SOME NEW HOMOGENEOUS EINSTEIN METRICS ON SYMMETRIC SPACES

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Abstract. We classify homogeneous Einstein metrics on compact irreducible symmetric spaces. In particular, we consider symmetric spaces with $\mathrm{rank}(M) > 1$, not isometric to a compact Lie group. Whenever there exists a closed proper subgroup G of $\mathrm{Isom}(M)$ acting transitively on M we find all G-homogeneous (non-symmetric) Einstein metrics on M.

1. Introduction

In this paper we look at compact symmetric spaces presented homogeneously, i.e. as $M = G/H$, where $G = \mathrm{Isom}_0(M)$ is simple, and we consider the cases where there exists a closed subgroup $G' \subset G$ which acts transitively on M. Denote by H' the isotropy subgroup in G', then $M = G'/H'$. Since G' is smaller than G, we expect more G'-invariant metrics on M than G-invariant metrics, and thus we can hope for non-symmetric G'-invariant Einstein metrics on our symmetric space M. We find the following.

Lemma 1.1. *Let M be a compact irreducible symmetric space of rank > 1, M not isometric to a compact Lie group with biinvariant metric. Let $G = Isom_0(M)$, and $M = G/H$. Then there exists a subgroup $G' \subset G$ acting transitively on M $\Leftrightarrow$*

1. $G = \mathrm{SO}(2n),\ H = \mathrm{U}(n),\ G' = \mathrm{SO}(2n-1),\ H' = \mathrm{U}(n-1)\ (n \geq 4)$;
2. $G = \mathrm{SU}(2n),\ H = \mathrm{Sp}(n),\ G' = \mathrm{SU}(2n-1),\ H' = \mathrm{Sp}(n-1)\ (n \geq 3)$;
3. $G = \mathrm{SO}(7),\ H = \mathrm{SO}(2)\,\mathrm{SO}(5),\ G' = \mathrm{G}_2,\ H' = \mathrm{U}(2)\ (\mathrm{U}(2) \subset \mathrm{SU}(3))$;
4. $G = \mathrm{SO}(8),\ H = \mathrm{SO}(3)\,\mathrm{SO}(5),\ G' = \mathrm{Spin}(7),\ H' = \mathrm{SO}(4)\ (\mathrm{SO}(4) \subset \mathrm{G}_2)$.

Theorem 1.2. *Among the compact irreducible symmetric spaces of rank > 1, not isometric to a Lie group with a biinvariant metric, only $G_2^+(\mathbb{R}^7)$, $G_3^+(\mathbb{R}^8)$, and $\mathrm{SO}(2n)/U(n)$, for $n \geq 4$, carry non-symmetric homogeneous Einstein metrics. The Grassmannians $G_2^+(\mathbb{R}^7)$ and $G_3^+(\mathbb{R}^8)$ each carry two and $\mathrm{SO}(2n)/\mathrm{U}(n)$ carries one; the only homogeneous Einstein metric on $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ is the symmetric metric.*

Let $\mathcal{M}_{G'}$ denote the space of G'-invariant metrics of volume one. Our results are summarized in the following table.

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The analogous results for symmetric spaces of rank 1 were studied in [Z] and for compact Lie groups with biinvariant metrics in [DA-Z].

Remark 1.3. The following result should be true for all compact irreducible symmetric spaces M, up to diffeomorphism: If G is a compact connected Lie group acting transitively and effectively on M, then G is conjugate to a subgroup of $\mathrm{Isom}_0(M)$. This would imply that Theorem 1.2 classifies *all* homogeneous Einstein metrics on compact irreducible symmetric spaces of rank > 1, not isometric to a Lie group. Such a result is well known for rank 1 symmetric spaces, but does not seem to be known for all symmetric spaces of rank > 1. Partial results can be found in [O2], [O3], [S], [T].

For example, in [O3, Thm. 1] Oniščik showed that if M is diffeomorphic to $G_{2k}(\mathbb{R}^n)$ for n even, $n > 5$, $1 < k < \frac{n-2}{2}$, or for n odd, $2 < k < \frac{n-3}{2}$, and if a compact connected Lie group G acts transitively and effectively on M, then G is conjugate to $\mathrm{SO}(n)$ with the standard action. Tsukada proved in [T] that if M is diffeomorphic to $G_{2k+1}(\mathbb{R}^{2n})$, and G is compact, connected, and simple, then if G acts transitively and effectively, the action of G is conjugate to the standard action.

2. Preliminaries

A Riemannian manifold (M, g) is Einstein if $\mathrm{Ric}(X, Y) = \lambda g(X, Y)$ for some constant λ, for all vector fields X, Y. We say a Riemannian manifold (M, g) is a symmetric space if for all $p \in M$ there exists an isometry $\sigma_p : M \to M$ such that $\sigma_p(p) = p$ and $(d\sigma_p)_p = -\mathrm{Id}$. Symmetric spaces make up a class of manifolds which includes spheres, projective spaces, and Grassmannians; their geometry is well understood. In fact, every symmetric space is homogeneous.

A manifold M is defined to be G-homogeneous if we have a Riemannian metric g and a closed subgroup $G \subset \mathrm{Isom}(M, g)$ such that for any p and $q \in M$, there exists a $g \in G$ with $g(p) = q$. We write $H_p = \{g \in G \mid g(p) = p\}$, called the isotropy subgroup corresponding to p. Notice H_p is compact, since $H_p \subset \mathrm{O(T_pM)}$. Via the map $g \mapsto g(p)$ we identify the two manifolds G/H_p and M. Any two isotropy subgroups H_p and H_q are conjugate: if $q = g(p)$, then $g^{-1} H_q g = H_p$, hence we will usually suppress the point p.

Given a homogeneous manifold G/H, where G is compact and H is closed, what metrics can we put on G/H so that G acts by isometries?

G/H	G'/H'	$\dim \mathcal{M}_{G'}$	no. Einstein
$\mathrm{SO}(2n)/\mathrm{U}(n)$	$\mathrm{SO}(2n-1)/\mathrm{U}(n-1)$	1	2
$\mathrm{SU}(2n)/\mathrm{Sp}(n)$	$\mathrm{SU}(2n-1)/\mathrm{Sp}(n-1)$	2	1
$\mathrm{SO}(7)/\mathrm{SO}(2)\,\mathrm{SO}(5)$	$\mathrm{G}_2/\mathrm{U}(2)$	2	3
$\mathrm{SO}(8)/\mathrm{SO}(3)\,\mathrm{SO}(5)$	$\mathrm{Spin}(7)/\mathrm{SO}(4)$	2	3

Just as a left-invariant metric on a Lie group is determined by any inner product on its Lie algebra, a G-invariant metric on G/H is determined by an inner product on $\mathfrak{g}/\mathfrak{h} \cong T_{[H]}(G/H)$, with the additional requirement that the inner product be $\mathrm{Ad}(H)$-invariant. We identify the quotient $\mathfrak{g}/\mathfrak{h}$ with an $\mathrm{Ad}(H)$-invariant complement $\mathfrak{p}$ to $\mathfrak{h}$ in $\mathfrak{g}$; compactness of H guarantees such a $\mathfrak{p}$ exists. If $\mathfrak{g}$ is semisimple then the Killing form is $\mathrm{Ad}(H)$-invariant and we can use it to define $\mathfrak{p} = \mathfrak{h}^{\perp}$. We want to consider all $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{p}$.

