with p_i and build X so that if Λ appears to be tracing out an amalgamator for p_i and θ_j, Λ does not settle on any row.

Suppose Λ rests on row k at level t. If Λ appears to be tracing out an amalgamator, then we build X so that Λ may not settle on row k and continue to trace out an amalgamator. Then, for Λ to continue tracing out an amalgamator, Λ must move from row k to a row l for $l > k$, *i.e.*, $\Lambda(t') = l$ for some $t' > t$. If Λ continues to trace out an amalgamator, we will ensure that Λ must move infinitely often, preventing Λ from settling. Hence we can describe the requirement $\mathbf{N}_e$ as, for all k,

$\mathbf{N}_{e,k}$**:** If Λ traces out an amalgamator for p_i and θ_j, then Λ does not settle on row k.

Note that we use "stage" and "level" to describe different concepts. We use the stage of a construction to obtain our computable approximation Λ_s to Λ. "Level" refers to the length of time Λ_s has been tracing out an amalgamator (as in Definition 3.3.3).

3.3.3 Two Examples

Before venturing into the construction, we start with two examples in order to give some intuition for how the positive and negative requirements interact and how they can be satisfied. Some details are left out here but will be fully described in the proof. We fix a 1-type $p_i(x)$ and a 2-ary formula $\theta_j(x,y)$ consistent with p_i on which to satisfy $\mathbf{N}_{e,k}$ for all k.

Example 1

Suppose that at stage s of the construction there exists some $t \leq s$ such that Λ_s traces out an amalgamator through level t for p_i and θ_j. Furthermore suppose that Λ_s rests on row k at level t. Since p_k extends p_i and θ_j through level t, we wish to force Λ off row k of X to satisfy $\mathbf{N}_{e,k}$. To do this, we want to extend row k so that p_k is inconsistent with p_i. Then Λ cannot remain on row k if g_e is an MEF regardless of whether Λ_s is a good approximation of Λ through level t.

Suppose that no homogeneity markers rest on row k. Then there are no constraints from the positive requirements on how we build row k. By the flexibility of the theory and the fact that at stage s only finitely much of rows k and i have been filled, we can find some unary relation P_l that has not appeared in any formula in p_k or p_i by this stage in the construction. We extend X so $p_k(x,y)$ and $p_i(x)$ disagree on the formula $P_l(x)$ and hence p_k cannot extend p_i. Then $\mathbf{N}_{e,k}$ is forever satisfied because if Λ settles on row k, Λ cannot trace out an amalgamator. In this case, it is easy to satisfy $\mathbf{N}_{e,k}$ because no homogeneity marker rests on row k.

Example 2

Now suppose that row k (on which Λ_s is resting at level t) has a homogeneity marker H resting on it. In the most extreme case, suppose that H is the marker that requires that we build an amalgamator for p_i and θ_j. In this case, we say row k is *dependent* on row i.

We wish to satisfy $\mathbf{N}_{e,k}$ by making p_k inconsistent with $p_i \cup \{\theta_j\}$. But marker H requires that row k be built so that p_k extends $p_i \cup \{\theta_j\}$. Thus our need to satisfy a homogeneity condition directly conflicts with $\mathbf{N}_{e,k}$. We ensure that p_i and θ_j have an amalgamator in the $\mathbf{0}$-basis X but that this amalgamator row cannot be found $\mathbf{d}$-monotonically.

Suppose Λ traces out an amalgamator (otherwise $\mathbf{N}_{e,k}$ is satisfied automatically). To resolve the tension between the positive and negative requirements, we exploit our assumption that Λ satisfies the formula monotonicity property, *i.e.*, condition 2b. in Definition 3.3.3. We also allow homogeneity marker H to move finitely often to a different row. We will find two possible consistent extensions for row k that differ on some formula $\theta_{split^k_k}$. Since rows i and k and formula θ_j contain only finitely much information at stage s, they have not commented on some binary relation R_l. By the flexibility of the theory, there are two ways to extend row k so that one extension contains $R_l(x, y)$ and the other contains $\neg R_l(x, y)$ and both extensions are consistent with p_i and θ_j.

At this stage we extend row k in one direction of the splitting and we build on an empty row $k' > k$ the other direction (*i.e.*, one row contains $R_l(x, y)$ and the other contains $\neg R_l(x, y)$). Let $\theta_{split^k_k}$ be the formula $R_l(x, y)$ in the effective enumeration of all formulas, and let $split^{k'}_k = split^k_k$. We say that row k and k' *split* at $\theta_{split^k_k}$. We call rows k and k' and the types they correspond to a *splitting* of the partial type corresponding to row k at stage s, and we call this process *building a splitting* of row k in marker H on rows k and k'.

Then we will see which extension (if any) Λ_s traces out. If Λ_s traces out an amalgamator through level $split^k_k$, either Λ_s will move off of rows k and k' by level $split^k_k$ or Λ_s will decide whether the type it is tracing out will include $R_l(x, y)$ or $\neg R_l(x, y)$, *i.e.*, which direction of the splitting $split^k_k$ it will follow. At that point we will correctly make row i inconsistent with the row Λ_s chose and move the marker H onto the other row. In either case, if Λ_s correctly approximates Λ through level t, $\mathbf{N}_{e,k}$ will be satisfied forever (since Λ respects formula monotonicity and is monotonically increasing). Moreover, we continue to have a row on which to satisfy H. If Λ_s does not correctly approximate Λ through level t, we repeat this procedure at the same splitting formula. Eventually the approximation of Λ will be correct and the strategy will succeed.

We will show later that only finitely many Λ can move a given homogeneity marker such as H and that each such Λ can only move a single marker finitely many times.

Thus each homogeneity marker moves only finitely often as desired.

The strategy of setting and monitoring splittings can be generalized to deal with any case where both row k and Λ depend on row i.

3.4 Enacting the Strategy

In the second example above, we saw how a row in the matrix of types X can be dependent on another row via a homogeneity marker. We can think of these dependent rows as being generated (according to the homogeneity markers) from other rows. In the next section we formalize this notion of dependency. For a homogeneity marker H, we denote the row on which H rests at stage s by h_s.

3.4.1 Row Dependencies from Homogeneity Markers

Definition 3.4.1. Suppose that H rests on row k at stage s and that H is a homogeneity marker made to satisfy some closure condition for row j. Then row k is *directly dependent* on row j at stage s. A row without a homogeneity marker at stage s is an *independent row* at stage s.

Definition 3.4.2. We define $R_{k,s}$ the *dependency graph for row k at stage s* as follows. Let the rows of X denote the nodes in a directed graph G. We include an edge from row i to row j in G if row i is directly dependent on row j at stage s. Let $R_{k,s}$ be the subgraph of G consisting of all nodes and edges that are on some path in G beginning at row k. We say row k is *generated from* or *depends on* row j if row j is a node in $R_{k,s}$ and $j \neq k$.

Our construction will ensure that if row k depends on row j, then $k > j$. The nodes in $R_{k,s}$ with no outgoing edges will be independent rows at stage s. Recall that the closure conditions for row 0, the row corresponding to the theory T, will be satisfied without using homogeneity markers. Thus for $k > 0$, no $R_{k,s}$ contains row 0 as a node.

Definition 3.4.3. We inductively define the *rank of a row k at stage s* as follows:

(i) $rank_s(k) = 0$ if k is an independent row at stage s, and

(ii) $rank_s(k) = \max_{j \in R_{k,s}, j \neq k} rank_s(j) + 1$ otherwise.

Rank is well defined by the definition of $R_{k,s}$ and by our construction assumption that if k depends on j, then $k > j$.

Dependency graphs help us determine whether we will act as in the first example above or the second. Suppose Λ_e appears to be tracing out an amalgamator for p_{i_e} and θ_{j_e}. If Λ_e rests on row k and row $i_e \notin R_{k,s}$, we will show that there is a way to consistently extend rows i and k while respecting the homogeneity marker on row k so that p_i and p_k are inconsistent. We will also prove that we can enact the splitting strategy described in the second example if $i_e \in R_{k,s}$.

3.4.2 A Simple Complete Decidable Theory

We define a flexible complete decidable theory T that will be the theory of our counterexample $\mathcal{A}$. We utilize the flexibility of the theory to satisfy requirements **P** and **N** simultaneously.

The Theory T

The language of T is $L = \{P_0, P_1, P_2, P_3, \ldots, R_0, R_1, R_2, R_3, \ldots\}$ where P_i is a unary relation symbol and R_i is a binary relation symbol for all $i \in \omega$. Let $L_s = \{P_0, \ldots, P_{s-1}, R_0, \ldots, R_{s-1}\}$ for $s \geq 0$; note $L_0 = \emptyset$. Let $T = \bigcup_{s \in \omega} T_s$ where T_s is the set of sentences in L_s defined below.

The theory T_s consists of the following axiom schema:

$$Ax_s \quad (\forall x_0 \ldots \forall x_{n-1})[\delta'(x_0, x_1, \ldots, x_{n-1}) \rightarrow (\exists x_n)\delta(x_0, x_1, \ldots, x_{n-1}, x_n)],$$

where δ is a consistent conjunction of atomic formulas and negations of atomic formulas in L_s and δ' is a subformula of δ.

Theorem 3.4.4. *The set $T = \bigcup_{s \in \omega} T_s$ is a complete decidable theory, and T admits quantifier elimination.*

This follows from the next series of lemmas.

Lemma 3.4.5. *For $s \geq 0$, T_s is consistent.*

Proof. The theory T_0 is consistent. Assume $s > 0$. Let $\mathcal{A}_0$ be a model in L_s of T_0. We construct a model $\mathcal{A}$ of T_s. Let $\theta_1, \theta_2, \ldots$ be an enumeration of all axioms Ax_s in which each axiom appears infinitely often. Let θ_1 be of the form

$$(\forall x_0 \ldots \forall x_{n-1})[\delta'(x_0, x_1, \ldots, x_{n-1}) \rightarrow (\exists x_n)\delta(x_0, x_1, \ldots, x_{n-1}, x_n)].$$

We extend $\mathcal{A}_0$ to $\mathcal{A}_1$ in such a way that $\mathcal{A}_1$ satisfies the matrix of θ_1 on all n-tuples of $\mathcal{A}_0$. Let $A_1 = A_0 \cup \{a\}$ where $a \notin A_0$.

Suppose $\mathcal{A}_0 \vDash \delta'(a_0, a_1, \ldots, a_{n-1})$ for $a_0, a_1, \ldots, a_{n-1} \in A_0$. Extend the definitions of the predicates in L_s to the set $\{a_0, a_1, \ldots, a_{n-1}, a\}$ so that

$$\mathcal{A}_1 \vDash \delta(a_0, a_1, \ldots, a_{n-1}, a).$$

Continuing similarly, we construct a chain of models

$$\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$$

Let $\mathcal{A} = \bigcup_{s \geq 0} \mathcal{A}_s$. Then $\mathcal{A} \vDash T_s$.

□

Lemma 3.4.6. *For $s \geq 0$, T_s is $\aleph_0$-categorical.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be countable models of T_s. We show that they are isomorphic. Assume f is a finite (partial) isomorphism from $\mathcal{A}$ to $\mathcal{B}$ and $dom(f) = \{a_0, a_1, \ldots, a_{n_1}\}$. Let $\delta'(x_0, x_1, \ldots, x_{n-1})$ be the finite diagram of $\mathcal{A}$ determined by $dom(f)$, let a be an element of $A \setminus dom(f)$, and $\delta(x_0, x_1, \ldots, x_{n-1}, x_n)$ be the finite diagram of $\mathcal{A}$ determined by $dom(f) \cup \{a\}$. Now $\mathcal{B} \vDash \delta'(f(a_0), f(a_1), \ldots, f(a_{n-1}))$. Thus there is a $b \in B$ such that

$$\mathcal{B} \vDash \delta(f(a_0), f(a_1), \ldots, f(a_{n-1}), b).$$

Then $f_1 = f \cup \{(a,b)\}$ is a finite isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Symmetrically, we can extend f to include a given $b \in B$ in the range of an extension of f. □

Lemma 3.4.7. *For $s \geq 0$, T_s is complete and decidable.*

Proof. T_s has no finite models since $\{x_i \neq x_j \mid i \neq j \ \& \ i,j \leq n\}$ belongs to a finite diagram of T_s for all n. Since T_s is $\aleph_0$-categorical, by the Los-Vaught Test, it is complete. Since T_s is complete and computably axiomatizable, T_s is decidable. □

Lemma 3.4.8. *For $s \geq 0$, T_s admits elimination of quantifiers.*

Proof. We show that T_s is submodel complete and hence admits quantifier elimination. Let $\mathcal{A}, \mathcal{B} \models T_s$ and $\mathcal{D} \subseteq \mathcal{A}, \mathcal{B}$. We show $\mathcal{A}$ and $\mathcal{B}$ satisfy the same existential sentences in L_s with parameters from $\mathcal{D}$. Let $\mathcal{A} \models \theta(d_0, \ldots, d_{n-1}, a_0, \ldots, a_{m-1})$, where $d_0, \ldots, d_{n-1} \in D$ and $a_0, \ldots, a_{m-1} \in A \setminus D$. Extend the identity function on $\{d_0, \ldots, d_{n-1}\}$ to a finite isomorphism f from $\mathcal{A}$ to $\mathcal{B}$ such that $a_0, \ldots, a_{m-1} \in dom(f)$ as in the categoricity lemma above. Then

$$\mathcal{B} \models (\exists x_0 \ldots \exists x_{m-1})\, \theta(d_0, \ldots, d_{n-1}, x_0, \ldots, x_{m-1}).$$

□

3.4.3 Satisfying Positive Requirements

Let X_s be the stage s approximation of the **0**-basis X we are building for the homogeneous model $\mathcal{A}$, and let $QF(\theta)$ be the least-indexed quantifier-free formula equivalent to θ. We say a formula θ is *decided* in row k at stage s if the partial type p corresponding to row k in X_s includes θ or $\neg\theta$. We consider each homogeneity closure requirement, and we assume that at each stage s of the construction we have the following invariants.

Invariants on X

- Each row in X_s is consistent with T.
- For all $k > 0$, only finitely many formulas in row k of X_s have been decided.
- For all $k > 0$, if $\theta \in p_k$ at stage s, then all the atomic formulas in $QF(\theta)$ are decided in row k at stage s.

  Suppose row k is directly dependent on row i via homogeneity condition H at stage s. Let ρ denote the conjunction of all literals included in row k at stage s.

- If $H = Perm_{i,m}$ or $H = Sub_{i,m}$, then row k is the correct permutation of variables or subtype of the type in row i.
- If $H = Tr_{i,j}$ and p_i and θ_j are consistent, then

$$T \cup p_i(\bar{x}) \vdash (\exists y)[\rho(\bar{x}, y) \wedge QF(\theta_j(\bar{x}, y))]. \tag{3.1}$$

- Suppose $H = Tap_{i,j}$, and p_i and p_j share the free variables $\bar{x}$. Assume that there exists some $q(\bar{x})$ extending the subtypes in $\bar{x}$ of the types corresponding to rows i and j at stage s. Then $T \cup p_i(\bar{x}, y) \vdash (\exists z)\rho(\bar{x}, y, z)$ and $T \cup p_j(\bar{x}, z) \vdash (\exists y)\rho(\bar{x}, y, z)$.

  (Row k can be extended to satisfy the requirement corresponding to H.)

We show that we can computably build X by finite extensions while ensuring that these invariants are maintained.

Lemma 3.4.9. *Suppose X_s satisfies the invariants above, and let row k of X_s correspond to partial type $p_k(\bar{x})$. Let F be a finite set of formulas in $\bar{x}$. Then there exists a finite extension X_{s+1} of X_s so that all the formulas in F are decided in row k and the above invariants are maintained. Moreover, only the rows in $R_{k,s}$ are extended in creating X_{s+1}, and X_{s+1} is uniformly computable from X_s.*

Proof. We prove the lemma by induction on the rank of row k at stage s. Let ρ be the conjunction of all the literals included in row k at stage s.

Suppose $rank_s(k) = 0$. Then row k has no homogeneity marker resting on it. We show how to extend k to decide some formula θ. First extend row k to include all the (positive) atomic subformulas of $QF(\theta)$ that are not decided in row k and exclude their negations. This is a consistent extension by definition of T and the consistency of row k of X_s. Check whether $T \cup \{\rho'\} \vdash QF(\theta)$ or $T \cup \{\rho'\} \vdash \neg QF(\theta)$ where ρ' is the conjunction of ρ and the included atomic formulas. Since all literals in $QF(\theta)$ are decided, one of the statements holds. Include or exclude θ in row k respectively. Note that all invariants are maintained and that this is a computable extension. To decide a finite set of formulas, repeat this process.