A homogeneous space $M = G/H$ is said to be isotropy irreducible if the isotropy action, denoted $\chi : H \to \mathrm{GL}(T_pM)$, or equivalently $\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{p})$, is an irreducible representation of H. When this is the case, the G-invariant metric on G/H is unique, up to scaling, and it is Einstein. When G/H is a symmetric space with G simple and $G = \mathrm{Isom}_0(G/H)$, then G/H is an irreducible symmetric space. (In fact, the only irreducible symmetric space with G not simple is $(K \times K)/\Delta K$, for K a compact simple Lie group, and $(K \times K)/\Delta K$ is isometric to K with a biinvariant metric.)

In 1962 A.L. Oniščik classified all simple compact Lie algebras $\mathfrak{g}$ with subalgebras $\mathfrak{g}'$ and $\mathfrak{g}''$, such that $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}''$. In terms of transitive group actions, let G be the simply connected compact Lie group corresponding to $\mathfrak{g}$ and let G', G'' be subgroups corresponding to $\mathfrak{g}'$, $\mathfrak{g}''$, respectively, then $G/G' = G''/(G' \cap G'')$ and $G/G'' = G'/(G' \cap G'')$. When G/G' or G/G'' is a symmetric space, Oniščik's list tells us when a subgroup of G still acts transitively. Here are the symmetric spaces on his list [O1]: (See the appendix of this paper for the non-symmetric homogeneous spaces on his list.)

$$
\begin{aligned}
&\mathrm{SO}(2n)/\mathrm{SO}(2n-1) &&= \mathrm{U}(n)/\mathrm{U}(n-1) &&= S^{2n-1}\\
&\mathrm{SO}(2n)/\mathrm{SO}(2n-1) &&= \mathrm{SU}(n)/\mathrm{SU}(n-1) &&= S^{2n-1}\\
&\mathrm{SO}(4n)/\mathrm{SO}(4n-1) &&= \mathrm{Sp}(n)/\mathrm{Sp}(n-1) &&= S^{4n-1}\\
&\mathrm{SO}(4n)/\mathrm{SO}(4n-1) &&= \mathrm{Sp}(n)\,\mathrm{U}(1)/\mathrm{Sp}(n-1)\,\mathrm{U}(1) &&= S^{4n-1}\\
&\mathrm{SO}(4n)/\mathrm{SO}(4n-1) &&= \mathrm{Sp}(n)\,\mathrm{Sp}(1)/\mathrm{Sp}(n-1)\,\mathrm{Sp}(1) &&= S^{4n-1}\\
&\mathrm{SO}(7)/\mathrm{SO}(6) &&= \mathrm{G}_2/\mathrm{SU}(3) &&= S^{6}\\
&\mathrm{SO}(8)/\mathrm{SO}(7) &&= \mathrm{Spin}(7)/\mathrm{G}_2 &&= S^{7}\\
&\mathrm{SO}(16)/\mathrm{SO}(15) &&= \mathrm{Spin}(9)/\mathrm{Spin}(7) &&= S^{15}\\
&\mathrm{SU}(2n)/\mathrm{U}(2n-1) &&= \mathrm{Sp}(n)/\mathrm{Sp}(n-1)\,\mathrm{U}(1) &&= \mathbb{C}P^{2n}\\
&\mathrm{SO}(2n)/\mathrm{U}(n) &&= \mathrm{SO}(2n-1)/\mathrm{U}(n-1) &&= \text{spec. orth. cx. str. on } \mathbb{R}^{2n}\\
&\mathrm{SU}(2n)/\mathrm{Sp}(n) &&= \mathrm{SU}(2n-1)/\mathrm{Sp}(n-1) &&= \text{spec. orth. quat. str. on } \mathbb{C}^{2n}\\
&\mathrm{SO}(7)/\mathrm{SO}(2)\,\mathrm{SO}(5) &&= \mathrm{G}_2/\mathrm{U}(2) &&= G_2^+(\mathbb{R}^7)\\
&\mathrm{SO}(8)/\mathrm{SO}(3)\,\mathrm{SO}(5) &&= \mathrm{Spin}(7)/\mathrm{SO}(4) &&= G_3^+(\mathbb{R}^8).
\end{aligned}
$$

Each of these symmetric spaces in the left-hand presentation is an irreducible symmetric space. Up to scaling, each has exactly one Einstein metric, the symmetric metric, homogeneous with respect to the left-hand presentation. However, with respect to the right-hand presentation, only the sixth and seventh symmetric spaces are isotropy irreducible.

The first nine examples are discussed in [Z]. In this paper we consider the last four examples in the table. Of these, the first is originally described in [W-Z]. On $\mathrm{SU}(2n)/\mathrm{Sp}(n)$, the only homogeneous Einstein metric is the original one. However, the last two spaces each carry two new Einstein metrics, homogeneous with respect to the right-hand presentation.

For any homogeneous space $M = G/H$, with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ on the Lie algebra level, we parametrize the space of G-invariant metrics on M by decomposing $\mathfrak{p}$ into its $\mathrm{Ad}(H)$-irreducible subspaces, $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots \oplus \mathfrak{p}_k$. If the $\mathfrak{p}_i$'s are pairwise inequivalent representations, a G-homogeneous metric is determined by an inner product on $\mathfrak{p}$ of the form $\langle\ ,\ \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \perp \cdots \perp x_k Q|_{\mathfrak{p}_k}$, for Q an $\mathrm{Ad}(H)$-invariant inner product and $x_i > 0$ for all i. If $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are equivalent for some i and j, then $\langle \mathfrak{p}_i, \mathfrak{p}_j \rangle$ does not necessarily vanish; however, in each of the examples in this paper we have $\mathfrak{p}_i \not\simeq \mathfrak{p}_j$ for all $i \neq j$.

Asume G/H is compact, and let $S(g)$ denote the scalar curvature of g. Einstein metrics are the critical points of the total scalar curvature functional

$$T(g) = \int_M S(g) d\mathrm{vol}_g$$

on the space $\mathcal{M}$ of Riemannian metrics of volume one [Ber], [H]. Let $\mathcal{M}_G$ denote the set of all G-invariant metrics of volume one on M. Notice that on $\mathcal{M}_G$, $T(g) \simeq S(g)$. Furthermore, critical points of $T|_{\mathcal{M}_G}$ are precisely G-invariant Einstein metrics of volume one [Bes, p.121].

If for our homogeneous space G/H, every homogeneous metric is diagonal, i.e., $\langle\ ,\ \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \perp \cdots \perp x_k Q|_{\mathfrak{p}_k}$, with $x_i > 0$ for all i, then we use equation (1.3) for the scalar curvature given in [W-Z].

$$S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \binom{k}{i\,j} \frac{x_k}{x_i x_j}.$$

In their formula, for each i, $-B|_{\mathfrak{p}_i} = b_i Q|_{\mathfrak{p}_i}$, where B denotes the Killing form, and $d_i = \dim(\mathfrak{p}_i)$; the triple $\binom{k}{i\,j} = \sum Q([X_\alpha, X_\beta], X_\gamma)^2$, summed over $\{X_\alpha\}$, $\{X_\beta\}$, and $\{X_\gamma\}$: Q-orthonormal bases for $\mathfrak{p}_i$, $\mathfrak{p}_j$, and $\mathfrak{p}_k$, respectively. Notice $\binom{k}{i\,j}$ is symmetric in all three entries.

We are now ready to prove the theorem.

3. SO(2N)/U(N)

We begin with the symmetric space $\mathrm{SO}(2n)/\mathrm{U}(n)$. Consider the space of orthogonal complex structures on $\mathbb{R}^{2n}$, and let M_0 be the connected component containing J_0, the complex structure represented by $\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$ with respect to the standard basis on $\mathbb{R}^{2n}$. We will show that $M_0 \cong \mathrm{SO}(2n)/\mathrm{U}(n)$.

Let $J \in M_0$. Since J is an orthogonal complex structure $\|Jv\| = \|v\|$ and $J^2 = -\mathrm{Id}$. We construct an orthonormal basis $\{v_\alpha\}$ for $\mathbb{R}^{2n}$ such that for $1 \leq i \leq n$, $Jv_i = v_{n+i}$ and $Jv_{n+i} = -Jv_i$. Let $v_1 = e_1$, and let $v_{n+1} = Jv_1$. We have $\langle v_1, Jv_1\rangle = \langle Jv_1, J^2v_1\rangle = -\langle v_1, Jv_1\rangle$, hence $\{v_1, v_{n+1}\}$ is an orthonormal basis for a J-invariant subspace. Let v_2 be any unit vector in $\mathrm{span}\{v_1, v_{n+1}\}^\perp$, and let $v_{n+2} = Jv_2$. Continue up to v_n, v_{2n}. With respect to the basis $\{v_\alpha\}$, $J = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$. Let P be the change of basis transformation from the standard basis to $\{v_\alpha\}$, then $J = PJ_0P^{-1}$. The hypothesis that J be in the connected component containing J_0 corresponds exactly to the fact that P must be in $\mathrm{SO}(2n)$. Via conjugation, $\mathrm{SO}(2n)$ acts transitively on M_0.

The isotropy subgroup of J_0 is the set of all $P \in \mathrm{SO}(2n)$ such that $PJ_0 = J_0P$. If we identify $\mathbb{R}^n \oplus \mathbb{R}^n \cong \mathbb{C}^n$ via $(u, v) \mapsto u + iv$ then J_0 is multiplication by i and hence $PJ_0 = J_0P$ implies $P \in \mathrm{SO}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n)$. Thus $M_0 \cong \mathrm{SO}(2n)/\mathrm{U}(n)$, where we have $\mathrm{U}(n)$ embedded in $\mathrm{SO}(2n)$ in the following way: $A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. It is well known that $\mathrm{SO}(2n)/\mathrm{U}(n)$ is an irreducible symmetric space [W, p.287]. Let μ_n denote the standard complex n-dimensional representation of $\mathrm{U}(n)$; the isotropy representation of $\mathrm{U}(n)$ is $[\wedge^2\mu_n]_{\mathbb{R}}$. Since $\wedge^2\mu_n$ is unitary, $[\wedge^2\mu_n]_{\mathbb{R}}$ is the irreducible real representation whose complexification is isomorphic to the direct sum of $\wedge^2\mu_n$ and its dual.

If we look at the low dimensional examples, for $n \leq 4$, we find $\mathrm{SO}(4)/\mathrm{U}(2) = S^2$, $\mathrm{SO}(6)/\mathrm{U}(3) = \mathbb{C}P^3$, and $\mathrm{SO}(8)/\mathrm{U}(4) = G_2^+(\mathbb{R}^8)$. For $n \geq 4$ the rank of the symmetric space is greater than one.

Notice that in our description above we let $v_1 = e_1$, so the subgroup $\mathrm{SO}(2n-1) \subset \mathrm{SO}(2n)$ fixing $\mathrm{span}\{e_1\}$ also acts transitively on M_0. The isotropy subgroup of $\mathrm{SO}(2n-1)$ corresponding to J_0 is $\mathrm{U}(n-1) \subset \mathrm{SO}(2n-2)$, where $\mathrm{SO}(2n-2)$ is the subgroup fixing e_1 and e_{n+1}.

On the Lie algebra we have, for $X, Y \in \mathfrak{gl}(n-1,\mathbb{R})$,

$$\mathfrak{u}(n-1) \cong \left\{ \begin{pmatrix} X & 0 & -Y \\ 0 & 0 & 0 \\ Y & 0 & X \end{pmatrix} \middle| X=-X^t,\ Y=Y^t \right\} \subset \mathfrak{so}(2n-1);$$

$$\text{therefore, } \mathfrak{p} \cong \left\{ \begin{pmatrix} X & v & Y \\ -v^t & 0 & -w^t \\ Y & w & -X \end{pmatrix} \middle| X=-X^t,\ Y=-Y^t,\ v,w\in\mathbb{R}^{n-1} \right\}.$$

We find that $\mathfrak{p}$ decomposes into the sum of two irreducible representations. In fact $\mathrm{U}(n-1) \subset \mathrm{SO}(2n-2) \subset \mathrm{SO}(2n-1)$ gives rise to the following fibration:

$$\mathrm{SO}(2n-2)/\,\mathrm{U}(n-1) \to \mathrm{SO}(2n-1)/\,\mathrm{U}(n-1) \to \mathrm{SO}(2n-1)/\,\mathrm{SO}(2n-2) \cong S^{2n-2}.$$