Suppose $rank_s(k) = n + 1$ and that the lemma holds for all rows of rank less than or equal to n. Suppose marker $Tr_{i,j}$ rests on row k, row i corresponds to an l-type p_i, $rank_s(i) = n$, and θ_j is an $(l + 1)$-formula.

First suppose row k is empty. By induction, extend row i to decide all the atomic subformulas of $QF(\,(\exists y)\,[\theta_j(\bar{x}, y)]\,)$ if they are not already decided. If this extension implies $(\exists y)\,[\theta_j(\bar{x}, y)]$, then p_i and θ_j are consistent. We extend row k to include θ_j and consistently decide all the atomic formulas in $QF(\theta_j)$ and extend row i so that this extension ρ of row k proves $QF(\theta_j)$ and row i proves $(\exists y)[\rho(\bar{x}, y) \wedge QF(\theta_j)]$. This extension is computable since T is decidable. Otherwise p_i and θ_j are not consistent, and we can treat row k as in the rank zero case.

Assuming p_i and θ_j are consistent and row k is nonempty, we show how to extend row k to decide some formula θ. Let S be the set of all the (positive) atomic subformulas of $QF(\theta)$ that are not subformulas of ρ. By the invariants, $T \cup p_i(\bar{x}) \vdash (\exists y)[\rho(\bar{x}, y) \wedge QF(\theta_j)]$, and $T \cup \{\rho\} \vdash \varphi$ for all $\varphi \in p_k$. Let $S_{\bar{x}}$ be all the formulas in $\bar{x}$, *i.e.*, the variables of $p_i(\bar{x})$, in S. Let $S_y = S \setminus S_{\bar{x}}$, *i.e.*, atomic formulas in variables including y in S.

By inductive hypothesis, we can finitely (and computably) extend row i and rows dependent on row i to decide the atomic formulas in $S_{\bar{x}}$ and continue satisfying the invariants. Extend row k to decide the atomic formulas in $S_{\bar{x}}$ the same way row i did. Let $\nu_{\bar{x}}$ be the conjunction of the literals included in row i with subformulas in $S_{\bar{x}}$. Let

$\rho' = \rho \wedge \nu_{\bar{x}}$. By the axioms of T, we have $T \cup p_i \vdash (\exists y)[\rho'(\bar{x}, y) \wedge QF(\theta_j)]$. Extend row k to include all the positive atomic formulas in S_y, and include or exclude θ accordingly. Since the relations in S_y have not been mentioned in row k, this extension of row k is consistent by the axioms of T. Let ρ'' denote the conjunction of all literals included in row k at this point. Once again, by the axioms of T, $T \cup p_i \vdash (\exists y)[\rho''(\bar{x}, y) \wedge QF(\theta_j)]$ so the invariants are maintained.

The case where marker $Tap_{i,j}$ rests on row k and rows i and j have rank at most n is similar, and the cases where markers $Perm_{l,m}$ or $H = Sub_{l,m}$ rest on row k are straightforward. □

3.4.4 Defeating Λ_e on row k if $i_e \notin R_{k,s}$

Here we show that if Λ_e rests on row k at stage s and $i_e \notin R_{k,s}$, then we can extend rows k and i_e so that p_k and p_{i_e} are inconsistent, and X_{s+1} satisfies the invariants in §3.4.3. Then if $i_e \notin R_{k,s}$, we can immediately extend the **0**-basis X so that Λ_e cannot settle on row k and trace out an amalgamator for p_{i_e} and θ_{j_e}. We show this lemma for independent rows i since all i_e in the construction will be independent.

Lemma 3.4.10. *If row $i \notin R_{k,s}$, $rank_s(i) = 0$, and X_s satisfies the invariants in §3.4.3, then there exists a finite extension X_{s+1} of X_s satisfying the invariants in which p_i and p_k are inconsistent. Moreover, X_{s+1} extends only rows i and rows in $R_{k,s}$ and is uniformly computable from X_s.*

Proof. Choose the least unary relation P_l such that P_l is not mentioned in any of the rows in $R_{k,s}$ or in row i. Use Lemma 3.4.9 to finitely extend row k to decide P_l on x_0 (the first variable in p_k). This requires only finitely extending the rows in $R_{k,s}$. Since $i \notin R_{k,s}$ and $rank_s(i) = 0$, we can finitely extend row i to decide P_l in the opposite manner to row k on x_0. This is a consistent extension by definition of T. Note that all invariants still hold, and this is a computable extension.

□

3.4.5 The Splitting Strategy for Defeating Λ_e on k if $i_e \in R_{k,s}$

We now formally develop the splitting strategy described in the second example in §3.3.3. Suppose Λ_e rests on row k at stage s and $i_e \in R_{k,s}$, *i.e.*, row k is dependent at stage s on row i_e. Then we may not be able to make p_{i_e} inconsistent with p_k *and* respect the homogeneity marker on row k.

We show that we can compute two incompatible extensions of row k so that both extensions respect the homogeneity marker H on row k. We extend row k to one of the extensions and build the other on an empty row $k' > k$. We wait for Λ_e to decide the splitting (it must if Λ_e is tracing out an amalgamator). We then make the extension Λ_e is following inconsistent with p_{i_e} and satisfy H on the other extension.

If H is $Tr_{i,j}$ or $Tap_{i,j}$, the theory has enough flexibility to set a splitting directly in row k. However, if H is $Perm_{j,l}$ or $Sub_{j,l}$, row k must be built according to row j. In this case, we cannot directly build a splitting in row k. Instead we must create a splitting in the rows on which it depends. Then these splittings will "percolate up" to provide a splitting in row k.

The following definitions and lemma describe between which variables a splitting can be made.

Definition 3.4.11. Let p be the partial type corresponding to row k in X_s. Suppose H is a homogeneity marker resting on row k at stage s.

1. Marker H is a *marker in variables $\bar{x}$ at stage s* if $\bar{x}$ is the set of free variables in p. Recall, by assumption, that n-types in X have free variables $x_0, x_1, \ldots, x_{n-1}$.

2. Let x and y be variables in H. We say H *allows a splitting between variables x and y* if there exists some formula $\theta(x, y)$ so that q_1 and q_2 are partial types such that $p \subseteq q_i$, we have $\theta(x, y) \in q_1$ and $\neg\theta(x, y) \in q_2$, both q_1 and q_2 respect the homogeneity condition H, and q_1 and q_2 are uniformly computable from X_s and $R_{k,s}$. We say that q_1 and q_2 *split* p at θ. We will choose a particular θ satisfying the above and denote its index in our enumeration of formulas as $split^k_k = split^{k'}_k$.

Lemma 3.4.12. *Let H rest on row k at stage s.*

1. *Suppose $H = Tr_{i,j}$ is the homogeneity marker for type $p_i(\bar{x})$ and formula $\theta_j(\bar{x}, y)$. Then H allows a splitting at stage s between any variable in $\bar{x}$ and y.*

2. *Suppose $H = Tap_{i,j}$ is the homogeneity marker for type $p_i(\bar{x}, y)$ and $p_j(\bar{x}, z)$. Then H allows a splitting at stage s between variables y and z.*

Note if H is $Perm_{j,m}$ or $Sub_{j,m}$, H does not allow any (direct) splittings between any variables at any stage.

Proof. Suppose $H = Tr_{i,j}$, and let p_k be the partial type generated by row k at stage s. Suppose $p_i(\bar{x})$ and $\theta_j(\bar{x}, y)$ have not proved themselves inconsistent by stage s. Let x be a variable in $\bar{x}$. Let R_l be the least two-ary relation such that R_l is not mentioned in θ_j or any partial n-type corresponding to a row in X_s for $n \geq 1$. Then $R_l(x, y)$ and $\neg R_l(x, y)$ are both consistent with p_i, p_k, and θ_j by definition of T. Let q_1 be $p_k \cup \{R_l(x, y)\}$ and q_2 be $p_k \cup \{\neg R_l(x, y)\}$. These finite extensions, which an be found effectively, will satisfy the invariants in §3.4.3 by the same argument as in Lemma 3.4.9. Let $split_k^k$ and $split_k^{k'}$ equal the least index j such that $\theta_j(x, y) = R_l(x, y)$. Then q_1 and q_2 split p_k at $\theta_{split_k^k}$. The other case is similar.

□

Lemma 3.4.12 only shows that splittings can be made in some homogeneity markers H and between certain variables. We develop a method to build splittings in other situations that relies on Lemma 3.4.12.

The next lemma states that when row k is dependent on row i, we will be able to implement either the direct diagonalization strategy (like in Example 1) or the splitting strategy (like in Example 2). We need some definitions first.

Definition 3.4.13. Let $l \in R_{k,s}$. We say that *row k has a forced subtype $p(\bar{x})$ of row l at stage s* if $p(\bar{x})$ is a subtype of row k and the homogeneity markers relating the rows in $R_{k,s}$ require that $p(\bar{x})$ is a subtype of row l under some permutation of variables.

Let $p(x,y)$ be a subtype of row k. Suppose that $p(x,y)$ is a forced subtype of row $l \in R_{k,s}$ at stage s and that $p(x,y)$ corresponds to the subtype $\hat{p}(\hat{x},\hat{y})$ of row l under some permutation of variables. Suppose a splitting in variables $\hat{x}$ and $\hat{y}$ is built on rows l and l'. By construction, there is a row k' related (via homogeneity markers) to row l' in the same way row k is related to row l. Then row k and row k' form a splitting of $p(x,y)$ in variables x and y. We say the splitting on rows l and l' *generates the splitting* of $p(x,y)$ on rows k and k'. If we require a splitting of row k as above to satisfy $\mathbf{N}_{e,k}$, we denote the index of the formula θ that splits rows l and l' by $split^{l}_{e,k} = split^{l'}_{e,k}$ and the index of the corresponding formula that splits rows k and k' by $split^{k}_{e,k} = split^{k'}_{e,k}$. We supress the index e referring to $\mathbf{N}_{e,k}$ in this notation if the particular requirement is not specified.

Lemma 3.4.14. *Suppose H rests on row k corresponding to a 2-type $p(x,y)$ at stage s and row i_e is in $R_{k,s}$. Given X_s, there exists an extension X_{s+1} of X_s that satisfies the invariants in §3.4.3 such that one of the following cases holds. Moreover, X_{s+1} and determining the case that holds are uniformly computable in X_s.*

- *The partial type $p(x,y)$ in X_{s+1} is inconsistent with p_{i_e} but respects marker H.*

- *Row k and some row k' in X_{s+1} split the partial type p at the formula indexed by $split^{k}_{k} = split^{k}_{k'}$. If H is a homogeneity marker resting on row k at stage s, then H rests on k at stage $s+1$, and H', a copy of H, rests on row k'. Moreover, for any n, the basis X_{s+1} can be chosen so that the index $split^{k}_{k} = split^{k}_{k'}$ is greater than n.*

- *There exists a row $l \in R_{k,s}$ such that row l and some row l' in X_{s+1} split the partial type q corresponding to row l of X_s at the formula indexed by $split^{l}_{k} = split^{l'}_{k}$. Let row k' be the row related to row l' as row k is related to row l in X_{s+1}. This splitting generates a splitting of the partial type p on row k. A homogeneity marker L rests on row l at stages s and $s+1$, and L', a copy of L, is placed on l' at stage $s+1$. For any row m on the shortest path from row k to row l in $R_{k,s}$, let row m' denote the row related to row l' in the same way row m is related to row l, and denote the index of the formula at which rows m*

and m' split by $split_k^m = split_k^{m'}$. (Formula $\theta_{split_k^m}$ corresponds to $\theta_{split_k^l}$ via the relationship described between rows m and l given in $R_{k,s}$.) Finally, for any n, the basis X_{s+1} can be chosen so that for every row m as above, the index $split_k^m = split_k^{m'}$ is greater than n.

Proof. In the construction, row i_e will be a 1-type for all e. Hence, if Λ_e is an MEF, Λ_e may only rest on rows that correspond to 2-types. By definition of $p_{i_e}(x)$ and $\theta_{j_e}(x, y)$, we also ensure that the 1-subtypes $r_1(x)$ and $r_2(y)$ of any 2-type amalgamating p_{i_e} and θ_{j_e} are distinct.

Suppose H is $Tr_{i,j}$ or $Tap_{i,j}$. Then Lemma 3.4.12 shows there exist incompatible extensions $q_1(x, y)$ and $q_2(x, y)$ of $p(x, y)$ that respect H. We finitely extend row k to correspond to q_1, and we build q_2 on the first unmarked empty row k' of X_s not attended to by stage s. Then we place a marker H', a copy of H, on row k'. Let the resulting matrix be X_{s+1}. By the proof of Lemma 3.4.12, for any n, we can make a splitting in rows k and k' such that $split_k^k = split_{k'}^k > n$

Now suppose $H = Perm_{j,m}$ or $H = Sub_{j,m}$. Suppose row k is a forced 2-type of an independent row l in $R_{k,s}$ at stage s. Then $l \neq i_e$ because row l corresponds to an n-type for $n > 1$. Since row l is independent, we can make a consistent extension of row l that causes row i_e to be inconsistent with the extension of row k generated by the extension of row l (as in Lemma 3.4.10).

If row k is not a forced 2-type of some independent row l, we can create a splitting between x and y in some row of $R_{k,s}$ with marker L equal to $Tr_{i,j}$ or $Tap_{i,j}$ using Lemma 3.4.12. We say the *depth* of some row $n \in R_{k,s}$ is the length of the shortest path from row k to row n. Let row l be the row of least depth in the dependency tree $R_{k,s}$ for which such a splitting can be made. Row l exists because k is not a forced 2-type of an independent row. Specifically, if we trace the ancestry of the 1-subtypes of row k through $R_{k,s}$, row l is the least depth node where these types were joined into a higher arity (at least 2) type. (We ensure that the 1-subtypes of row k are distinct by the definition of θ_{j_e}.) Let q be the partial type generated by row l in X_s, and let the free variables $\hat{x}$ and $\hat{y}$ in q correspond to x and y in p (via the relationships prescribed by $R_{k,s}$).

By Lemma 3.4.12, there exist incompatible extensions q_1 and q_2 of q that split in variables $\hat{x}$ and $\hat{y}$ and respect L. Finitely extend row l to correspond to q_1 and build q_2 on the first fresh row l' of X_s. Place a marker L', a copy of marker L, on row l'. Let row k' be related to row l' in the same way as row k is related to row l in $R_{k,s}$. The splitting in rows l and l' between variables $\hat{x}$ and $\hat{y}$ generates a splitting in rows k and k' in variables x and y. Since we will computably keep track of where the homogeneity makers rest, we will be able to compute row k' from l'.

By the proof of Lemma 3.4.12, for any n, we can make a splitting in row l and row l' such that $split_k^l = split_k^{l'} > n$. There are only finitely many rows on the shortest path in $R_{k,s}$ from row k to row l. For any such row m, the 2-subtype of row m corresponding to the 2-subtype $q_1(\hat{x}, \hat{y})$ on row l is a forced 2-type of row l. Hence, for any n, by knowing $R_{k,s}$, we can create a splitting of row l at a large enough index such that for every row m on the shortest path from row k to row l, $split_k^m = split_k^{m'} > n$. □

3.5 Construction

We put together the above strategic modules and lemmas to construct the desired counterexample.

3.5.1 Main Construction

Let $\mathbf{d} \in \Delta_2^0$ be a low$_2$ degree. We will build a $\mathbf{0}$-basis (and hence a $\mathbf{d}$-uniform basis) X for a homogeneous model $\mathcal{A}$ that has no $\mathbf{d}$-monotone extension function.

Let the function $g(e, x, s)$ be a computable approximation to a listing of all $\mathbf{d}$-computable functions (as defined in §3.3.1).

$$\Lambda_{e,s}(t) = g_{e,s}(\langle i_e, j_e, t\rangle) = g(e, \langle i_e, j_e, t\rangle, s) \tag{3.2}$$

be the stage s approximation to $\Lambda_e(t) = \lim_s g_{e,s}(\langle i_e, j_e, t\rangle)$. Let $f(i, j, t)$ be some $\mathbf{d}$-computable function, a potential MEF for X. Then there exists an e such that $\Lambda_e(t) = f(i_e, j_e, t)$. We satisfy the requirements previously discussed.

Construction.

Stage 0**:**

On row 0 we place the infinite string corresponding to the complete decidable theory T described in §3.4.2.