Both base and fibre are irreducible symmetric spaces. Let $\mathfrak{p}_1$ denote the Ad $\mathrm{SO}(2n-2)$-invariant complement to $\mathfrak{so}(2n-2)$ in $\mathfrak{so}(2n-1)$; in our fibration $\mathfrak{p}_1$ corresponds to the tangent space of the base. The representation of $\mathrm{U}(n-1)$ on $\mathfrak{p}_1$ is the restriction of the standard representation of $\mathrm{SO}(2n-2)$ on $\mathfrak{p}_1 \cong \mathbb{R}^{2n-2}$ to $\mathrm{U}(n-1)$, which is $[\mu_{n-1}]_{\mathbb{R}}$, again irreducible. Let $\mathfrak{p}_2$ denote the Ad $\mathrm{U}(n-1)$-invariant complement to $\mathfrak{u}(n-1)$ in $\mathfrak{so}(2n-2)$; $\mathfrak{p}_2$ corresponds to the tangent space of the fibre. This representation of $\mathrm{U}(n-1)$ on $\mathfrak{p}_2$ is the irreducible isotropy representation of the fibre symmetric space, $[\wedge^2\mu_{n-1}]_{\mathbb{R}}$ [W, p.287]. We have $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

The dimensions of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are $2(n-1)$ and $(n-1)(n-2)$, respectively. We see that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are clearly inequivalent representations of $\mathrm{U}(n-1)$ for $n \neq 4$, and for $n=4$, while μ_3 and $\wedge^2\mu_3$ are equivalent representations on $\mathrm{SU}(3)$, they are inequivalent on the center of $\mathrm{U}(3)$. We apply Schur's lemma to know that $\langle \mathfrak{p}_1, \mathfrak{p}_2\rangle$ and $\mathrm{Ric}(\mathfrak{p}_1,\mathfrak{p}_2)$ must vanish. Thus any $\mathrm{SO}(2n-1)$-homogeneous metric on M_0 must be of the form $\langle\ ,\ \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2}$, where $Q(X,Y) = -\frac{1}{2}\operatorname{tr}(XY)$ and $x_1, x_2 > 0$. We express the scalar curvature in terms of x_1 and x_2 using [W-Z, (1.3)]. The equation is

$$S = \frac{1}{2}\sum_i \frac{d_i b_i}{x_i} - \frac{1}{4}\sum_{i,j,k} \binom{k}{i\,j} \frac{x_k}{x_i x_j}.$$

Since for $\mathfrak{so}(k)$, $-B(X,Y) = (k-2)\operatorname{tr}(XY)$, we have $b_1 = b_2 = 2(2n-3)$. From the fibration it follows that $[\mathfrak{p}_1,\mathfrak{p}_1] \subset \mathfrak{u}(n-1)\oplus\mathfrak{p}_2$, $[\mathfrak{p}_2,\mathfrak{p}_2]\subset\mathfrak{u}(n-1)$, hence the only nonzero triple (up to rearrangements) is $\binom{2}{11}$.

Let E_{ij} denote the skew-symmetric matrix in $\mathfrak{so}(2n-1)$ with 1 in the ij^{th} entry and -1 in the ji^{th} entry, and zeros everywhere else.

$$\mathfrak{p}_1 = \operatorname{span}\{-E_{in},\ E_{n,n+i} \mid 1 \le i \le n-1\}$$
$$\mathfrak{p}_2 = \operatorname{span}\{E_{ij} - E_{n+i,n+j},\ E_{i,n+j} - E_{j,n+i} \mid 1 \le i < j \le n-1\}.$$

We find $\binom{2}{11} = 2(n-1)(n-2)$, and then we substitute into the scalar curvature equation to find S. Let $\tilde{S}$ be the equation for S with the boundary constraint that volume $= 1$:

$$S = \frac{2(n-1)(2n-3)}{x_1} + \frac{2(n-1)(n-2)^2}{x_2} - \frac{(n-1)(n-2)x_2}{2x_1^2}$$
$$\tilde{S} = S - \lambda({x_1}^{2(n-1)}{x_2}^{(n-1)(n-2)} - 1).$$

We find the partial derivatives of $\tilde{S}$:

$$\frac{\partial \tilde{S}}{\partial x_1} = \frac{-2(n-1)(2n-3)}{x_1^2} + \frac{(n-1)(n-2)x_2}{x_1^3} - 2(n-1)\lambda x_1^{2n-3}x_2^{(n-1)(n-2)}$$
$$\frac{\partial \tilde{S}}{\partial x_2} = \frac{-2(n-1)(n-2)^2}{x_2^2} - \frac{(n-1)(n-2)}{2x_1^2} - (n-1)(n-2)\lambda x_1^{2(n-1)}x_2^{(n-1)(n-2)-1}.$$

Setting both equations equal to zero is equivalent to the following equation:

$$\frac{n-1}{2}x_2 + 2(n-2)\frac{x_1^2}{x_2} = (2n-3)x_1.$$

We find that the solutions are $x_1 = \frac{1}{2}x_2$, and $x_1 = \frac{(n-1)}{2(n-2)}x_2$. The second solution is a (non-symmetric) $\mathrm{SO}(2n-1)$-invariant Einstein metric, discovered earlier in [W-Z, §3,Ex.6]. The first solution is the $\mathrm{SO}(2n)$-invariant symmetric metric, but this is not obvious until we see how to compare them.

Let $\tilde{\mathfrak{p}}$ denote the Q-orthogonal complement to $\mathfrak{u}(n)$ in $\mathfrak{so}(2n)$:

$$\tilde{\mathfrak{p}} = \mathrm{span}\{E_{ij} - E_{n+i,n+j}, E_{i,n+j} - E_{j,n+i} \mid 1 \le i < j \le n\}.$$

We must project $\mathfrak{p}$ to $\tilde{\mathfrak{p}}$. We take a basis element of $\mathfrak{p}_1$: $-E_{in}$. Under the embedding of $\mathfrak{so}(2n-1)$ in $\mathfrak{so}(2n)$, $-E_{in} \mapsto -E_{i+1,n+1}$. Next we write $-E_{i+1,n+1}$ as the sum of an element in $\mathfrak{u}(n)$ and an element in $\tilde{\mathfrak{p}}$:

$$-E_{i+1,n+1} = -\frac{1}{2}(E_{i+1,n+1} + E_{1,n+i+1}) - \frac{1}{2}(E_{i+1,n+1} - E_{1,n+i+1}).$$

This shows that an element of norm $= \sqrt{x_1}$ is sent to an element of norm $= \frac{1}{\sqrt{2}}$. A basis element of $\mathfrak{p}_2$ is $\frac{1}{\sqrt{2}}(E_{ij} - E_{n+i,n+j})$, which the embedding sends to $\frac{1}{\sqrt{2}}(E_{i+1,j+1} - E_{n+i+1,n+j+1})$, already in $\tilde{\mathfrak{p}}$; hence an element of norm $= \sqrt{x_2}$ is sent to an element of norm $= 1$. The symmetric metric on $\mathrm{SO}(2n)/\mathrm{U}(n)$ is given by the restriction of Q to $\tilde{\mathfrak{p}}$. Hence it corresponds to $\frac{2}{1} = \frac{x_2}{x_1}$, i.e., $x_1 = \frac{1}{2}x_2$.