Create the countably many homogeneity markers, and effectively place them on rows $\{4j+1\}_{j\in\omega}$ so that $R_{k,0}$ is computable for any row k. (Recall that we do not use markers to satsify closure conditions involving row 0, the 0-type T.)

Let $i_e = 4e+2$ for all e. Let j_e be the first index such that θ_{j_e} is $\neg P_e(y)$ for all e. Place on row i_e for all $e \in \omega$ the finite data that corresponds to the partial 1-type containing $\neg P_i(x)$ for all $i < e$ and $P_e(x)$ and formulas $x_0 = x_0$ and $x_1 \neq x_1$. We call these formulas the *coding formulas for* e. Thus any 2-type $p(x, y)$ amalgamating $p_{i_e}(x)$ and $\theta_{j_e}(x, y)$ has distinct 1-subtypes. Also the inclusion of $x_0 = x_0$ and $x_1 \neq x_1$ guarantees that row i_e corresponds to a 1-type. Let e^* denote the maximum of the indices of the formulas included in row i_e at this stage, the formula θ_{j_e}, and the formulas $x_i = x_i$ for $i \in \{0, 1\}$ and $x_2 \neq x_2$.

Let $\{\theta_{w(j)}(x_0)\}_{j\in\omega}$ be an effective enumeration of all 1-ary formulas consistent with T. On row $4e+3$ place the finite data that corresponds to the partial 1-type containing $\theta_{w(e)}(x_0)$ and all the literals in $QF(\theta_{w(e)}(x_0))$ needed to imply $\theta_{w(e)}$. (By including these rows, we satsify the homogeneity closure conditions involving row 0, the 0-type T.) For all e, row $4e+4$ remains empty and has no homogeneity marker. Call this matrix X_0.

Stage $s+1$**:**

We are given X_s in which θ_j decided in all rows i for $i, j \leq s$ and satisfies the invariants in §3.4.3. The matrix X_s also has infinitely many empty unmarked rows with index greater than any row attended to at a previous stage.

<u>$\mathbf{N}_{e,\tilde{k}}$ Requires Attention</u>

We say $\mathbf{N}_{e,\tilde{k}}$ *requires attention* at stage $s+1$ if either the following primary or redecision conditions for attention hold. Requirement $\mathbf{N}_{e,\tilde{k}}$ satisfies the *primary conditions for attention* if the following conditions are satisfied.

1. $\Lambda_{e,s+1}(t') = \tilde{k}$ for some t' such that $e^* < t' < s+1$,

   ($\Lambda_{e,s+1}$ is resting on row $\tilde{k}$, which corresponds to a 2-type, and row $\tilde{k}$ contains the coding formulas for e, ensuring that row $\tilde{k}$ corresponds to a 2-type and that only one Λ_e can require attention when resting on row $\tilde{k}$.)

2. $\Lambda_{e,s+1}$ traces out an amalgamator for p_{i_e} and θ_{j_e} through level t'.

   ($\Lambda_{e,s+1}$ appears to be an MEF.)

3. Row $\tilde{k}$ is consistent with row i_e and θ_{j_e}.

   (If not Λ_e will not be able to permanently rest on row $\tilde{k}$ if it is an MEF, *i.e.*, $\mathbf{N}_{e,\tilde{k}}$ is satisfied forever.)

4. We are not monitoring Λ_e on row $\tilde{k}$ on behalf of $\mathbf{N}_{e,\tilde{k}}$.

Requirement $\mathbf{N}_{e,\tilde{k}}$ satisfies the *redecision conditions for attention* if there is some row k that satisfies the conditions below.

1′ We are monitoring a decision on behalf of $\mathbf{N}_{e,\tilde{k}}$ for redecision at $split = split^k_{e,\tilde{k}}$ on row k.

2′. The most recent splitting that was decided on behalf of $\mathbf{N}_{e,\tilde{k}}$ was built on rows $\hat{k}$ and $\hat{k}'$ at $split^{\hat{k}}_{e,\tilde{k}} = split$. The splitting on rows $\hat{k}$ and $\hat{k}'$ was last decided at stage s' where $s' < s+1$. At stage s', row $\Lambda_{e,s'}(split)$ agreed with row $\hat{k}'$ through (and at) index $split$.

3′. At stage $s+1$, $\Lambda_{e,s+1}$ traces out an amalgamator for p_{i_e} and θ_{j_e} through level $split$, and row $\Lambda_{e,s+1}(split)$ agrees with row $\hat{k}$ and row k through (and at) index $split$.

   Moreover,

   (a) We have not yet attended to this redecision on row k on behalf of $\mathbf{N}_{e,\tilde{k}}$, and

   (b) Requirement $\mathbf{N}_{e,\tilde{k}}$ has not been reset since stage s'.

Attending to $\mathbf{N}_{e,\tilde{k}}$

Suppose $\langle e, \tilde{k}\rangle$ is the least number such that $\mathbf{N}_{e,\tilde{k}}$ requires attention. We attend to $\mathbf{N}_{e,\tilde{k}}$. First suppose $\mathbf{N}_{e,\tilde{k}}$ satisfies the primary conditions for attention.

Suppose $i_e \notin R_{\tilde{k},s}$, *i.e.*, row $\tilde{k}$ is not dependent on row i_e. Then by Lemma 3.4.10, we can finitely extend X_s so that row $\tilde{k}$ will be inconsistent with row i_e and this extension satisfies the invariants. Then $\mathbf{N}_{e,\tilde{k}}$ is satisfied forever and will never act again. If $i_e \in R_{\tilde{k},s}$ and by Lemma 3.4.14, we can make an extension to X_s so that row $\tilde{k}$ is inconsistent with row i_e, do so. Then $\mathbf{N}_{e,\tilde{k}}$ is satisfied forever and will never act again.

Otherwise, by Lemma 3.4.14, we can set a splitting against Λ_e on row $\tilde{k}$ and some row k'. This splitting is either built directly into rows $\tilde{k}$ and k' or is generated by a splitting on rows l and l'. Take row l to be the row of least depth in $R_{\tilde{k},s}$ where a splitting in the appropriate variables can be made, and take row l' to be an unmarked empty row with l' greater than the index of any row attended to so far. Without loss of generality, we assume the splitting is built in rows l and l'. By Lemma 3.4.14, if marker L rests on row l at stage s, we place a marker L', a copy of marker L, on row l'. Furthermore, row k', marked by some H', is related to row l' in the same way row $\tilde{k}$, marked by H, is related to row l.

By Lemma 3.4.14, we may make the splitting in rows l and l' such that for each row $m \in R_{\tilde{k},s}$ on the path of least length from row $\tilde{k}$ to row l and for each row $m' \in R_{k',s}$, the row corresponding to row m, we can choose the indices of the splitting formulas to satisfy the following splitting priorities.

- $split^m_{e,\tilde{k}} > split^m_{d,\tilde{n}}$ if $split^m_{d,\tilde{n}}$ exists on row m and $\langle e, \tilde{k}\rangle > \langle d, \tilde{n}\rangle$
- $split^m_{e,\tilde{k}} = split^{m'}_{e,\tilde{k}} > e^*$

If there exists an active splitting index $split^l_{d,\tilde{n}}$ on row l where $\langle e, \tilde{k}\rangle < \langle d, \tilde{n}\rangle$, reset $\mathbf{N}_{d,\tilde{n}}$, and delete all of its associated splittings. Notice that no splittings have been built into any row m' at this stage by choice of row l'.

Now suppose $\mathbf{N}_{e,\tilde{k}}$ satisfies the redecision conditions for attention above for some row k. Then row k is half of a splitting that splits at formula $\theta_{split^k_{e,\tilde{k}}}$. Let row l be the row of least depth in $R_{k,s}$ where the splitting at $split^k_{e,\tilde{k}}$ can be made, and suppose the formula indexed by $split^l_{e,\tilde{k}}$ in row l corresponds to the formula indexed by $split^k_{e,\tilde{k}}$ in row k. Suppose row l is marked by marker L. Let row l' be an empty unmarked row greater than all rows attended to so far, and let row k' be the row related to l' in the same way row k is related to l. As in the last case, construct a splitting on row l and row l' at $split^l_{e,\tilde{k}} = split^{l'}_{e,\tilde{k}}$ corresponding to the index $split^k_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$. Mark the row l' with a homogeneity marker L' as above.

We now monitor Λ_e on $split^k_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$ on rows k and k', where row k' is the row related to row l' in the same way row k is related to row l.

<u>Deciding Splittings</u>

Suppose we are monitoring Λ_e on $split^k_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$ on rows k and k' on behalf of $\mathbf{N}_{e,\tilde{k}}$. (Note k may equal $\tilde{k}$.) The splitting $split^k_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$ on rows k and k' *is decided* if:

- $\Lambda_{e,s+1}$ traces out an amalgamator for p_{i_e} and θ_{j_e} through level $split^k_{e,\tilde{k}}$.
- Row $\Lambda_{e,s+1}(split^k_{e,\tilde{k}})$ agrees with either row k or k' through (and at) $split^k_{e,\tilde{k}}$.

  ($\Lambda_{e,s+1}$ acts like an MEF on row i_e and formula θ_{j_e} through level $split^k_{e,\tilde{k}}$.)

Let $\langle e, \tilde{k}\rangle$ be the least number such that the splitting $split^k_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$ is decided. Suppose row $\Lambda_{e,s}(split^k_{e,\tilde{k}})$ agrees with row k through $split^k_{e,\tilde{k}}$. As above, we suppose the splitting on rows k and k' is generated by a splitting in rows l and l'. Then we move the marker L on row l to row l' (overwriting the marker L' on row l'). We say that Λ_e *kicks* L on behalf of $\mathbf{N}_{e,\tilde{k}}$. As in Lemma 3.4.10, we make row k inconsistent with row i_e via an appropriate extension of row l, and delete the splitting indices on behalf of $\mathbf{N}_{e,\tilde{k}}$ previously generated by row l. This action respects the homogeneity conditions since row l is no longer dependent on any other row. Thus we can make row k and i_e inconsistent on some P_i not yet mentioned in any row other than row 0 as in the above-mentioned lemmas. We now monitor this decision on behalf of $\mathbf{N}_{e,\tilde{k}}$ for redecision at $split^{k'}_{e,\tilde{k}}$ on row k'.

Reset any requirement $\mathbf{N}_{d,\tilde{n}}$ such that $\langle d, \tilde{n}\rangle > \langle e, \tilde{k}\rangle$ and there exists an index for an active splitting $split^{l}_{d,\tilde{n}}$ on row l at this stage. In other words, reset all active splittings for the requirements that satisfy these conditions. Now consider $\langle d, \tilde{n}\rangle$ such that $\langle d, \tilde{n}\rangle < \langle e, \tilde{k}\rangle$. Suppose we are monitoring splitting $split^{n}_{d,\tilde{n}} = split^{n'}_{d,\tilde{n}}$ on rows n and n' on behalf of $\mathbf{N}_{d,\tilde{n}}$ where this splitting is generated by $split^{u}_{d,\tilde{n}}$ and $split^{u'}_{d,\tilde{n}}$. (We act similarly if we are monitoring a decision on behalf of $\mathbf{N}_{d,\tilde{n}}$ for redecision at $split^{n}_{d,\tilde{n}}$ on row n.) Let $\mathcal{P}$ denote the shortest path in $R_{n,s}$ between row n and row u, and let $\mathcal{P}'$ be the corresponding path in $R_{n',s}$ between row n' and row u'. We consider two cases.

First, suppose row $l \in \mathcal{P}$. Then, by how we construct splittings, $p(x_0)$, the subtype of row n that corresponds to variable x_0, is a forced 1-type of row l. Since row l has no marker, it is an independent row. Thus, for each such $\langle d, \tilde{n}\rangle < \langle e, \tilde{k}\rangle$, we can extend row l so that this extension causes $p(x_0)$ to be inconsistent with p_{i_d}. Then $\mathbf{N}_{d,\tilde{n}}$ is satisfied forever.

To ensure that homogeneity markers settle, we continue to monitor any splittings associated with these $\mathbf{N}_{d,\tilde{n}}$. For each row m between row n and row l in $\mathcal{P}$, let row m' be the row that is related to row l' in the same way row m is related to row l in $R_{k,s}$. Note that by the construction of splittings, $split^{m}_{d,\tilde{n}} < split^{m}_{e,\tilde{k}}$. Define $split^{m'}_{d,\tilde{n}} = split^{m}_{d,\tilde{n}}$ for each such m'. Let row n'' be the row related to row l' in the same way row n is related to row l, and monitor the splitting $split^{n''}_{d,\tilde{n}} = split^{n'}_{d,\tilde{n}}$ on rows n'' and n'. We call this *shifting the splitting* on behalf of $\mathbf{N}_{d,\tilde{n}}$ on rows n and n' to rows n'' and n'. Note that row l' was not marked with any homogeneity markers when $split^{l'}_{e,\tilde{k}}$ was built on it. Since $\mathbf{N}_{e,\tilde{k}}$ was never reset since the splitting was built, no splittings previously existed on row l' below $split^{l'}_{e,\tilde{k}}$.

Second, suppose row $l \in \mathcal{P}'$. We similarly shift the the splitting on behalf of $\mathbf{N}_{d,\tilde{n}}$ on rows n and n' to rows n'' and n' as above, but we are unable to satisfy $\mathbf{N}_{d,\tilde{n}}$ forever. If l is not in $\mathcal{P}$ and $\mathcal{P}'$, row l does not impact any splittings for $\mathbf{N}_{d,\tilde{n}}$ for $\langle d, \tilde{n}\rangle < \langle e, \tilde{k}\rangle$. Thus we take no action on behalf of such splittings.

If $\Lambda_{e,s+1}$ follows the direction of the splitting that row k' follows, we act symmetrically to above.

Satisfying Homogeneity Conditions

Extend X_{s+1} so that θ_j is decided in all rows i for $i, j \leq s+1$ and satisfies the invariants in §3.4.3 using Lemma 3.4.9.

End Construction.

3.5.2 Verification.

We show that each homogeneity closure requirement is satisfied and that there is no **d**-MEF for the **0**-basis of types X we have built. We first show that $\mathbf{N}_{e,\tilde{k}}$ is satisfied for all $e, \tilde{k} \in \omega$. We then show that all homogeneity markers eventually settle, *i.e.*, are kicked from the row they are resting on at a given stage only finitely often. Each homogeneity requirement will be satisfied if it has a homogeneity marker that eventually settles.

Lemma 3.5.1. *For all $e, \tilde{k} \in \omega$, $\mathbf{N}_{e,\tilde{k}}$ requires attention or has a splitting decided on its behalf only finitely often.*

Proof. Assume the statement is true for all $\mathbf{N}_{d,\tilde{n}}$ for $\langle d, \tilde{n}\rangle < \langle e, \tilde{k}\rangle$. Choose the least stage $\hat{s}$ after which there is no $\langle d, \tilde{n}\rangle < \langle e, \tilde{k}\rangle$ such that $\mathbf{N}_{d,\tilde{n}}$ requires attention or has any splittings decided on its behalf. Suppose $\mathbf{N}_{e,\tilde{k}}$ requires attention at some stage $s' \geq \hat{s}$. Suppose $\mathbf{N}_{e,\tilde{k}}$ satisfies the primary conditions for attention at stage s'. Then $\mathbf{N}_{e,\tilde{k}}$ receives attention at stage s'. The argument is the same if $\mathbf{N}_{e,\tilde{k}}$ satisfies the redecision conditions for attention at stage s' or if $\mathbf{N}_{e,\tilde{k}}$ never requires attention after stage s' but decides a splitting after stage s'.

If, in receiving attention, $\mathbf{N}_{e,\tilde{k}}$ is satisfied forever without setting a splitting, then $\mathbf{N}_{e,\tilde{k}}$ will never require attention or have a splitting decided on its behalf again. Otherwise, we set a splitting at index $split = split^{\tilde{k}}_{e,\tilde{k}}$ in rows $\tilde{k}$ and k' on behalf of $\mathbf{N}_{e,\tilde{k}}$ and monitor this splitting. Since $\mathbf{N}_{e,\tilde{k}}$ is never reset after stage s', by definition of the redecision conditions for attention, $\mathbf{N}_{e,\tilde{k}}$ can only require attention at some stage greater than s' if $\mathbf{N}_{e,\tilde{k}}$ must redecide the splitting at index $split$. In this case, when $\mathbf{N}_{e,\tilde{k}}$ requires attention after stage s', the splitting monitored on behalf of $\mathbf{N}_{e,\tilde{k}}$

at this later stage is again indexed by *split*. Thus after stage s', if we are monitoring a splitting on behalf of $\mathbf{N}_{e,\tilde{k}}$ on rows k and k', this splitting occurs at index *split*.

We claim that we decide a splitting on behalf of $\mathbf{N}_{e,\tilde{k}}$ only finitely many times after stage s'. By the splitting priorities, actions taken on behalf of $\mathbf{N}_{d,\tilde{n}}$ for $\langle d,\tilde{n}\rangle > \langle e,\tilde{k}\rangle$ must respect the splittings on behalf of $\mathbf{N}_{e,\tilde{k}}$. Moreover, these requirements cannot reset $\mathbf{N}_{e,\tilde{k}}$. Let s'' be a stage greater than s' such that for all $s \geq s''$, the approximation $\Lambda_{e,s}(x)$ equals $\Lambda_e(x)$ for all $x \leq split$. Suppose $\mathbf{N}_{e,\tilde{k}}$ decides the splitting at index *split* at some stage $s > s''$. Since $\Lambda_{e,s}$ equals Λ_e through index *split*, by the redecision conditions for attention, $\mathbf{N}_{e,\tilde{k}}$ cannot require attention after stage s. Then no more splittings can be decided on behalf of $\mathbf{N}_{e,\tilde{k}}$ after stage s.

□

Lemma 3.5.2. *Requirement $\mathbf{N}_{e,\tilde{k}}$ is satisfied for all $e,\tilde{k} \in \omega$.*

Proof. Assume that $\mathbf{N}_{d,\tilde{n}}$ is satisfied for all $\langle d,\tilde{n}\rangle < \langle e,\tilde{k}\rangle$. Moreover, suppose that all such requirements $\mathbf{N}_{d,\tilde{n}}$ do not require attention or have any splittings decided on their behalf after stage s'. Suppose Λ_e traces out an amalgamator for p_{i_e} and θ_{i_e}, and $\lim_{t\to\infty}\Lambda_e(t) = \tilde{k}$. Let $t' > e^*$ be the least stage such that $\Lambda_e(t') = \tilde{k}$. Then $\Lambda_e(t) = \tilde{k}$ for all $t \geq t'$. Take $s'' \geq max\{s',t'\}$ such that $\Lambda_{e,s}(x) = \Lambda_e(x)$ for all $x \leq t'$ and $s \geq s''$.

At stage s'', requirement $\mathbf{N}_{e,\tilde{k}}$ will require and receive attention via the primary conditions for attention or we will already be monitoring some splitting on behalf of $\mathbf{N}_{e,\tilde{k}}$ on rows k and k'. Suppose $\mathbf{N}_{e,\tilde{k}}$ receives attention at this stage. We will discuss the other case below. If possible, we extend $X_{s''}$ so that row $\tilde{k}$ is inconsistent with row i_e. This is a contradiction since we assumed Λ_e traces out an amalgamator for p_{i_e} and θ_{j_e}, and $\lim_{t\to\infty}\Lambda_e(t) = \tilde{k}$. Thus we set a splitting at index $split^{\tilde{k}}_{e,\tilde{k}} = split^{k'}_{e,\tilde{k}}$ on row $\tilde{k}$ and some empty unmarked row k'. Let $s''' \geq s''$ be a stage by which $\Lambda_e(x) = \Lambda_{e,s}(x)$ for all $x \leq split^{\tilde{k}}_{e,\tilde{k}}$ and all $s \geq s'''$. Then $\Lambda_{e,s}(split^{\tilde{k}}_{e,\tilde{k}}) = \tilde{k}$ for all $s \geq s'''$. By stage s''', our construction will have decided the splitting indexed by $split^{\tilde{k}}_{e,\tilde{k}}$ correctly. Since $\Lambda_{e,s'''}(split^{\tilde{k}}_{e,\tilde{k}}) = \tilde{k}$, we made row $\tilde{k}$ inconsistent with row i_e in $X_{s'''}$ when deciding this splitting. Hence Λ_e does not trace out an amalgamator for p_{i_e} and θ_{i_e}, a contradiction.

Suppose we are already monitoring a splitting on behalf of $\mathbf{N}_{e,\tilde{k}}$ on rows k and k' at stage s''. If k or k' equals $\tilde{k}$, the argument above holds. Otherwise, the splitting in rows k and k' was originally set in row $\tilde{k}$ and some row $\tilde{k}'$. Since k and k' are unequal to $\tilde{k}$, the splitting on row $\tilde{k}$ was shifted at some point to another row when some splitting was decided. When this shift occurred, row $\tilde{k}$ was made inconsistent with row i_e by how splittings are decided, a contradiction. □

We now show that all the homogeneity closure conditions are satisfied.

Lemma 3.5.3. *Let H be any homogeneity marker placed on X at stage* 0*. Let h_s denote the row H rests on at stage s. Then* $\lim_{s\to\infty} h_s$ *exists.*

Proof. Let H be any homogeneity marker that placed on X at stage 0. Suppose H is a marker on behalf of a subtype or permutation closure requirement. Then the marker H is never moved by construction since we do not set splittings in such markers.

Suppose that H is any other kind of homogeneity marker. We have $h_s \neq h_{s+1}$ if and only if marker H is kicked off of row h_s onto row h_{s+1} when a splitting was decided on behalf of some $\mathbf{N}_{e,\tilde{k}}$ at stage $s+1$. We show H is kicked on behalf of only finitely many $\mathbf{N}_{e,\tilde{k}}$. Then since only finitely many splittings are decided on behalf of each $\mathbf{N}_{e,\tilde{k}}$ by Lemma 3.5.1, H is kicked only finitely many times and the theorem holds. For a contradiction, suppose that H is kicked on behalf of infinitely many $\mathbf{N}_{e,\tilde{k}}$.

We define a sequence of $\langle e_i, \tilde{k}_i\rangle$ and s_i inductively. Let $\langle e_1, \tilde{k}_1\rangle$ be the least $\langle e, \tilde{k}\rangle$ such that H is kicked on behalf of $\mathbf{N}_{e,\tilde{k}}$, and let s_1 be the greatest stage at which this occurs. Suppose we are given $\langle e_i, \tilde{k}_i\rangle$ and s_i for $i < n$ such that H is kicked on behalf of $\mathbf{N}_{e_i,\tilde{k}_i}$ at stage s_i, and H is not kicked on behalf of $\mathbf{N}_{e,\tilde{k}}$ for $\langle e, \tilde{k}\rangle \leq \langle e_i, \tilde{k}_i\rangle$ after stage s_i for all $i < n$. Let $\langle e_n, \tilde{k}_n\rangle$ be the least $\langle e, \tilde{k}\rangle$ such that H is kicked on behalf of $\mathbf{N}_{e,\tilde{k}}$ after stage s_{n-1}, and let s_n be the greatest stage at which H is kicked on behalf of this requirement. For all n, when H was kicked at stage s_n, all splittings associated with $\mathbf{N}_{e,\tilde{k}}$ with $\langle e, \tilde{k}\rangle > \langle e_n, \tilde{k}_n\rangle$ that could interfere with the splittings on behalf of $\mathbf{N}_{e_n,\tilde{k}_n}$ are reset. Then from this stage forward, all splittings created for $\mathbf{N}_{e,\tilde{k}}$ with $\langle e, \tilde{k}\rangle > \langle e_n, \tilde{k}_n\rangle$ satisfy $split^l_{e,\tilde{k}} > split^l_{e_n,\tilde{k}_n}$. Thus, $p_{h_{s_i}} \restriction split^{h_{s_i}}_{e_i,\tilde{k}_i} \subset p_{h_s}$

for all $s \geq s_i$. Let $p_\infty = \bigcup_{i\in\omega} p_{h_{s_i}} \restriction split^{h_{s_i}}_{e_i,\tilde{k}_i}$. Since $split^{h_{s_i}}_{e_i,\tilde{k}_i} < split^{h_{s_i}}_{e_{i+1},\tilde{k}_{i+1}}$ for all i, $\lim_{s\to\infty} split^{h_{s_i}}_{e_i,\tilde{k}_i} = \infty$. Thus p_∞ is a complete n-type where n is determined by the homogeneity closure requirement associated with H.

Since H was kicked on behalf of $\mathbf{N}_{e_i,\tilde{k}_i}$ at stage s_i in deciding a splitting built on some rows k and k', row k was being built as a forced 2-type of row h_{s_i-1}. For H to be kicked, row k contains the coding formulas for e_i. Hence row h_{s_i-1} contains these coding formulas in some of its variables. Since any splitting index is taken greater than the indices of these coding formulas, p_∞ contains these coding formulas in some of its variables. There exist only finitely many 2-types that correspond to a permutation of a subtype of p_∞. Thus there are only finitely many e such that Λ_e can kick H.

Suppose $q(x,y)$ is one of the finitely many 2-types that is a permutation of a subtype $\hat{q}(\hat{x},\hat{y})$ of p_∞ where $\hat{x}$ and $\hat{y}$ denote the variables in p_∞ that correspond to the variables x and y in q. There are only finitely many pairs $(\hat{x},\hat{y})$ of such variables in p_∞

We say that we *set a splitting in H in variables $\hat{x}$ and $\hat{y}$ on behalf of* $\mathbf{N}_{e,\tilde{k}}$ *at stage* s if $\mathbf{N}_{e,\tilde{k}}$ receives attention at stage s, and at this stage we build a splitting on rows k and k' in variables x and y that is generated by a splitting in variables $\hat{x}$ and $\hat{y}$ on rows l and l' where row l is marked by H and row l' is marked by H', a copy of H. Let $\langle e'_1, \tilde{k}'_1\rangle$ be the least value such that we set a splitting in H in some pair of variables $\hat{x}$ and $\hat{y}$ on behalf of $\mathbf{N}_{e'_1,\tilde{k}'_1}$. Let s' be the least stage at which we set this splitting in H. Let $split = split^{l}_{e'_1,\tilde{k}'_1}$. Note that $split \leq split^{h_{s_0}}_{e_1,\tilde{k}_1}$ where $e_1, \tilde{k}_1$, and s_1 are defined in the beginning of the proof of this lemma. Hence by choice of $\langle e'_1, \tilde{k}'_1\rangle$, whenever H is kicked, the splitting at *split* is preserved. Let $s'' > s'$ be a stage such that for all $s \geq s''$, $\Lambda_{e,s}(x) = \Lambda_e(x)$ for all $x \leq split$ and no $\langle d, \tilde{n}\rangle < \langle e'_1, \tilde{k}'_1\rangle$ acts at stage s. Suppose H rests on row $\hat{l}$ at stage s'' and that row $\hat{k}$ is related to row $\hat{l}$ in the same way row k was related to row l at stage s'. Suppose row $\Lambda_e(split)$ disagrees with row *hatk* through (and including) *split*. By the note above, this disagreement is preserved whenever H is kicked, *i.e.*, p_∞ extends this disagreement. If row $\Lambda_e(split)$ agrees with row $\hat{k}$ through (and including) *split*, by the redecision conditions for attention and the choice of s'', we set a splitting in H in $\hat{x}$ and $\hat{y}$ on rows $\hat{l}$ and $\hat{l}'$. Let row $\hat{k}'$ be the row related to row $\hat{l}'$ in the same way $\hat{k}$ is related to $\hat{l}$. Then by choice of s'', we kick H off of row $\hat{l}$ onto row $\hat{l}'$ so that row $\Lambda_e(split)$ disagrees with row $\hat{k}$ at index $split$. As in the first case, this disagreement is preserved whenever H is kicked after this stage. Therefore, after stage s'', no $\mathbf{N}_{e,\tilde{k}}$ receives attention via a splitting in H in variables $\hat{x}$ and $\hat{y}$ unless it already required such attention by stage s''. Only finitely many $\mathbf{N}_{e,\tilde{k}}$ require such attention by stage s''. Since each $\mathbf{N}_{e,\tilde{k}}$ requires attention or decides splittings finitely often, there exists a stage s''' after which Λ_e does not kick H while deciding a splitting in H in variables $\hat{x}$ and $\hat{y}$.

We continue inductively, defining $\langle e_2', \tilde{k}_2' \rangle$ to be the least value such that there exists or we set a splitting in H in some pair of variables *other than* $\hat{x}$ and $\hat{y}$ on behalf of $\mathbf{N}_{e_2', \tilde{k}_2'}$ after stage s'''. As above, we can show that there exists a stage after which Λ_e does not kick H while deciding a splitting in H in either these pairs of variables. Since there are only finitely many pairs of variables in p_∞, by repeating this argument, we see that there exists a stage after which Λ_e does not kick H. (Remember Λ_e can only kick H while deciding some splitting in H in some pair of variables in the row marked by H). Since there are only finitely many e such that Λ_e kicks H, marker H is kicked only finitely often. Equivalently $\lim_{s\to\infty} h_s$ exists.

□

Theorem 3.5.4. *Let* $\mathbf{d} \leq \mathbf{0}'$ *be low*$_2$ *There exists a homogeneous model* $\mathcal{A}$ *with a* $\mathbf{0}$*-basis with no* $\mathbf{d}$*-decidable copy.*

Proof. By Lemma 3.5.3, each original homogeneity marker comes to rest on a given row. By construction then, all the homogeneity conditions are satisfied. As the construction is effective, we have built a homogeneous model $\mathcal{A}$ with a $\mathbf{0}$-basis. By Lemma 3.5.2, no $\mathbf{d}$-function can be an MEF for this basis. Thus by the relativized Effective and Monotone Extension Function Theorems, $\mathcal{A}$ has no $\mathbf{d}$-decidable copy.

□

CHAPTER 4

REVERSE MATHEMATICS OF HOMOGENEOUS MODELS

4.1 Another Perspective

As we have seen, the computability theoretic results on homogeneous models turn out to be the same as the analogous results in the prime model case. A natural question is whether there is some intrinsic structural connection between prime and homogeneous models. Since countable models are prime if and only if they are atomic classically, we used the terms prime and atomic interchangeably. Now we want to explore the distinctions between these definitions as well as their connections with homogeneous models from a proof-theoretic perspective. Reverse mathematics provides one way to study these ideas. Reverse mathematics deems two mathematical statements to have the same proof theoretic strength if each statement can be used to prove the other over some weak base system of axioms, typically the *Recursive Comprehension Axioms* (RCA_0). For this reason, we are interested in the reverse mathematical strength of statements about homogeneous models and, in particular, the relative strength of statements about the existence of specific homogeneous models and the existence of atomic models.

We begin this section by examining basic classical properties of homogeneous models from a reverse mathematics perspective. Then we consider the strength of the statement, "Every theory T has a homogeneous model," which had been examined in the computable setting by Csima, Harizanov, Hirschfeldt, and Soare in [5]. Finally we compare the strength of statements on the existence of specific homogeneous models and on the existence of atomic models. The atomic model statement is studied extensively by Hirschfeldt, Shore, and Slaman in [13].

4.1.1 Basic Notions

We briefly detail some of the basic axiom systems commonly studied in reverse mathematics. For a complete introduction to the field, please see [29], the standard reference. Each system uses the language of second order arithmetic, *i.e.*, the first order language of arithmetic with set variables and the set membership symbol $\in$ added. All of these standard systems include the following induction axiom I_0 for sets X and P_0, the axioms that say ω is an ordered semiring.

I_0: $(0 \in X \wedge (\forall n)[\, n \in X \rightarrow n+1 \in X\,]\,) \rightarrow (\forall n)[\, n \in X\,]$

We use RCA_0 as our base set of axioms. The axiom system RCA_0 consists of a weak form of induction and also set existence axioms that are just strong enough to prove the existence of computable sets.

RCA_0 (Recursion Comprehension Axioms): RCA_0 includes I_0 and P_0 and the collections of axioms Δ^0_1-CA and $\mathrm{I}\Sigma_1$.

Δ^0_1-CA: $((\forall n)\ [\varphi(n) \leftrightarrow \psi(n)\,]) \rightarrow (\exists X)\ (\forall n)\ [n \in X \leftrightarrow \varphi(n)]$ for all Σ^0_1 formulas φ and Π^0_1 formulas ψ in which X is not a free variable.

$\mathrm{I}\Sigma_1$: $(\varphi(0) \wedge (\forall n)\,[\varphi(n) \rightarrow \varphi(n+1)\,]) \rightarrow (\forall n)\ [\varphi(n)]$ for all Σ^0_1 formulas φ.

WKL_0 stands for the axiom system that includes RCA_0 and the principle known as Weak König's Lemma. This system is strongly related to degrees of Peano arithmetic.