To see that these metrics are distinct, we can compare the scale-invariant product $(S)^{\frac{d}{2}}(V)^{\frac{1}{2}}$, where S is the scalar curvature, V is the volume, and d is the dimension of M. The first metric has $S = \frac{2n(n-1)^2}{x_2}$ and $V = x_1^{2(n-1)}x_2^{(n-1)(n-2)}$, so that

$$(S)^{\frac{n(n-1)}{2}}(V)^{\frac{1}{2}} = 2^{\frac{(n-1)(n-2)}{2}}(n(n-1)^2)^{\frac{n(n-1)}{2}}.$$

The second metric has $S = \frac{2n(n-2)(n^2-n-1)}{(n-1)x_2}$, and $V = (\frac{n-1}{2(n-2)})^{2(n-1)} x_2^{n(n-1)}$, hence

$$(S)^{\frac{n(n-1)}{2}} (V)^{\frac{1}{2}} = \left(\frac{2n(n-2)(n^2-n-1)}{n-1} \right)^{\frac{n(n-1)}{2}} \left(\frac{n-1}{2(n-2)} \right)^{n-1}.$$

4. SU(2n)/Sp(n)

Our next example is the symmetric space $M = \mathrm{SU}(2n)/\mathrm{Sp}(n)$, an analogue of the previous example. This is the set of special orthogonal quaternionic structures on $\mathbb{C}^{2n}$. We identify $\mathbb{R}^{4n} \cong \mathbb{C}^{2n}$ via a fixed orthogonal complex structure I on $\mathbb{R}^{4n}$. An orthogonal quaternionic structure on $\mathbb{C}^{2n}$ is given by $J \in \mathrm{SO}(4n)$ such that $J^2 = -\mathrm{Id}$ and $IJ = -JI$. As a map from $\mathbb{C}^{2n}$ to itself, J is complex anti-linear, i.e., $J(\lambda v) = \bar{\lambda} J(v)$. We show that the set of all orthogonal quaternionic structures can be written homogeneously as $\mathrm{U}(2n)/\mathrm{Sp}(n)$, and we call the submanifold $M = \mathrm{SU}(2n)/\mathrm{Sp}(n)$ the set of special orthogonal quaternionic structures. We first observe that if $I = \begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 \end{pmatrix}$ and if we identify $\mathbb{C}^n \oplus \mathbb{C}^n = \mathbb{H}$ via $(u, v) \mapsto u + jv$, then multiplication by j on $\mathbb{C}^{2n} = \mathbb{R}^{4n}$ becomes

$$J_0 = \begin{pmatrix} 0 & \mathrm{Id}_n & 0 & 0 \\ -\mathrm{Id}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id}_n \\ 0 & 0 & \mathrm{Id}_n & 0 \end{pmatrix},$$

which is the standard orthogonal quaternionic structure. We want to show that $\mathrm{U}(2n)$ acts transitively on $\{J \in \mathrm{SO}(4n) \mid J^2 = -\mathrm{Id} \text{ and } IJ = -JI\}$, and the isotropy subgroup is $\mathrm{Sp}(n)$. Since $\mathrm{U}(2n) = \mathrm{GL}(2n, \mathbb{C}) \cap \mathrm{SO}(4n)$, A is unitary when $A \in \mathrm{SO}(4n)$ and $AI = IA$, and for A unitary, AJA^{-1} is a quaternionic structure if J is: $(AJA^{-1})(AJA^{-1}) = -\mathrm{Id}$ and $(AJA^{-1})I = AJIA^{-1} = -AIJA^{-1} = -I(AJA^{-1})$. Furthermore, IJ is also a quaternionic structure:

$$(IJ)(IJ) = -I(IJ)J = -(-\mathrm{Id})^2, \text{ and } (IJ)I = -I(IJ).$$

Given an orthogonal quaternionic structure J, we construct a unitary basis of $\mathbb{C}^{2n}$ in which J is represented by the matrix J_0. Let $v_1 = e_1$, the first element of the standard basis. Let $v_{n+1} = Jv_1$, $v_{2n+1} = Iv_1$, and $v_{3n+1} = IJv_1$. Clearly $\{v_1, Jv_1, Iv_1, IJv_1\}$ is an orthonormal basis for a 4-plane invariant under I, J, and IJ. Choose v_2 to be any unit vector in the orthogonal complement, and repeat the process above, continuing up to v_n, v_{2n}, v_{3n}, v_{4n}. Notice this is a unitary basis for $\mathbb{R}^{4n}$, since $v_{2n+i} = Iv_i$ for all $1 \le i \le 2n$.

The isotropy subgroup of $\mathrm{U}(2n)$ corresponding to J_0 is all $A \in \mathrm{U}(2n)$ such that $AJ_0 = J_0A$. I.e., A commutes with I, J, and IJ: A is quaternionic linear. We embed $\mathrm{Sp}(n) \subset \mathrm{U}(2n)$ via $A + jB \mapsto \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}$. The image of this embedding is contained in $\mathrm{SU}(2n)$, and we now restrict ourselves to the orbit of $\mathrm{SU}(2n)$, which is the symmetric space of special orthogonal quaternionic structures on $\mathbb{R}^{4n}$, or $\mathrm{SU}(2n)/\mathrm{Sp}(n)$. This symmetric space is irreducible; up to scaling the symmetric metric is the unique $\mathrm{SU}(2n)$-invariant metric, and it is Einstein. Notice that when $n=2$, $\mathrm{SU}(4)/\mathrm{Sp}(2)=S^5$; for $n\geq 3$ the rank of the symmetric space is greater than one.

Let $\tilde{\mathfrak{p}}$ be the orthogonal complement to $\mathfrak{sp}(n)$ in $\mathfrak{su}(2n)$ with respect to the inner product $Q(X,Y)=-\frac{1}{2}\operatorname{tr}(XY)$.

$$\text{We have } \mathfrak{su}(2n)=\left\{\begin{pmatrix} X & Y\\ -\overline{Y}^t & Z\end{pmatrix} \;\middle|\; X,Z\in\mathfrak{u}(n),\ Y\in\mathfrak{gl}(n,\mathbb{C}),\ \operatorname{tr}Z=-\operatorname{tr}X\right\}$$

$$\text{and } \mathfrak{sp}(n)\cong\left\{\begin{pmatrix} X & -\overline{Y}\\ Y & \overline{X}\end{pmatrix} \;\middle|\; X\in\mathfrak{u}(n),\ Y=Y^t\in\mathfrak{gl}(n,\mathbb{C})\right\},$$

$$\text{hence } \tilde{\mathfrak{p}}=\left\{\begin{pmatrix} X & \overline{Y}\\ Y & -\overline{X}\end{pmatrix} \;\middle|\; X\in\mathfrak{su}(n),\ Y=Y^t\in\mathfrak{gl}(n,\mathbb{C}),\ \text{and } \operatorname{tr}X=0\right\}.$$

The isotropy representation of $\mathrm{Sp}(n)$ on $\tilde{\mathfrak{p}}$ is $[\wedge^2\nu_n-\mathrm{Id}]_{\mathbb{R}}$, where ν_n is the standard representation of $\mathrm{Sp}(n)$ on $\mathbb{H}^n\cong\mathbb{C}^{2n}$. (The representation $\wedge^2\nu_n$ is the sum of a complex $(2n+1)(n-1)$-dimensional irreducible representation and a one-dimensional trivial representation. We denote by $\wedge^2\nu_n-\mathrm{Id}$ the non-trivial summand.) We write $[\wedge^2\nu_n-\mathrm{Id}]_{\mathbb{R}}$ for the real representation whose complexification is $\wedge^2\nu_n-\mathrm{Id}$.

The subgroup $\mathrm{SU}(2n-1)\subset\mathrm{SU}(2n)$ fixing e_1 acts transitively on M, just as in the previous example. The isotropy subgroup of $\mathrm{SU}(2n-1)$ corresponding to J_0 is

$$H=\left\{\begin{pmatrix} A & 0 & -\overline{B}\\ 0 & 1 & 0\\ B & 0 & \overline{A}\end{pmatrix}\in\mathrm{SU}(2n-1) \;\middle|\; A+jB\in\mathrm{Sp}(n-1,\mathbb{C})\right\}$$

$=\mathrm{Sp}(n-1)\subset\mathrm{SU}(2n-2)$ fixing e_{2n+1}. In $\mathfrak{su}(2n-1)$, for $X,Y\in\mathfrak{gl}(n-1,\mathbb{C})$,

$$\mathfrak{h}=\mathfrak{sp}(n-1)=\left\{\begin{pmatrix} X & 0 & -\overline{Y}\\ 0 & 0 & 0\\ Y & 0 & \overline{X}\end{pmatrix} \;\middle|\; X\in\mathfrak{u}(n-1),\ Y=Y^t\right\}.$$

We denote by $\mathfrak{p}$ the Q-orthogonal complement to $\mathfrak{sp}(n-1)$ in $\mathfrak{su}(2n-1)$:

$$\mathfrak{p}=\left\{\begin{pmatrix} X & -\bar{u}^t & \overline{Y}\\ u & z & v^t\\ Y & -\bar{v} & -\overline{X}\end{pmatrix} \;\middle|\; X\in\mathfrak{u}(n-1),\ Y=-Y^t,\ u,v\in\mathbb{C}^{n-1},\ z=-2\operatorname{tr}X\right\}.$$

We have the following fibration of our symmetric space, which tells us how to decompose $\mathfrak{p}$ into irreducible $\mathrm{Ad}\,\mathrm{Sp}(n-1)$-invariant subrepresentations:

$$\mathrm{SU}(2n-2)/\mathrm{Sp}(n-1)\to\mathrm{SU}(2n-1)/\mathrm{Sp}(n-1)\to\mathrm{SU}(2n-1)/\mathrm{SU}(2n-2)=S^{4n-3}.$$

From the fibration we see that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}'$, where $\mathfrak{p}_1$ is tangent to the fibre and the Ad $\mathrm{Sp}(n-1)$ action on $\mathfrak{p}_1$ is $[\wedge^2 \nu_{n-1} - \mathrm{Id}]_{\mathbb{R}}$, and $\dim(\mathfrak{p}_1) = (2n-1)(n-2)$. The subspace $\mathfrak{p}'$ is tangent to the base, and Ad $\mathrm{SU}(2n-2)$ acts on $\mathfrak{p}'$ by $[\mu_{2n-2}]_{\mathbb{R}} \oplus \mathrm{Id}$, which when restricted to $\mathrm{Sp}(n-1)$ is $[\nu_{n-1}]_{\mathbb{R}} \oplus \mathrm{Id}$. That is, $\mathfrak{p}' = \mathfrak{p}_2 \oplus \mathfrak{p}_3$; $\dim(\mathfrak{p}_2) = 4(n-1)$ and $\dim(\mathfrak{p}_3) = 1$ (the Ad $\mathrm{Sp}(n-1)$ action on $\mathfrak{p}_3$ is trivial). The set of elements listed below gives a Q-orthogonal basis for $\mathfrak{p}$. We write E_{ij} for the skew-symmetric $(2n-1) \times (2n-1)$ matrix with 1 in the ij^{th} entry and -1 in the ji^{th} entry, and zeros elsewhere. We denote by F_{ij} the symmetric $(2n-1) \times (2n-1)$ matrix with 1 in both the ij^{th} and ji^{th} entries.

$$\begin{aligned}
\mathfrak{p}_1 &= \mathrm{span}\{(E_{kl} - E_{n+k,n+l}), i(F_{kl} + F_{n+k,n+l}), \\
&\qquad (E_{k,n+l} - E_{l,n+k}), i(F_{k,n+l} - F_{l,n+k}) \mid 1 \le k < l \le n-1\} \\
&\quad \oplus \mathrm{span}\{i(F_{kk} - F_{n-1,n-1} + F_{n+k,n+k} - F_{2n-1,2n-1}) \mid 1 \le k < n-1\} \\
\mathfrak{p}_2 &= \mathrm{span}\{E_{nk}, iF_{nk}, E_{n,n+k}, iF_{n,n+k} \mid 1 \le k \le n-1\} \\
\mathfrak{p}_3 &= \mathrm{span}\{\mathrm{diag}(\eta, \ldots, \eta, -2(n-1)\eta, \eta, \ldots, \eta)\}, \text{where } \eta = \frac{i}{\sqrt{(2n-1)(n-1)}}.
\end{aligned}$$

Since $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$ are inequivalent irreducible representations of $\mathrm{Sp}(n-1)$, any $\mathrm{SU}(2n-1)$-invariant metric on M must take the form

$$\langle\ ,\ \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \perp x_3 Q|_{\mathfrak{p}_3}, \text{ with } x_i > 0 \text{ for } i = 1,2,3.$$

To find all $\mathrm{SU}(2n-1)$-invariant Einstein metrics on M, we solve for the critical points of the scalar curvature equation in terms of x_1, x_2, and x_3 (restricting to unit volume). As in the previous case we use the formula given in [W-Z, (1.3)]:

$$S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \binom{k}{i\,j} \frac{x_k}{x_i x_j}.$$