WKL_0 (Weak König's Lemma): This system consists of RCA_0 and Weak König's Lemma, the statement that every infinite tree in $2^{<\omega}$ has an infinite path.

Finally, the last main system we consider here is ACA_0, which corresponds to the existence of the halting problem.

ACA_0 (Arithmetic Comprehension Axioms): This system consists of RCA_0 and the axioms $(\exists X)\,(\forall n)\,[\, n \in X \leftrightarrow \varphi(n)\,]$ for every arithmetic formula φ in which X is not a free set variable.

The system ACA_0 implies WKL_0 (see Theorem III.7.2 in [29]), and both imply RCA_0 by definition. We assume all theories under consideration are complete and consistent and that all languages and models are countable. A predicate S is a *full satisfaction predicate for* $\mathcal{A}$ if for any formula $\theta(\bar{c})$ with constants from $\mathcal{A}$, S of $\theta(\bar{c})$ holds if and only if $\theta(\bar{c})$ holds in $\mathcal{A}$. In reverse mathematics, a model $\mathcal{A}$ of T includes a function interpreting the terms in the model and the full satisfaction predicate for $\mathcal{A}$. For a formal definition of a model in reverse mathematics, see Definition II.8.3 in [29].

We now examine how basic homogeneity principles related to these axiom systems.

4.2 Classical Homogeneity Concepts

We recall two classical definitions of a homogeneous model.

Definition 4.2.1. (i) $\mathcal{M} \vDash T$ is *homogeneous* if whenever $\bar{a} \subseteq M$ and $f : A \to M$ is partial elementary, and $c \in M$, there exists a partial elementary $f^* \supseteq f$ where $f^* : A \cup \{c\} \to M$.
(ii) $\mathcal{M} \vDash T$ is *strongly homogeneous* if for all $\bar{a}, \bar{b} \subseteq M$ realizing the same type, there exists an automorphism Φ of $\mathcal{M}$ with $\Phi(a_i) = b_i$.

Classically these definitions are the same, as one can use the one point extension definition to find an automorphism using a back and forth construction. This construction, however, cannot be carried out in RCA_0. Note that RCA_0 can prove that if $\mathcal{M}$ is strongly homogeneous then $\mathcal{M}$ is homogeneous by appropriately restricting the given automorphism.

Proposition 4.2.2. *The following are equivalent over* RCA_0*:*

1. ACA_0
2. *$\mathcal{M}$ is homogeneous implies $\mathcal{M}$ is strongly homogeneous.*
3. *If $\mathcal{M}$ and $\mathcal{N}$ are homogeneous and realize the same types, then $\mathcal{M} \cong \mathcal{N}$.*

Proof. ACA_0 implies the first homogeneity statement because the classical construction of the desired automorphism from one point extensions can be carried out in ACA_0. In particular, ACA_0 can recognize partial elementary maps. ACA_0 implies the second statement since ACA_0 can recognize when a tuple from $\mathcal{M}$ and a tuple from $\mathcal{N}$ realize the same type. Then the classical back and forth construction of the isomorphism can be done.

The second homogeneity statement together with RCA_0 implies ACA_0 since in [13], Hirschfeldt, Shore, and Slaman built an example of two atomic models of the same theory (in RCA_0) so that any isomorphism between them computes $\mathbf{0}'$. Since atomic models are homogeneous (in RCA_0 by Proposition 4.2.4) and they realize exactly the principal types in the theory, the second statement implies that an isomorphism between them exists.

We see the first homogeneity statement together with RCA_0 implies ACA_0 by using an extended version of the example built by Hirschfeldt, Shore, and Slaman in [13]. We add an equivalence relation to split the domain in half, and then build a homogeneous (actually atomic) model with one half of the domain like model $\mathcal{A}$ in the example and the other half like $\mathcal{B}$.

The theory T

The language of T has unary predicates R_i and $R_{i,j}$ for $i, j \in \omega$ and one binary predicate E. Let Φ_i list the Turing machines.

We will define T in terms of the following set S of axioms. Include the axioms below for all $i, s \in \omega$.

1. Axioms asserting E is an equivalence relation with two infinite equivalence classes.
2. Axioms asserting that the R_i define infinite, pairwise disjoint sets and that R_i has infinite intersection with both equivalence classes of E.
3. $R_{i,s}(x) \to R_i(x)$.

4. $R_{i,s}(x) \to R_{i,s+1}(x)$.

5. If $\Phi_i(i)$ does not converge in fewer than s many steps, then the axiom $\neg R_{i,s}(x)$ is included.

6. If $\Phi_i(i)$ converges in exactly s many steps, then there are axioms asserting that there are infinitely many x in each equivalence class such that $R_{i,s}(x)$ and infinitely many x in each equivalence class such that $\neg R_{i,s}(x)$.

7. If $\Phi_i(i)$ converges in exactly s many steps, then the axioms $\neg R_{i,s}(x) \to \neg R_{i,t}(x)$ are included for all $t > s$.

This set of axioms S is computable and hence provably exists in RCA_0.

Lemma 4.2.3. *The set S admits computable quantifier elimination,* i.e.*, for every formula $\varphi(x)$, we can uniformly compute a quantifier-free formula $\psi(x)$ (where $\psi(x)$ could be one of the propositional symbols* T *for true or* F *for false) such that $S \vdash \varphi(x) \leftrightarrow \psi(x)$.*

We prove this result momentarily. Let T be the deductive closure of the computable set S. Since S admits computable quantifier elimination, every sentence is provably equivalent to T or F under S. Therefore, T is Δ^0_1-definable, and it is provable in RCA_0 that T is a complete theory.

Proof of Lemma 4.2.3. It suffices to provide a uniformly computable procedure that finds a quantifier-free formula $\psi(\bar{x})$ that is S-equivalent to a given existential formula $(\exists \bar{y})\varphi(\bar{x}, \bar{y})$. (The usual induction to show quantifier elimination in general can be done in RCA_0 using $\mathrm{I}\Sigma_1$.) We may further assume that $\varphi(\bar{x}, \bar{y})$ is a conjunction of atomic formulas and negations of atomic formulas. In this case, all atomic formulas are of the form $R_i(w)$, $R_{i,s}(w)$, or $E(w, z)$ where $w, z \in \bar{x}\bar{y}$.

We first ensure that $(\exists \bar{y})\varphi(\bar{x}, \bar{y})$ passes the following consistency checks. If it fails any of the following checks, then it is inconsistent with S and is equivalent to F.

Consistency Checks

- Formula $\varphi(\bar{x}, \bar{y})$ does not contain both $R_i(w)$ and $\neg R_i(w)$ or both $R_{i,s}(w)$ and $\neg R_{i,s}(w)$, or both $E(w, z)$ and $\neg E(w, z)$ for any $w, z \in \bar{x}\bar{y}$.

- Formula $\varphi(\bar{x},\bar{y})$ does not contradict that E is an equivalence relation. For example φ does not include both $E(w,z)$ and $\neg E(w,z)$ or the following literals $E(u,w)$, $E(w,z)$, and $\neg E(u,z)$ for any $u,w,z \in \bar{x}\bar{y}$.

- Formula $\varphi(\bar{x},\bar{y})$ is consistent with Axiom 2, *i.e.*, φ does not include $R_i(w)$ and $R_j(w)$ for any $i,j \in \omega$ and $w \in \bar{x}\bar{y}$.

- Formula $\varphi(\bar{x},\bar{y})$ is consistent with Axioms 3 and 4, *i.e.*, φ does not include both $R_{i,s}(w)$ and $\neg R_i(w)$ or both $R_{i,s}(w)$ and $\neg R_{i,s+1}(w)$ for any $i,s \in \omega$ and $w \in \bar{x}\bar{y}$.

- Formula $\varphi(\bar{x},\bar{y})$ is consistent with Axiom 5. Specifically if $\Phi_{i,s}(i)\uparrow$ for some pair $i,s \in \omega$, then φ does not include $R_{i,s}(w)$ for any $w \in \bar{x}\bar{y}$.

- Formula $\varphi(\bar{x},\bar{y})$ is consistent with Axiom 7. Suppose $\neg R_{i,s}(w)$ and $R_{i,t}(w)$ are in $\varphi(\bar{x},\bar{y})$ with $s < t$ and $i,s,t \in \omega$. Then $\Phi_i(i)$ converges in exactly n steps with $s < n \leq t$.

If $\varphi(\bar{x},\bar{y})$ satisfies the above consistency checks, let $\psi(\bar{x})$ be the φ with all literals involving $w \in \bar{y}$ being replaced with T. This new formula ψ is quantifier-free and S-equivalent to $(\exists\bar{y})\varphi(\bar{x},\bar{y})$.

□

We define a homogeneous model $\mathcal{A}$ of T. Since T admits quantifier elimination, to define a model $\mathcal{A}$ of T it suffices to describe an atomic diagram consistent with T. Define E so that one equivalence class of $\mathcal{A}$ is $\{\langle i,n,j\rangle \mid j \text{ even}, i,n \in \omega\}$ and the other is $\{\langle i,n,j\rangle \mid j \text{ odd}, i,n \in \omega\}$. Let R_i consist of $\{\langle i,n,j\rangle \mid n,j \in \omega\}$. To define $R_{i,s}$, see whether $\Phi_i(i)$ converges in s steps. If not, $R_{i,s}$ is empty. If so, suppose $\Phi_i(i)$ converges at step $t \leq s$. Put $\{\langle i,j,2n\rangle \mid 0 \leq j \leq t \,\&\, n \in \omega\}$ into $R_{i,t}$. Similarly put $\{\langle i,j,2n+1\rangle \mid 0 \leq j \leq t \,\&\, n \in \omega\}$ into $\neg R_{i,t}$. Then effectively split the rest of $\{\langle i,j,n\rangle \mid j,n \in \omega\}$ into two sets. Put one set in $R_{i,t}$ and the other in $\neg R_{i,t}$. Then let the elements of $R_{i,s}$ be the elements of $R_{i,t}$ and the elements of $\neg R_{i,s}$ be the elements of $\neg R_{i,t}$.

We show $\mathcal{A}$ is homogeneous. Suppose f is a partial elementary map sending tuple $\bar{a}$ to tuple $\bar{b}$. Let $c \in A$. Since f is partial elementary, $\bar{a}$ and $\bar{b}$ realize the same type $\mathbb{T}(\mathcal{A})$. To show $\mathcal{A}$ is homogeneous, it suffices to show there exists a $d \in A$ such that $(\bar{a}, c)$ and $(\bar{b}, d)$ realize the same type in $\mathbb{T}(\mathcal{A})$. Since T admits quantifier elimination, it is enough to find a $d \in A$ such that $(\bar{a}, c)$ and $(\bar{b}, d)$ satisfy the same literals in $\mathcal{A}$. By assumption we know $\bar{a}$ and $\bar{b}$ satisfy the same literals in $\mathcal{A}$. Since $c \in A$, $c = \langle i, n, j\rangle$ for some $i, n, j \in \omega$. Let a_1 and b_1 correspond to the first element of $\bar{a}$ and $\bar{b}$ respectively. Determine whether $E(a_1, c)$ holds in $\mathcal{A}$. If so, choose some d such that $R_i(d)$ and $E(b_1, d)$ hold in $\mathcal{A}$ and $R_{i,s}(d)$ holds if and only if $R_{i,s}(c)$ holds. Such a d exists by construction of $\mathcal{A}$. We can determine non-uniformly whether $R_{i,s}(d)$ holds if and only if $R_{i,s}(c)$ holds because any 1-type with $R_i(x)$ as an element is principal. If $\Phi_i(i)\uparrow$, then (by Axiom 5), $p(x)$ contains $\neg R_{i,s}(x)$ for all x, and $p(x)$ is generated by $R_i(x)$. If $\Phi_s(i)\downarrow$ in exactly s steps, then $p(x)$ is generated either by the formula $R_{i,s}(x)$ or $R_i(x) \wedge \neg R_{i,s}(x)$ (by Axioms 6 and 7). If $\neg E(a_1, c)$ holds in $\mathcal{A}$, similarly choose some d such that $R_i(d)$ and $\neg E(b_1, d)$ hold in $\mathcal{A}$ and $R_{i,s}(d)$ holds if and only if $R_{i,s}(c)$ holds. Since E is an equivalence relation, $(\bar{a}, c)$ and $(\bar{b}, d)$ satisfy the same literals in $\mathcal{A}$. Thus $\mathcal{A}$ is homogeneous.

Since $\mathcal{A}$ is homogeneous, $\mathcal{A}$ is strongly homogeneous by assumption. Hence, for all $\bar{a}, \bar{b} \subseteq A$ realizing the same type, there exists an automorphism σ of $\mathcal{A}$ with $\sigma(a_i) = b_i$. We show that using σ we can compute $\mathbf{0}'$. Let a, b be two elements of A that satisfy the same 1-type but are not E-related, and let σ be the automorphism given by the definition. We determine whether $\Phi_i(i)$ converges as follows. By definition $\mathcal{A} \models R_i(\langle i, i, 0\rangle)$. Let $\langle i, k, l\rangle$ equal $\sigma(\langle i, i, 0\rangle)$. Note that since $\neg E(a, b)$, the number l is odd. We show that $\Phi_i(i)\downarrow$ if and only if $\Phi_{i,\max(i,k)}(i)\downarrow$. Suppose $\Phi_{i,i}(i)\uparrow$ but $\Phi_i(i)\downarrow$ in exactly t steps. Then $t > i$, so by construction, $\mathcal{A} \models R_{i,t}(\langle i, i, 0\rangle)$. Then $\mathcal{A} \models R_{i,t}(\langle i, k, l\rangle)$ since σ is an automorphism. Since $\{\langle i, j, 2n+1\rangle \mid 0 \leq j \leq t \;\&\; n \in \omega\}$ is contained in $\neg R_{i,t}$ for all $n \in \omega$, we have $k > t$, as desired. Thus, we can compute $\mathbf{0}'$ using the principle that $\mathcal{A}$ homogeneous implies $\mathcal{A}$ strongly homogeneous. By a similar argument, ACA_0 follows from this principle. □

Proposition 4.2.4. *The following are provable in RCA_0.*

1. *$\mathcal{M}$ atomic implies $\mathcal{M}$ homogeneous.*

2. *$\mathcal{M}$ prime implies $\mathcal{M}$ homogeneous.*

3. *$\mathcal{M}$ saturated implies $\mathcal{M}$ homogeneous.*

Proof. Suppose $\bar{a} \to \bar{b}$ is elementary and $c \in M$. Since $\mathcal{M}$ is atomic, some $\psi(\bar{x}, y)$ isolates $tp^{\mathcal{M}}(\bar{a}, c)$. Now $\mathcal{M} \models (\exists y)\psi(\bar{a}, y)$ so $\mathcal{M} \models (\exists y)\psi(\bar{b}, y)$. Take $d \in M$ such that $\mathcal{M} \models \psi(\bar{b}, y)$. Then $(\bar{a}, c) \to (\bar{b}, d)$ is elementary.

Hirschfeldt, Shore, and Slaman showed $\mathcal{M}$ prime implies $\mathcal{M}$ atomic holds in RCA_0. This result combined with the first statement gives the second statement above.

Suppose $\mathcal{M}$ is saturated, *i.e.*, for all $\bar{a} \subseteq M$ and $p \in S_n^{\mathcal{M}}(\bar{a})$, the type p is realized in $\mathcal{M}$. Let $\bar{a} \subseteq M$ and $f : \bar{a} \to M$ be partial elementary. Let $b \in M$. Consider

$$\Gamma = \{\psi(f(\bar{a}), v) : \mathcal{M} \models \psi(\bar{a}, b)\},$$

which is Δ_0^0-definable in $\mathcal{M}$. The set Γ is finitely consistent since f is partial elementary, and Γ is closed under logical deduction. Thus by weak Lindenbaum's Lemma (which holds in RCA_0 by Lemma II.8.5 in [29]), Γ is consistent. Then as $\mathcal{M}$ is saturated, there exists $c \in M$ realizing Γ. Then $f \cup \{(b, c)\}$ is a partial elementary extension of f. □

4.3 $\mathbf{WKL_0}$ and Existence of Homogeneous Models

We will prove the reverse mathematical analogue to Csima, Harizanov, Hirschfeldt, and Soare's computability result in [5] that the PA degrees exactly characterize the degrees that are capable of deciding a homogeneous model in any complete decidable theory. MacIntyre and Marker in [19] first proved that the PA degrees are capable of deciding a homogeneous model of any complete decidable theory. The next result corresponds to MacIntyre and Marker's result.