In our example we have $b_i = 4(2n-1)$ for all i, since for $\mathfrak{su}(k)$, $-B(X,Y) = 2k\,\mathrm{tr}(XY)$. And $d_1 = (2n-1)(n-1)$, $d_2 = 4(n-1)$, $d_3 = 1$. We find that

$$\begin{aligned}
&[\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{sp}(n-1), && [\mathfrak{p}_1, \mathfrak{p}_2] \subset \mathfrak{p}_2, \\
&[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{sp}(n-1) + \mathfrak{p}_1 + \mathfrak{p}_3, && [\mathfrak{p}_1, \mathfrak{p}_3] = 0, \\
&[\mathfrak{p}_3, \mathfrak{p}_3] = 0, && [\mathfrak{p}_2, \mathfrak{p}_3] \subset \mathfrak{p}_2.
\end{aligned}$$

Therefore $\binom{1}{2\,2}$, $\binom{3}{2\,2}$ (and rearrangements) are the only nonzero triples. We compute $\binom{1}{2\,2} = 4(2n-1)(n-2)$ and $\binom{3}{2\,2} = 4(2n-1)$. We now have the equation for the scalar curvature of M in terms of x_1, x_2, and x_3.

$$S = (2n-1) \left(\frac{4(n-1)(n-2)}{x_1} + \frac{8(n-1)}{x_2} - \frac{x_3}{x_2^2} - \frac{(n-2)x_1}{x_2^2} \right).$$

We normalize for volume 1 metrics: $\tilde{S} = S - \lambda(x_1^{d_1} x_2^{d_2} x_3 - 1)$.

$$\frac{\partial \tilde{S}}{\partial x_1} = -\frac{4(2n-1)(n-1)(n-2)}{x_1^2} - \frac{(2n-1)(n-2)}{x_2^2} - (2n-1)(n-2)\lambda x_1^{d_1-1} x_2^{d_2} x_3$$
$$\frac{\partial \tilde{S}}{\partial x_2} = \frac{8(2n-1)(n-1)}{x_2^2} + \frac{2(2n-1)((n-2)x_1 + x_3)}{x_2^3} - 4(n-1)\lambda x_1^{d_1} x_2^{d_2-1} x_3$$
$$\frac{\partial \tilde{S}}{\partial x_3} = -\frac{(2n-1)}{x_2^2} - \lambda x_1^{d_1} x_2^{d_2}.$$

Setting $\frac{\partial \tilde{S}}{\partial x_1} = \frac{\partial \tilde{S}}{\partial x_2} = \frac{\partial \tilde{S}}{\partial x_3} = 0$ simultaneously is equivalent to

$$4(n-1)x_2^2 + x_1^2 = (2n-1)x_1 x_3 = 2(2n-1)x_1 x_2 - \frac{2n-1}{2n-2}((n-2)x_1^2 + x_1 x_3).$$

There is only one solution; it is $x_2 = \frac{1}{2}x_1$ and $x_3 = \frac{n}{2n-1}x_1$, unique up to scaling. This is not a new metric, rather it is the symmetric metric, which we knew must solve our equations. (It is $\mathrm{SU}(2n)$-invariant, hence $\mathrm{SU}(2n-1)$-invariant.) Thus the only homogeneous Einstein metric on $M = \mathrm{SU}(2n-1)/\operatorname{Sp}(n-1) \cong \mathrm{SU}(2n)/\operatorname{Sp}(n)$ is the symmetric metric.

5. $G_2^+(\mathbb{R}^7)$

The Grassmann manifold of oriented two-planes through the origin in $\mathbb{R}^7$ is generally written homogeneously $G_2^+(\mathbb{R}^7) \cong \mathrm{SO}(7)/\mathrm{SO}(2)\,\mathrm{SO}(5)$. It is irreducible: the symmetric metric is not only Einstein, it is the only $\mathrm{SO}(7)$-invariant metric. We will show that this Grassmannian manifold can also be written homogeneously as $\mathrm{G}_2/\mathrm{U}(2)$ and we find it carries two non-symmetric G_2-invariant Einstein metrics.

First we must see how $\mathrm{G}_2 \subset \mathrm{SO}(7)$, following [M, p.190]. We identify $\mathbb{R}^8$ with the Cayley numbers, or Octonians, the normed division algebra $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon$. Then G_2 is the set of automorphisms of $\mathbb{O}$. Any automorphism of the Cayley numbers must take 1 to itself and must preserve the inner product, so elements of G_2 also preserve $\mathrm{Im}(\mathbb{O})$, the space of imaginary Cayley numbers, the orthogonal complement to 1. In this way we see $\mathrm{G}_2 \subset \mathrm{SO}(7)$. To see that G_2 acts transitively on $G_2^+(\mathbb{R}^7)$, we use the following observation [M, p.186].

Lemma 5.1. *Given three imaginary orthogonal unit octonians: v_1, v_2, and $v_3 \in \{v_1, v_2, v_1 v_2\}^\perp$, there exists a unique automorphism A of $\mathbb{O}$ with $A(i) = v_1$, $A(j) = v_2$, and $A(\varepsilon) = v_3$.*

Using the lemma we take any $P = \mathrm{span}\{v, w\}$ to the oriented two-plane $P_0 = \mathrm{span}\{i, j\}$, where we take v and w an orthonormal basis for P. We must find the isotropy subgroup fixing P_0. It will be useful to recall the following well known fact.

Lemma 5.2. *The quotient* $\mathrm{G}_2/\mathrm{SU}(3) \cong S^6$.

Proof. By the previous lemma, G_2 acts transitively on $S^6(1) \subset \mathrm{Im}(\mathbb{O})$. We need to show that the isotropy subgroup H_v of G_2 corresponding to any $v \in S^6$ is $\mathrm{SU}(3)$. We observe first that $A(v) = v$ implies the map L_v is a complex structure on $\mathbb{O}$ and on $V = \mathrm{span}\{1, v\}^\perp$. This shows that $H_v \subset \mathrm{U}(V) \cong \mathrm{U}(3)$. Furthermore, $\dim(H_v) = 8$. Since G_2 and S^6 are connected and simply connected, H_v is connected.