Theorem 4.3.1. *WKL_0 implies the statement "Any complete theory T has a homogeneous model."*

Proof. Let $C = \bigcup_{i \in \omega} C_i$ where $C_i = \{c_{j,k} \mid j \leq i \;\&\; k \in \omega\}$ for each $i \in \omega$. Let $\{\theta_i\}_{i \in \omega}$ be an effective enumeration of all formulas in $\mathcal{L}$ such that if θ_i is a subformula of θ_j, then $i \leq j$. Following the classical construction of a homogeneous elementary extension of any model of T, we will define a set of homogeneity axioms H in the language $L \cup C$ and show that $T_{C_0} \cup H$ is finitely satisfiable, where T_{C_0} are axioms related to T. Then by WKL_0 (see Theorem IV.3.3 in [29]), the set $T_{C_0} \cup H$ has a model $\mathcal{N}$. We will show a model related to $\mathcal{N}$ is homogeneous. We first define T_{C_0}. Under RCA_0, the theory T has a model $\mathcal{M}$. Assign $c_{0,i}$ to the i^{th} element of the universe of $\mathcal{M}$ under the natural ordering. Let T_{C_0} denote the set of formulas in $L \cup C_0$ with no free variables that hold in $\mathcal{M}$ under this assignment of the constants in C_0. Note that $T \subseteq T_{C_0}$. We now define H.

For each $i \in \omega$, we define H_i as follows. Effectively list all the triples $(\bar{a}, \bar{b}, d)$ where $\bar{a}$ and $\bar{b}$ are tuples contained in C_i of the same arity and $d \in C_i$. Then let H_i consist of all formulas of the form for all n and m:

$$\Psi^i_{m,n} = [\bigwedge_{k=1}^{n} \theta_k(\bar{a}_m, d_m) \wedge (\exists y) \bigwedge_{k=1}^{n} \theta_k(\bar{b}_m, y)] \rightarrow \bigwedge_{k=1}^{n} \theta_k(\bar{b}_m, c_{i+1,m}).$$

(The formula θ_k is included in the above conjunctions only if its arity matches that of the tuple of constants being substituted into it).

Then let $H = \bigcup_{i \in \omega} H_i$. We show $T_{C_0} \cup H$ is finitely satisfiable. Let $\Delta \subseteq T_{C_0} \cup H$ be finite. We show Δ has a model (under RCA_0) by defining the constants in C mentioned in Δ so that $\mathcal{M}$ with these definitions satisfies Δ. Now as Δ is finite, Δ contains finitely many constants $\{c_{i_k,j_k}\}_{k=1}^{n} \subseteq C$ where $i_k \leq i_{k+1}$ and if $i_k = i_{k+1}$, $j_k < j_{k+1}$. As before, define $c_{0,i}$ to be the i^{th} element of the universe of $\mathcal{M}$ under the natural ordering. Suppose $c_{i,j} \in C$ and $c_{i,j}$ is not mentioned in any formula in Δ. Then define $c_{i,j}$ to be the least element of $\mathcal{M}$ under the natural ordering.

It remains to define the constants $\{c_{i_k,j_k}\}_{k=1}^{n}$. We define these constants inductively. Suppose the constants $\{c_{i_k,j_k}\}_{k=1}^{m'-1}$ for $m'-1 < n$ have been defined. To define $c_{i_{m'},j_{m'}} \in C_{l+1}$, let n' be the largest n such that $\Psi^l_{j_{m'},n}$ is in Δ and the first part of the implication $\Psi^l_{j_{m'},n}$ holds in $\mathcal{M}$. As $(\exists y) \bigwedge\!\!\!\bigwedge_{k=1}^{n'} \theta_k(\bar{b}_{j_{m'}}, y)$ holds in $\mathcal{M}$, define $c_{i_{m'},j_{m'}}$ to be the element of $\mathcal{M}$ that witnesses this existential. Note that this definition ensures $\Psi^l_{j_{m'},n}$ holds for all $n \leq n'$. Repeat this process finitely many times to define $\{c_{i_k,j_k}\}_{k=1}^n$

By construction, $\mathcal{M}$ with the constants C defined as above satisfies Δ. Thus $T_{C_0} \cup H$ is finitely satisfiable and hence satisfiable by some $\mathcal{N}$ under WKL_0. Consider the partial nonprincipal 1-type $\{x \neq c_{i,j} \mid c_{i,j} \in C\}$. Hirschfeldt, Slaman, and Shore in [13] note that the following theorem of Millar holds in RCA_0.

Theorem 4.3.2 (Millar [25]). *Let T be a complete decidable theory. Let S_0 be a computable set of complete types of T. Let S_1 be a computable set of nonprincipal partial types of T. There exists a decidable model of T omitting all nonprincipal types in S_0 and all partial types in S_1.*

Thus we can obtain a model $\mathcal{N}'$ of $T_{C_0} \cup H$ omitting the nonprincipal partial type $\{x \neq c_{i,j} \mid c_{i,j} \in C\}$. We claim $\mathcal{N}' \models T$ (restricted to the language L) is homogeneous. Let f be a partial elementary map taking $\bar{a}$ to $\bar{b}$ and let d be some element in $\mathcal{N}'$. Let i denote the least j such that for each element of $\bar{a}, \bar{b}, d$ there is a constant in C_j attached to this element in $\mathcal{N}'$. Then suppose the triple $(\bar{a}, \bar{b}, d)$ is first listed as the m^{th} triple in C_i. Then for all n

$$\mathcal{N}' \models \Psi^i_{m,n} = [\bigwedge\!\!\!\bigwedge_{k=1}^{n} \theta_k(\bar{a}_m, d_m) \bigwedge\!\!\!\bigwedge (\exists y) \wedge_{k=1}^{n} \theta_k(\bar{b}_m, y)] \rightarrow \bigwedge\!\!\!\bigwedge_{k=1}^{n} \theta_k(\bar{b}_m, c_{i+1,m})$$

since $\mathcal{N}' \models T_{C_0} \cup H$ in the language $L \cup C$.

Since f is a partial elementary map,

$$\mathcal{N} \models \bigwedge\!\!\!\bigwedge_{k=1}^{n} \theta_k(\bar{a}_m, d_m) \leftrightarrow (\exists y) \bigwedge\!\!\!\bigwedge_{k=1}^{n} \theta_k(\bar{b}_m, y)$$

holds for all n. Hence, we have for all n, $\mathcal{N} \models \bigwedge\!\!\!\bigwedge_{k=1}^{n} \theta_k(\bar{a}_m, d_m)$ if and only if $\mathcal{N} \models \bigwedge\!\!\!\bigwedge_{k=1}^{n} \theta_k(\bar{b}_m, c_{i+1,m})$. Thus $f \cup (d_m, d)$ is a partial elementary map that extends f where $d = c^{\mathcal{N}'}_{i+1,m}$. □

In [5], Csima, Harizanov, Hirschfeldt, and Soare proved that only the PA degrees are homogeneous bounding by constructing a complete decidable theory T with the property that any homogeneous model of T computes a PA degree. Specifically, any homogeneous model of T will compute a separating set for the c.e. set A of sentences provable from PA and the c.e. set B of sentences refutable from PA. To obtain the theorem below, we check that this theory T can be constructed in RCA_0.

Theorem 4.3.3. *The statement "Any theory T has a homogeneous model" together with RCA_0 implies WKL_0.*

Proof. Given an effective coding $\{\varphi_i\}_{i\in\omega}$ of all sentences in the language $\mathcal{L}$ consisting of D, P_i, E, and R_i for all $i \in \omega$ where D, P_i are unary predicates and E, R_i are binary predicates, we will show the theory T laid out in [5] exists, is consistent, and is complete in RCA_0. The following description of the axioms for T is taken exactly from [5]. Let $P_n^1\,y$ denote the formula $P_n\,y$, and let $P_n^0\,y$ denote the formula $\neg P_n\,y$. For $\sigma \in 2^{<\omega}$, let $P^\sigma y$ denote the formula $\bigwedge\!\!\!\bigwedge_{n=0}^{|\sigma|-1} P_n^{\sigma(n)} y$. We similarly define $R^\sigma xy$.

Axioms for T

Ax I. The relation E is an equivalence relation and splits the universe into two equivalence classes. That is,

$$(\forall x)Exx,$$
$$(\forall x, y)[Exy \to Eyx],$$
$$(\forall x, y, z)[Exy \wedge Eyz \to Exz], \quad \text{and}$$
$$(\exists x, y)[\neg Exy \wedge (\forall z)(Exz \vee Eyz)].$$

Ax II. For each $\sigma \in 2^{<\omega}$ there are infinitely many y in each equivalence class such that $P^\sigma y$ holds. That is, for each $\sigma \in 2^{<\omega}$ and $n \in \omega$,

$$(\forall x)(\exists y_0, \ldots, y_n)[\bigwedge\nolimits_{0\le i<j\le n}(y_i \neq y_j \wedge Ey_ix) \wedge \bigwedge\nolimits_{i=0,\ldots,n} P^\sigma y_i].$$

Ax III. No x, y in the same E-equivalence class are R_i related for any i. That is, for each $i \in \omega$,

$$(\forall x, y)[Exy \to \neg R_i xy].$$

Ax IV. Each R_i is symmetric. That is, for each $i \in \omega$,

$$(\forall x, y)[R_i xy \to R_i yx].$$

Ax V. The relation D holds of exactly two elements, one in each E-equivalence class. For any such element x, and any y, we have $\neg P_i x$ and $\neg R_i xy$ for every $i \in \omega$. That is,

$$(\exists x_0, x_1)[Dx_0 \wedge Dx_1 \wedge (\forall z)(Dz \to (z = x_0 \vee z = x_1)) \wedge \neg Ex_0 x_1]$$

and for each $i \in \omega$,

$$(\forall x)[Dx \to \neg P_i x] \quad \text{and}$$
$$(\forall x, y)[Dx \to \neg R_i xy].$$

Ax VI. For each $\sigma \in 2^\omega$ and each i, if $i \in A_{|\sigma|}$ then we have the following axiom:

$$(\forall x, y)[P^\sigma x \wedge P^\sigma y \wedge \neg Exy \to R_i xy],$$

while if $i \in B_{|\sigma|}$ then we have the following axiom:

$$(\forall x, y)[P^\sigma x \wedge P^\sigma y \to \neg R_i xy].$$

Ax VII. This axiom group essentially says that anything not ruled out by the previous axioms and describable by a quantifier-free formula must hold of some tuple of elements. For $\sigma, \tau, \mu \in 2^{<\omega}$, we say that μ *is compatible with* σ, τ if for every $i < |\mu|$ and l equal to the length of agreement of σ and τ (that is, the length of the longest common initial segment of σ and τ),

$$i \in A_l \Rightarrow \mu(i) = 1$$

and

$$i \in B_l \Rightarrow \mu(i) = 0.$$

We write $\sigma \mid \tau$ to mean that there is an $i < |\sigma|, |\tau|$ such that $\sigma(i) \neq \tau(i)$.

Let $\sigma_0, \ldots, \sigma_n, \tau, \mu_0, \ldots, \mu_n \in 2^{<\omega}$ be such that $\tau \mid \sigma_k$ for all $k \leq n$ and μ_k is compatible with σ_k, τ for all $k \leq n$. Then we have axioms saying that if $x_0, \ldots, x_n$ are in the same E-equivalence class and D does not hold of any of them, then there are infinitely many y in the other equivalence class such that $P^\tau y$ and $R^{\mu_k} x_k y$ for all $k \leq n$. That is, for each m we have the following axiom:

$$(\forall x_0, \ldots, x_n)[\bigwedge_{k=0,\ldots,n}(Ex_0x_k \wedge \neg Dx_k \wedge P^{\sigma_k}x_k) \rightarrow$$
$$(\exists y_0, \ldots, y_m)(\bigwedge_{j=0,\ldots,m}(\neg Ex_0y_j \wedge P^\tau y_j) \wedge \bigwedge_{\substack{k=0,\ldots,n \\ j=0,\ldots,m}} R^{\mu_k}x_ky_j)].$$

We denote this axiom by $Ax(\sigma_0, \ldots, \sigma_n, \tau, \mu_0, \ldots, \mu_n, m)$.

We show that T, the computably enumerable set of sentences that can be deduced from the axioms given above, exists as follows. We show T admits quantifier elimination in RCA_0. Then each sentence in $\mathcal{L}$ is (effectively) equivalent to one of the propositional formulas T or F under these axioms. The theory T exists in RCA_0 since it can be written as both a Σ^0_1 formula (T consists of all sentences deducible from the axioms) and a Π^0_1 formula (A sentence is not in T if its negation is deducible from the axioms). Moreover, T is complete in RCA_0.

Lemma 4.3.4. *The theory T admits quantifier elimination in RCA_0.*

Proof. In other words, we will show that for all formulas φ in the language $\mathcal{L}$ we can compute a quantifier-free formula ψ such that $T \vdash \varphi \leftrightarrow \psi$. Let $|\psi|$ denote the number of characters in ψ and let $QF(\psi)$ be the Δ^0_1 formula that holds if ψ is quantifier-free. We prove (using Σ^0_1 induction)

$$(\forall n, i)\ [(i < n\ \wedge\ |\varphi_i| < n) \rightarrow (\exists m)\,[QF(\varphi_m)\ \wedge\ T \vdash \varphi_i \leftrightarrow \varphi_m]]$$

Note that $T \vdash \varphi_i \leftrightarrow \varphi_m$ (where by T we mean the axioms of T) can be written as a Σ^0_1 formula. We can also assume that if $i < n$ then $|\varphi_i| < n$ and that if φ_i is a subformula of φ_j then $i \leq j$.

We assume the statement holds for n. We show it holds for $n+1$. Consider $\varphi = \varphi_k$ with $|\varphi_k| = n$. We wish to show there exists an m such that $QF(\varphi_m) \wedge T \vdash \varphi \leftrightarrow \varphi_m$. If φ is quantifier-free, we are done with $m = k$. Otherwise φ is a conjunction, disjunction, negation, or existential of one or more subformulas of length less than n. All these cases are easily handled except when φ has the form $(\exists x)\psi(x, \bar{y})$ where $|\psi| < |\varphi| = n$. We will consider this case now. By our assumptions on the enumeration of the formulas and induction, ψ is equivalent under T to a quantifier-free formula. Again for simplicity, we will rename ψ to be a quantifier-free formula in disjunctive normal form (a quantifier-free formula can be put in disjunctive normal form in RCA_0). Thus

$$T \vdash \varphi \leftrightarrow (\exists x)[\bigvee_{i<l} Lit_i(x, \bar{y})]$$

where Lit_i is a conjunction of literals in x and $\bar{y}$. Then

$$T \vdash (\exists x)[\bigvee_{i<l} Lit_i(x, \bar{y})] \leftrightarrow \bigvee_{i<l} [(\exists x) Lit_i(x, \bar{y})].$$

Thus it suffices to show (in RCA_0) that, for all $i < l$, there is a quantifier-free formula τ_i such that

$$T \vdash (\exists x) Lit_i(x, \bar{y}) \leftrightarrow \tau_i(\bar{y})$$

as then $T \vdash \varphi \leftrightarrow \bigvee_{i<l} \tau_i(\bar{y})$.

Suppose we are given $(\exists x) Lit_i(x, \bar{y})$ of the form above. Recall we denote θ^1 to denote the formula θ and θ^0 to denote the formula $\neg\theta$. The formula Lit_i is simply a conjunction of literals, so in RCA_0, $Lit_i(x, \bar{y})$ is equivalent to some subformula of a conjunction of the form(for all $y_i, y_j \in \bar{y}$ for $i \leq j$):

- $D^{d_x}(x) \wedge P^{\sigma_x}(x)$

- $D^{d_{y_i}}(y_i) \wedge P^{\sigma_{y_i}}(y_i)$
- $R^{\rho_{x,y_i}}(x, y_i) \wedge E^{n_{x,y_i}}(x, y_i) \wedge x =^{e_{x,y_i}} y_i$
- $R^{\rho_{y_i,y_j}}(y_i, y_j) \wedge E^{n_{y_i,y_j}}(y_i, y_j) \wedge y_i =^{e_{y_i,y_j}} y_j$.

where $d_u, n_{u,v}, e_{u,v} \in \{0, 1\}$ and $\sigma_u, \rho_{u,v} \in 2^{<\omega}$ for all u and v equaling x, y_i, y_j.

Define k such that P_k or R_k is mentioned in Lit_i, and if P_j or R_k is mentioned in Lit_i, $j \le k$.

In RCA_0, T proves that Lit_i is equivalent to the disjunction of all possible combination of conjunctions as above containing Lit_i as a subformula with $\sigma_u, \rho_{u,v} \in 2^{k+1}$. We call one such conjunction in this disjunction Tot, as it has the form above with corresponding formulas. As before, we reduce our problem to finding a quantifier-free formula which is equivalent under RCA_0 to $(\exists x)Tot(x, \bar{y})$.

Let $Tot'(\bar{y})$ be the conjunction of (for all $y_i, y_j \in \bar{y}$ and $i \le j$):

- $D^{d_{y_i}}(y_i) \wedge P^{\sigma_{y_i}}(y_i)$
- $R^{\rho_{y_i,y_j}}(y_i, y_j) \wedge E^{n_{y_i,y_j}}(y_i, y_j) \wedge y_i =^{e_{y_i,y_j}} y_j$.

We will ensure the quantifier-free formula we are searching out will be a conjunction of Tot' and another formula. Thus, we wish to find a quantifier-free formula θ such that under RCA_0,

$$T \cup \{(\exists \bar{y})Tot'(\bar{y})\} \vdash (\exists x)Tot(x, \bar{y}) \leftrightarrow \theta(x, \bar{y}).$$

We assume $Tot'(\bar{y})$ satisfies the rules of the theory (i.e., E is an equivalence relation with two infinite classes, R_i is symmetric and respects E and the compatibility rules, and the rules for D and equality are respected). These basic checks can be performed in RCA_0. (In particular the compatibility axiom Ax. VI can be checked since RCA_0 knows A_i and B_i for all $i \le k$.) We do the same basic consistency check for $Tot(x, \bar{y})$. If Tot does not pass, we can let θ (our quantifier-free formula as above) be F. We claim that θ equal to T will work in all remaining cases.

If $D(x)$ is a literal in $Tot(x, \bar{y})$, we can set θ to T by Ax. V. Somewhat similarly, Ax. V gives that the formula $Tot(x, \bar{y})$ with the literal $D(y_i)$ removed is equivalent to $Tot(x, \bar{y})$. Thus we can assume $\neg D(x)$ and $\neg D(y_i)$ for all $y_i \in \bar{y}$ are literals in $Tot(x, \bar{y})$.

Let $\bar{y}_0 \subset \bar{y}$ be the set of $y \in \bar{y}$ that such that $\neg Exy$ is a literal in Tot. (Hence $\bar{y}_0$ is contained in one equivalence class.) We use Ax. VII to show that θ equal to T is our desired quantifier-free formula. For each $y_i \in \bar{y}_0$, the subformula $R^{\rho x, y_i}(x, y_i)$ must satisfy the compatibility required by Ax. VI (or it would have failed the basic consistency checks). If $\sigma_x | \sigma_{y_i}$ where $P^{\sigma_x}(x)$ and $P^{\sigma_{y_i}}(y_i)$ are in Tot for all $y_i \in \bar{y}_0$, Ax. VII deduces $(\exists x)Tot(x, \bar{y})$. Hence setting θ equal to T works. If $\sigma_x = \sigma_{y_i}$ for some y_i, note that Ax. VI says that there exists an x such that

$$P^{\sigma^\frown 1x}(x) \wedge P^{\sigma^\frown 0y_i}(y_i) \wedge R^{\rho x, y_i}(x, y_i) \text{ holds}$$

or

$$P^{\sigma^\frown 0x}(x) \wedge P^{\sigma^\frown 1y_i}(y_i) \wedge R^{\rho x, y_i}(x, y_i) \text{ holds}$$

Thus also in this case, θ can be T. Then $\theta \wedge Tot'(\bar{y})$ is a quantifier-free formula equivalent to $Tot(x, \bar{y})$ under T. By taking the disjunction of all the quantifier-free formulas found for each disjunction Tot in Lit_i, we find a quantifier-free formula for Lit_i. Then take the disjunction of all these quantifier-free formulas (one for each Lit_i) to find a quantifier-free formula φ_m equivalent under T to φ. Thus the induction step holds, so by $\mathrm{I}\Sigma^0_1$ in RCA_0, T admits quantifier elimination. □

Lemma 4.3.5. *It is provable in RCA_0 that T is consistent.*

Proof. Since T admits quantifier elimination, it suffices to produce in RCA_0 an atomic diagram consistent with T. Let $\{\sigma_i\}_{i \in \omega}$ denote an (effective) enumeration of elements in $2^{<\omega}$ where $\sigma_0 = \emptyset$. Let D hold of $\langle 0, 0\rangle$ and $\langle 0, 1\rangle$.

For any $x \in \{\langle i, 2j\rangle\}_{j \in \omega}$ other than $\langle 0, 0\rangle$, let $P_k(x)$ hold if $\sigma_i(k) = 1$ and $k < |\sigma|$. For all other $k \in \omega$, let $\neg P_k(x)$ hold. For any $x \in \{\langle i, 2j+1\rangle\}_{j \in \omega}$ other than $\langle 0, 1\rangle$, let $P_k(x)$ hold if $\sigma_i(k) = 1$ and $k < |\sigma|$. For all other $k \in \omega$, let $P_k(x)$ hold. Given $x = \langle i, j\rangle$ and $y = \langle i', j'\rangle$, let $E(x, y)$ hold if and only if $j - j'$ is even. If $\neg E(x, y)$

holds or if $D(x)$ or $D(y)$ holds, then $\neg R_k(x, y)$ for all $k \in \omega$. We now must define the R_k relationship between elements x, y such that $x, y \notin D$ and $\neg E(x, y)$.

Suppose $x = \langle i, 4j \rangle$ and $y = \langle i', 4j' + 1 \rangle$ or $x = \langle i, 4j + 2 \rangle$ and $y = \langle i', 4j' + 3 \rangle$. Then let σ be the longest common initial segment of $\sigma_i{}^\frown 0^\infty$ and $\sigma_{i'}{}^\frown 1^\infty$. Note that σ can be found in RCA_0 because $|\sigma| < \max(|\sigma_i|, |\sigma_{i'}|)$. Let $f \in 2^\omega$ be such that

- if $k \in A_{|\sigma|}$ then $f(k) = 1$,
- if $k \in B_{|\sigma|}$ then $f(k) = 0$, and
- otherwise, $f(k) = 0$.

Then define $R_k(x, y)$ and $R_k(y, x)$ to hold if and only if $\tau(k) = 1$. By defining the R_i relationships in this way, we are ensuring that Ax. VI holds for these pairs of elements.

Now we define all R_i relationships between elements of the form $x = \langle i, 4j \rangle$ and $y = \langle i', 4j' + 3 \rangle$ and $x = \langle i, 4j + 2 \rangle$ and $y = \langle i', 4j' + 1 \rangle$. To ensure that Ax. VII holds, we need a more complicated definition. Let $d^i_s = \langle i, 4s + 2 \rangle$ and $e^i_s = \langle i, 4s + 3 \rangle$. Let $\{\bar{c}_k\}_{k \in \omega}$ be a listing of all finite tuples of elements $\langle i, j \rangle$ not including the elements in D, and let $\{\bar{\sigma}_k\}_{k \in \omega}$ be a listing of all finite tuples of finite strings in $2^{<\omega}$. We define the remaining R_i relationships in stages. At stage $s = \langle i, j, k, l \rangle$ we consider $\bar{c}_i$, σ_j, and $\bar{\sigma}_k$. Let $\bar{c}_i = \{x_a = \langle n_a, m_a \rangle\}_{a=0}^{t-1}$. Suppose all elements of $\bar{c}_i$ are in the same equivalence class (*i.e.*, the m_a are all even or all odd) and $|\bar{\sigma}_k| = t$. Suppose further that each element of $\bar{\sigma}_k$ is compatible with its corresponding element of $\{\sigma_{n_a}\}_{a=0}^{t-1}$ and σ_j and that $\sigma_{n_a} \mid \sigma_j$. (Recall $P^{\sigma_{n_a}}$ holds of $\langle n_a, m_a \rangle$.) If the m_a are all even, then define the next l unused elements of $\{e^j_a\}_{a>s}$, named $y_0, y_1, \ldots, y_{l-1}$, so $R^{\sigma_a}(x_a, y_b)$ holds where σ_a is the a^{th} element of $\bar{\sigma}_k$ and $\neg R_r(x_a, y_b)$ holds for $r \geq |\sigma_a|$ for $0 \leq a < t$ and $0 \leq b < l$. If the m_a are all odd, then define the next l unused elements of $\{d^j_a\}_{a>s}$ similarly.

If not all of the above conditions hold, define any undefined elements of $\{e^j_a\}_{a \leq s}$ and $\{d^j_a\}_{a<s}$ as we did to satisfy Ax. VI as above. By construction, this model satisfies Ax. VII of the theory. Since this model satisfies all the axioms, we have proved that T is consistent in RCA_0. $\square$

We now show that from any homogeneous model of T, we obtain a separating set for the sets A and B, completing our proof. Let $\mathcal{A}$ be a homogeneous model of T, and let a, b be the two elements of $D^{\mathcal{A}}$. Since T admits quantifier elimination and a and b satisfy the same 1-ary atomic formulas by Ax. V, a and b satisfy the same 1-type. In other words, the map f sending a to b is partial elementary. Let $c \in |\mathcal{A}| \setminus \{a, b\}$ be such that $\mathcal{A} \vDash E(a, c)$. Since $\mathcal{A}$ is homogeneous, there is a partial elementary map $f^* \supset f$ with domain $\{a, c\}$. Let $d = f^*(c)$. Since $\mathcal{A} \vDash E(a, c)$, we have $\mathcal{A} \vDash E(b, d)$. Since a and b are not E equivalent, neither are c and d. Since f^* is a partial elementary map, $P_i^{\mathcal{A}} c \iff P_i^{\mathcal{A}} d$ for all i. But then by Ax. VI, if $i \in A$ (*i.e.*, $i \in A_s$ for some s), then $R_i^{\mathcal{A}}(c, d)$, and if $i \in B$ (*i.e.*, $i \in B_s$ for some s), then $\neg R_i(c, d)$. Thus $\{i \mid R_i^{\mathcal{A}}(c, d)\}$ is a separating set for A and B. Hence, the statement "Any theory has a homogeneous model" implies WKL_0 over RCA_0. □

Corollary 4.3.6. *Over the system RCA_0, WKL_0 is equivalent to the statement "Any theory T has a homogeneous model."*

4.4 The Atomic and Homogeneous Model Theorems

As we have mentioned before, the theorems we have presented on degrees of copies of homogeneous models all have analogous counterparts for prime models. These results on prime models were the original motivation for studying the homogeneous case. Since all prime models are homogeneous, we hoped that results on homogeneous models might give the results on prime models as corollaries. Theorem 2.1.1, which says any homogeneous model with a $\mathbf{0}'$-basis has a low copy, does give the analogous result for prime models as a corollary (see Corollary 2.1.2). However, it was not clear how Theorems 2.2.3 and 2.3.2 on homogeneous models related to their prime counterparts. In order to develop a connection between the homogeneous and prime model cases, we approach these questions from the perspective of reverse mathematics. Hirschfeldt, Shore, and Slaman [13] already investigated the reverse mathematical strength of many classical model theoretic theorems on prime and atomic models. In particular, they studied the following statement, known as the Atomic Model Theorem (AMT).

AMT: A complete atomic theory has an atomic model.

In their work, they compared AMT to other weak combinatorial statements. For example, they showed that AMT is incomparable with WKL_0 over RCA_0. Moreover, consider the following statements:

CADS: Every infinite linear order has a subordering of order type ω, ω^*, or $\omega+\omega^*$.

SADS: Every infinite linear order of order type $\omega + \omega^*$ has a subordering of order type ω or ω^*.

Hirschfeldt, Shore, and Slaman showed that SADS is strictly stronger than AMT over RCA_0 in [13], but it remains unknown whether CADS implies AMT.

We wish to compare AMT with its homogeneous counterpart, but first we must properly describe this counterpart. Since we are working in reverse mathematics, we cannot assume that we have a specific homogeneous model of a theory T. (We want the existence of that model to be the conclusion of the statement.) However, the statement should assert the existence of a homogeneous models realizing a particular set of types. Hence, we assume that we have a set of types that could be the type spectrum of a homogeneous model of a theory T. Let the *homogeneity closure conditions* be the conditions given in §3.2.1 that are necessary and sufficient to classically guarantee that a basis of types $X \subseteq S(T)$ equals $\mathbb{T}(\mathcal{A})$ for some homogeneous model $\mathcal{A}$ of T. A *basis of types* here is a function $f(i,j)$ such that for any fixed i, $\{f(i,j)\}_{j\in\omega}$ codes a type p_i in $S(T)$. We also assume that each type in a basis is listed infinitely many times, *i.e.*, if p_i is a type coded by $\{f(i,j)\}_{j\in\omega}$, then there are infinitely many i' such that p_i equals the type coded by $\{f(i',j)\}_{j\in\omega}$. Note that as a set of sets, $S(T)$ cannot be defined in second order arithmetic. We use the notation $p_i \in S(T)$ as a shorthand for writing that p_i is a consistent type in T, *i.e.*, for every n, the formula $\theta_{p_i\restriction n}$ is consistent with T. We can then state the Homogeneous Model Theorem (HMT).

HMT: Let T be a theory, and let $X \subseteq S(T)$ be a basis of types. If X satisfies the homogeneity closure conditions, then there exists a homogeneous model $\mathcal{A}$ of T with $\mathbb{T}(\mathcal{A}) = X$.

This principle can be broken into two more statements, the MEF Existence Theorem (MEFE) and the MEF Theorem (MEFT). A *monotone extension function* g in this context is a function g that satisfies all the conditions of Definition 1.3.6 for a monotone extension function except possibly the condition that g is computable.

MEFE: Let T be a theory and $X \subseteq S(T)$ be a basis satisfying the homogeneity closure conditions. Then there is a monotone extension function g for X.

MEFT: Let T be a theory and $X \subseteq S(T)$ be a basis satisfying the homogeneity closure conditions. If X has a monotone extension function g, then there exists a homogeneous model $\mathcal{A}$ of T with $\mathbb{T}(\mathcal{A}) = X$.

We show that AMT implies MEFE over RCA_0. In other words, given a theory T and a basis X satisfying the homogeneity closure conditions, there exists an atomic theory $\hat{T}$ such that any atomic model of $\hat{T}$ can be used to define a monotone extension function for X.

Theorem 4.4.1. *AMT implies MEFE over RCA_0.*

Before proving this theorem, we discuss our approach. In §1.2.3, we saw that theories are intimately connected with trees. Rather than define an atomic theory directly from a fixed theory T and basis X, we will define an atomic extendible tree that gives rise to an atomic theory. Moreover, any enumeration of the isolated paths in this tree will correspond exactly with an enumeration of the principal types of the atomic theory. Let $\mathcal{L}$ be the language $\{P_i\}_{i\in\omega}$ where P_i is a unary predicate. For $\sigma \in 2^{<\omega}$, let $P^\sigma(y)$ denote the formula $\bigwedge_{n=0}^{|\sigma|-1} P_n^{\sigma(n)}(y)$ where $P_n^1(y)$ denotes the formula $P_n(y)$ and $P_n^0(y)$ denotes the formula $\neg P_n(y)$.

Definition 4.4.2. (Definition 4.4 in [6]) Let $\mathcal{L}$ be the language $\{P_i\}_{i\in\omega}$ where P_i is a unary predicate. Given an extendible tree $\mathcal{T} \subseteq 2^{<\omega}$, let $T(\mathcal{T})$ be the deductive closure in $\mathcal{L}$ of the following set $R(\mathcal{T})$ of axioms.

1. $(\exists^{>m}x)[P^\sigma(x)]$ for all $\sigma \in \mathcal{T}$ and $m \in \omega$.

2. $\neg(\exists x)[P^\sigma(x)]$ for any $\sigma \notin \mathcal{T}$.

Note that $R(\mathcal{T})$ is definable in RCA_0.

Lemma 4.4.3. *The following is provable in RCA_0. Let $\mathcal{T} \subseteq 2^{<\omega}$ be an extendible tree.*
(i) The theory $T(\mathcal{T})$ exists and is complete and consistent.
(ii) If the isolated paths in $\mathcal{T}$ are dense, then $T(\mathcal{T})$ is an atomic theory.
(iii) From an enumeration $\mathcal{Y}$ of the isolated paths in $\mathcal{T}$, we can find an enumeration Y of the principal types in $T(\mathcal{T})$, and vice versa.

Proof of (i). We show $R(\mathcal{T})$ admits quantifier elimination in RCA_0. As in Lemma 4.2.3, $T(\mathcal{T})$ is then Δ^0_1-definable and complete. It suffices to provide a uniformly computable procedure that finds a quantifier-free formula $\psi(\bar{x})$ that is $R(\mathcal{T})$-equivalent to a given existential formula $(\exists\bar{y})\varphi(\bar{x},\bar{y})$. As in Lemma 4.2.3, we may assume that $\varphi(\bar{x},\bar{y})$ is a conjunction of atomic formulas and negations of atomic formulas. In this case, all atomic formulas are of the form $P_i(w)$ for $w \in \bar{x}\cup\bar{y}$. Thus, $\varphi(\bar{x},y)$ has the form $\bigwedge\!\!\!\bigwedge_{w\in\bar{x}\cup\bar{y}} P^{\sigma_w}(w)$ where $\sigma_w \in 2^{<\omega}$. Then

$$R(\mathcal{T}) \vdash (\exists\bar{y})\varphi(\bar{x},\bar{y}) \leftrightarrow \bigwedge\!\!\!\bigwedge_{w\in\bar{x}} P^{\sigma_w}(w)$$

if $\sigma_y \in \mathcal{T}$ for all $y \in \bar{y}$. Otherwise, $R(\mathcal{T}) \vdash (\exists\bar{y})\varphi(\bar{x},\bar{y}) \leftrightarrow \mathrm{F}$. Hence $R(\mathcal{T})$ admits quantifier elimination in RCA_0.

To show $T(\mathcal{T})$ is consistent in RCA_0, we provide a model $\mathcal{A}$ of $T(\mathcal{T})$. Since $R(\mathcal{T})$ admits quantifier elimination and there are only unary relations, we need to say only what unary relations hold of any element to define $\mathcal{A}$. Let $\{\tau_i\}_{i\in\omega}$ be an enumeration of all elements in $\mathcal{T}$. Let $\omega = \{\langle i,j\rangle\}_{i,j\in\omega}$ be the domain of $\mathcal{A}$. For all i, j, define $P^{\tau_i}(\langle i,j\rangle)$ to hold in $\mathcal{A}$ (and $\neg P_k(\langle i,j\rangle)$ to hold in $\mathcal{A}$ for all $k \geq |\tau_i|$). Clearly $\mathcal{A}$ is a model of $T(\mathcal{T})$. Hence $T(\mathcal{T})$ is consistent.

Proof of (ii). Suppose the isolated paths in $\mathcal{T}$ are dense. Since $\mathcal{L}$ has only unary relations, any type in $T(\mathcal{T})$ is simply the conjunction of its 1-ary subtypes. To show $T(\mathcal{T})$ is an atomic theory, it suffices to show any 1-ary formula $\theta(x)$ consistent with $T(\mathcal{T})$ in $\mathcal{L}$ is contained in a principal type $p(x)$ of $T(\mathcal{T})$. Let $\theta(x)$ be a 1-ary (quantifier-free) formula consistent with $T(\mathcal{T})$. Then $\theta(x)$ is logically equivalent to a disjunction of conjunctions of literals $P_i(x)$ and $\neg P_i(x)$. Since $\theta(x)$ is consistent with $T(\mathcal{T})$, there is some $\sigma \in \mathcal{T}$ such that $P^\sigma(x)$ logically implies $\theta(x)$. Since isolated paths are dense in $\mathcal{T}$, there exists some isolated path $f \in [\mathcal{T}]$ extending σ. (Note that $f \in [\mathcal{T}]$ is simply shorthand for the formula $(\forall n)[\, f\upharpoonright n \in \mathcal{T}\,]$.) By definition of $T(\mathcal{T})$, f corresponds to a principal 1-type $p(x)$ containing $P^\sigma(x)$ and θ. Thus $T(\mathcal{T})$ is atomic.

Proof of (iii). If we have an enumeration of the isolated paths through $\mathcal{T}$, the above process of finding a principal type extending a formula consistent with $T(\mathcal{T})$ is uniform. Thus, an enumeration of the isolated paths in $\mathcal{T}$ gives rise to an enumeration of $S^P(T(\mathcal{T}))$. By definition of $T(\mathcal{T})$, the process of obtaining an enumeration of the isolated paths in $\mathcal{T}$ from one of $S^P(T(\mathcal{T}))$ is similar. □

We now prove that AMT implies MEFE in RCA_0. We show that given a theory and a basis of types satisfying the homogeneity closure conditions, we can build an atomic theory such that any atomic model of this new theory can be used to define a monotone extension function for the original basis of types.

Proof of Theorem 4.4.1. Let T be a theory and X a basis of types in $S(T)$ satisfying the homogeneity conditions. From X and T, we construct an extendible atomic tree $\mathcal{T}$ with the following key property. If $\hat{T} = T(\mathcal{T})$ as in Definition 4.4.2, then a monotone extension function g for X is definable from any atomic model of $\hat{T}$ in RCA_0.

Let p_i denote the type in $S(T)$ coded by the i^{th} row of X. We also assume that each row is listed infinitely many times in X. Given an n-type $p_i \in X$, an $(n+1)$-ary formula ψ consistent with p_i, and a stage s, we say p_j *amalgamates* p_i *and* ψ *at stage* s if p_j is an $n+1$ type and $p_j\upharpoonright s$ is consistent with p_i and ψ, *i.e.*, $\psi \wedge \theta_{p_j\upharpoonright s}$ is consistent with p_i. By the homogeneity closure conditions, there exists some p_j such that $p_j \supseteq p_i \cup \{\psi\}$. Thus, for any type p_i, formula ψ consistent with p_i, and stage s, there exists some type in X that amalgamates p_i and ψ at stage s. We define a family of extendible atomic trees $\mathcal{T}_{i,j}$, and let $\mathcal{T}$ be the downward closure of $\{0^{\langle i,j\rangle}1^\frown\sigma | \sigma \in \mathcal{T}_{i,j}\}$. Note $\mathcal{T}$ exists and is also atomic. We build these trees so that any atomic model of the corresponding theory can be used to define a monotone extension function for X in RCA_0.

Definition of $\mathcal{T}_{i,j}$

Let $\mathcal{T}_{i,j} = \cup_{s\in\omega} \mathcal{T}^s_{i,j}$ where $\mathcal{T}^s_{i,j}$ is defined in stages.

Stage $s = 0$**:** Let $\psi_0 = \theta_j$. Let $\mathcal{T}^0_{i,j} = \{1^k\}_{k\in\omega}$. If p_i and ψ_0 are inconsistent, this completes the construction of $\mathcal{T}_{i,j}$. Otherwise, we associate with each node $\tau = 1^k$ in $\mathcal{T}^0_{i,j}$ a pair $s_\tau = k$ and $t_\tau = -1$, and a formula $\psi_\tau = \theta_j$. We label all the nodes τ in $\mathcal{T}^0_{i,j}$ as active.

Stage $s + 1$**:**

We are given the tree $\mathcal{T}^s_{i,j}$. For every $\tau \in \mathcal{T}^s_{i,j}$, we have

1. s_τ *(the guide amalgamator being followed)*

2. t_τ *(the length of time the guide amalgamator has been followed)*

3. ψ_τ consistent with p_i and θ_j *(the current guide formula)*

where p_{s_τ} amalgamates p_i and ψ_τ at stage t_τ.

Let $\mathcal{T}^{s+1}_{i,j} \supseteq \mathcal{T}^s_{i,j}$. For every active node $\tau \in \mathcal{T}^s_{i,j}$, check whether p_{s_τ} amalgamates p_i and ψ_τ at stage $t_\tau + 1$. If so, add $\tau^\frown 0$ to $\mathcal{T}^{s+1}_{i,j}$, and let $s_{\tau^\frown 0} = s_\tau$, $t_{\tau^\frown 0} = t_\tau + 1$, and $\psi_{\tau^\frown 0} = \psi_\tau$. Deactivate τ, and activate $\tau^\frown 0$.

If p_{s_τ} does not amalgamate p_i and ψ_τ at stage $t_\tau+1$, add $\tau^\frown 1^{k+1}$ for all k to $\mathcal{T}^{s+1}_{i,j}$. For each $\tau' = \tau^\frown 1^{k+1}$, let $s_{\tau'} = k$, $t_{\tau'} = -1$, and $\psi_{\tau'} = \theta_{p_{s_\tau}\upharpoonright t_\tau} \wedge \psi_\tau$. Deactivate τ, and activate each τ'.

End Definition.

Lemma 4.4.4. *It is provable in* RCA_0 *that for all* i, j, $\mathcal{T}_{i,j}$ *is an extendible atomic tree. Thus,* $\mathcal{T}$ *is an extendible atomic tree.*

Proof. Let $\tau \in \mathcal{T}_{i,j}$. Note τ is active for exactly one stage r by construction. We show there exists an isolated path $f \in [\mathcal{T}_{i,j}]$ extending τ. Suppose p_{s_τ} amalgamates p_i and ψ_τ. Then by construction, for all stages $s > r$, the node $\tau^\frown 0^{s-r}$ is added to $\mathcal{T}_{i,j}$, and no other strings extending τ are added. Hence $f = \tau^\frown 0^\omega$ is an isolated path in $[\mathcal{T}_{i,j}]$ extending τ. Suppose p_{s_τ} does not amalgamate p_i and ψ_τ. Let t be the least stage such that p_{s_τ} does not amalgamate p_i and ψ_τ at stage $t+1$. Note that $l = t+1 \geq 0$ and $\tau' = \tau^\frown 0^l \in \mathcal{T}_{i,j}$ by construction. Furthermore, $s_{\tau'} = s_\tau$, $\psi_{\tau'} = \psi_\tau$, and $t_{\tau'} = t$. By construction, τ' is active at stage $r+l$. At stage $r+l$, node $\sigma_k = \tau'^\frown 1^{k+1}$ is added to $\mathcal{T}_{i,j}$ for all k, and $\psi_{\sigma_k} = \theta_{p_{s_\tau} \upharpoonright t} \wedge \psi_\tau$ and $s_{\sigma_k} = k$. Since X satisfies the homogeneity closure conditions, there exists some $p_m \in X$ such that p_m amalgamates p_i and $\theta_{p_\tau \upharpoonright t} \wedge \psi_\tau$. Since σ_m is active at stage $r+l$ and $p_{s_{\sigma m}}$ amalgamates p_i and ψ_{σ_m}, there exists an isolated path $f = \sigma_m{}^\frown 0^\omega$ in $[\mathcal{T}_{i,j}]$ extending τ as in the first case of this lemma. Hence $\mathcal{T}_{i,j}$ is an extendible atomic tree. By definition, $\mathcal{T}$ is an extendible atomic tree as well.

□

Lemma 4.4.5. *A monotone extension function g for X is definable in RCA_0 from any enumeration $\mathcal{Y}$ of the isolated paths of $\mathcal{T}$.*

Proof. Let $\mathcal{Y}$ be an enumeration of the isolated paths of $\mathcal{T}$. Since $\mathcal{T}$ is atomic, there exists some isolated path $\hat{f} \in \mathcal{Y}$ such that $\hat{f}$ extends $0^{\langle i,j\rangle \frown}1$. Then

$$f(k) = \hat{f}(k + \langle i,j \rangle + 1)$$

is an isolated path in $\mathcal{T}_{i,j}$. By construction of $\mathcal{T}_{i,j}$, any τ such that $\tau(|\tau|-1) = 1$ can be extended by two incomparable paths. Since f is isolated, f therefore has the form $\tau^\frown 0^\omega$ for some $\tau \in 2^{<\omega}$.

We define $g(i,j,0) = l$ such that f extends $\tau_0 = 1^{l\frown}0$ (Note $s_{\tau_0} = l$ and $t_{\tau_0} = 0$). Given $g(i,j,s)$ and τ_s, suppose f extends $\tau_{s+1} = \tau_s{}^\frown 0$. Then by construction, the guide amalgamator continues to be correct so we define $g(i,j,s+1) = s_{\tau_s} = s_{\tau_{s+1}}$. Otherwise, suppose f extends $\tau_s{}^\frown 1$. Take the first σ such that f extends σ and $\sigma = \nu^\frown \rho$ and $\rho = 01^l 0^{s+1}$ for some $l \geq 1$ and $\nu \in 2^{<\omega}$ such that $|\nu| \geq |t_s| - 2$.

(Such a σ exists since f has the form $\tau^\frown 0^\omega$ for some τ.) Then let $\tau_{s+1} = \sigma$ and $g(i,j,s+1) = l = s_{\tau_{s+1}}$. Because f extends $\tau_s{}^\frown 1$ for only finitely many stages s, $g(i,j,s)$ changes for only finitely many s. By construction of $\mathcal{T}_{i,j}$, the function $g(i,j,s)$ is a monotone extension function and exists in RCA_0 since RCA_0 is closed under primitive recursion and minimization. □

To complete the proof of Theorem 4.4.1, let $\hat{T} = T(\mathcal{T})$. Note that $\hat{T}$ is a complete atomic theory by Lemma 4.4.3. By AMT, $\hat{T}$ has an atomic model $\mathcal{A}$. Let $\{\bar{c}_k\}_{k\in\omega}$ be an enumeration of the n-tuples of $\mathcal{A}$ with each tuple listed infinitely often. Let Y be the basis of types such that row i equals

$$\{j \mid \bar{c}_i \text{ and } \theta_j \text{ have same arity } \& \ \mathcal{A} \vDash \theta_j(\bar{c}_i)\}.$$

Since $\mathcal{A}$ is atomic, Y is an enumeration of the principal types of $\hat{T}$. Then by Lemma 4.4.3, $\mathcal{T}$ has an enumeration $\mathcal{Y}$ of its isolated paths. Finally, Lemma 4.4.5 gives that X has a monotone extension function g as desired. □

It remains to show that MEFT can be proved in RCA_0. We conjecture this is true, and hence by combining this conjecture with Theorem 4.4.1, we obtain the following.

Conjecture 4.4.6. *AMT implies HMT in RCA_0.*

It remains unknown whether HMT implies AMT over RCA_0. Given the complexity of the negative result on **0**-basis homogeneous bounding, we expect it will be more difficult to determine and prove this comparison.

Implications for Computability Results

Theorem 4.4.1 relates the corresponding computability results on prime and homogeneous models. In fact, this theorem shows that two of the positive results on homogeneous models are corollaries of the corresponding results on prime models. For example, Theorem 2.1.1 states that every homogeneous model $\mathcal{A}$ with a **0**-basis has a copy decidable in a low degree. (The theorem actually holds for any homogeneous model $\mathcal{A}$ with a $\mathbf{0}'$-basis. That slightly stronger statement does not immediately appear to be a corollary of Theorem 4.4.1.) Csima [4] showed that every CD atomic theory T has a prime (*i.e.*, atomic) model decidable in a low degree (see Corollary 2.1.2). Let $\mathcal{A}$ be a homogeneous model with a $\mathbf{0}$-basis X. We can computably build a CD atomic theory $\hat{T}$ as in Theorem 4.4.1. By Csima's result, let $\mathcal{P}$ be a prime model of $\hat{T}$ decidable in a low degree $\mathbf{d}$. Then as in Theorem 4.4.1, we can build a $\mathbf{d}$-MEF for X. Then by the relativized EEF/MEF Theorem 1.3.8, $\mathcal{A}$ has a copy decidable in the low degree $\mathbf{d}$. Theorem 2.3.2 states that nonlow$_2$ degrees are $\mathbf{0}$-bounding. This result again follows from the analogous result for prime models in [6] as above.

Theorem 2.2.3 says that in the case that all the types in $S(T)$ are computable, if a homogeneous model has a $\mathbf{0}$-basis, then the model has a copy decidable in any non-zero degree. This theorem does not follow as a corollary from Theorem 4.4.1 and Hirschfeldt's analogous result for prime models in [12] because even if the theory of the homogeneous model has only computable types, the atomic theory we construct in Theorem 4.4.1 may not.

4.5 Further Directions in Reverse Mathematics

Our first goal is to show that MEFT is provable in RCA_0, and our second goal is to determine whether HMT implies AMT over RCA_0 (we conjecture it does). We would also like to understand how these statements relate to other well-known combinatorial principles. For example, Cohesive ADS (CADS) is the principle that every infinite linear order has a subset S of order type ω, ω^*, or $\omega+\omega^*$. In [13], the authors showed that SADS implies AMT over RCA_0. It remains unknown, however, whether CADS implies AMT over RCA_0. Furthermore, we would like to mine other classical model theoretic concepts (for example submodel completeness) for their reverse mathematical content.

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