Consider the homomorphism $\det : H_v \to S^1$; it must be either trivial or onto. If it is trivial, $H_v \cong \mathrm{SU}(3)$. If it is onto, then let H' denote the kernel. Then H' is a normal subgroup of H_v of dimension seven, and if H'_0 is the connected component of the identity, $\mathrm{rank}(H'_0) \leq \mathrm{rank}(\mathrm{SU}(3)) = 2$. But the only compact connected seven-dimensional Lie groups, up to finite cover, are T^7, $S^3 \times T^4$, and $S^3 \times S^3 \times S^1$, and each of these has rank > 2:

$$\mathrm{rank}(T^7) = 7, \quad \mathrm{rank}(S^3 \times T^4) = 5, \quad \mathrm{rank}(S^3 \times S^3 \times S^1) = 3.$$

This shows $H' = H_v$, and thus $H_v \cong \mathrm{SU}(3)$. □

We are now ready to describe the isotropy subgroup H corresponding to the oriented two-plane P_0. Since G_2 and $G_2^+(\mathbb{R}^7)$ are connected and simply connected, we know H is connected. If we have $A \in \mathrm{G}_2$ such that $A(P_0) = P_0$ (with orientation), then $A(i) = i\cos\theta - j\sin\theta$, $A(j) = i\sin\theta + j\cos\theta$, thus $A(k) = A(i)A(j) = k$. The isotropy subgroup $H \subset \{A \in \mathrm{G}_2 \mid A(k) = k\} \cong \mathrm{SU}(3)$, and since the ij-plane is a complex line with respect to our complex structure L_k, H must preserve this complex line and the complex two-plane perpendicular to it. Hence $H \subset S(\mathrm{U}(1)\,\mathrm{U}(2)) \subset \mathrm{SU}(3)$. A dimension count shows $H = S(\mathrm{U}(1)\,\mathrm{U}(2)) \cong \mathrm{U}(2)$.

If we look on the Lie algebra level, and we take $\{i, j, k, \varepsilon, i\varepsilon, j\varepsilon, k\varepsilon\}$ as our basis for $\mathrm{Im}(\mathbb{O})$, then using the automorphism property we can write $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ in the following way:

$$\begin{aligned}\mathfrak{g}_2 = \mathrm{span}\{&E_{12} + E_{56}, E_{47} + E_{56}, E_{45} + E_{67}, E_{46} - E_{57}, E_{46} + E_{57} + 2E_{13},\\ &E_{67} - E_{45} + 2E_{23}, E_{14} - E_{27} - 2E_{36}, E_{15} + E_{26} - 2E_{37}, E_{16} - E_{25} + 2E_{34},\\ &E_{17} + E_{24} + 2E_{35}, E_{14} + E_{27}, -E_{15} + E_{26}, E_{16} + E_{25}, -E_{17} + E_{24}\}.\end{aligned}$$

We use the inner product on $\mathfrak{g}_2$ given by $Q(X,Y) = -\frac{1}{2}\,\mathrm{tr}(XY)$, in which the basis for $\mathfrak{g}_2$ above is orthogonal. The subalgebra $\mathfrak{h}$ corresponds to $H \cong \mathrm{U}(2)$:

$$\mathfrak{u}(2) \cong \mathrm{span}\{2E_{12} + E_{56} - E_{47}, E_{47} + E_{56}, E_{45} + E_{67}, E_{46} - E_{57}\}.$$

The isotropy representation of $\mathrm{U}(2)$ is the action of $\mathrm{Ad}\,\mathrm{U}(2)$ on $\mathfrak{p}$, the Q-orthogonal complement of $\mathfrak{u}(2)$ in $\mathfrak{g}_2$. We can use the following fibrations to decompose $\mathfrak{p}$ into its irreducible $\mathrm{Ad}\,\mathrm{U}(2)$ representations: First,

$$\mathbb{C}P^2 \cong \mathrm{SU}(3)/\,\mathrm{U}(2) \to \mathrm{G}_2\,/\,\mathrm{U}(2) \to \mathrm{G}_2\,/\,\mathrm{SU}(3) \cong S^6.$$

The tangent space to the base is isomorphic to $[\mu_3]_{\mathbb{R}}$; when restricted to U(2) it gives $[(\mu_1 \hat{\otimes} \mathrm{Id}) \oplus (\mathrm{Id} \hat{\otimes} \mu_2)]_{\mathbb{R}} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. Thus, $\mathfrak{p}_1 = [\mu_1 \hat{\otimes} \mathrm{Id}]_{\mathbb{R}}$, $\mathfrak{p}_2 = [\mathrm{Id} \hat{\otimes} \mu_2]_{\mathbb{R}}$. The tangent space to the fibre is $\mathfrak{p}_3 = [\mu_1 \hat{\otimes} \mu_2]_{\mathbb{R}}$. We have $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$.

$$\mathfrak{p}_1 = \mathrm{span}\{E_{46} + E_{57} + 2E_{13}, E_{67} - E_{45} + 2E_{23}\},$$
$$\mathfrak{p}_2 = \mathrm{span}\{E_{14} - E_{27} - 2E_{36}, E_{15} + E_{26} - 2E_{37}, E_{16} - E_{25} + 2E_{34}, E_{17} + E_{24} + 2E_{35}\},$$
$$\mathfrak{p}_3 = \mathrm{span}\{E_{14} + E_{27}, E_{26} - E_{15}, E_{16} + E_{25}, E_{24} - E_{17}\}.$$

We have a second fibration of our manifold: we claim $\mathrm{U}(2) \subset \mathrm{SO}(4) \subset \mathrm{G}_2$. Before showing this, we briefly discuss the embedding $\mathrm{SO}(4) \subset \mathrm{G}_2$ and the irreducible symmetric space $\mathrm{G}_2 / \mathrm{SO}(4)$.

Lemma 5.3. *The quotient* $\mathrm{G}_2 / \mathrm{SO}(4)$ *is the space of quaternionic subalgebras of the Cayley numbers,* $\mathbb{O}$.

Proof. We have $\mathrm{SO}(4) \cong (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\{(q_1, q_2) \simeq (-q_1, -q_2)\}$, and it acts on $\mathbb{O} \cong \mathbb{H} \oplus \mathbb{H}\varepsilon$ by

$$(q_1, q_2) : a + b\varepsilon \mapsto q_1 a \bar{q}_1 + (q_2 b \bar{q}_1)\varepsilon.$$

A calculation shows that $\mathrm{SO}(4) \subset \mathrm{G}_2$, and this embedding of SO(4) in G_2 can also be described as the subgroup of G_2 which leaves the subalgebra $\mathbb{H} \cong \mathrm{span}\{1, i, j, k\}$ invariant. □

Since U(2) is the subgroup of G_2 preserving the plane spanned by i and j, elements of U(2) take 1 to itself and k to itself, hence they preserve $\mathrm{span}\{1, i, j, k\}$. This also shows $\mathrm{U}(2) \subset \mathrm{SO}(4) \cap \mathrm{SU}(3)$. We also have $\mathrm{SO}(4) \cap \mathrm{SU}(3) \subset \mathrm{U}(2)$: under G_2, $1 \mapsto 1$; under SO(4), $\mathrm{span}\{1, i, j, k\} \mapsto \mathrm{span}\{1, i, j, k\}$; under SU(3), $k \mapsto k$. Thus $\mathrm{SO}(4) \cap \mathrm{SU}(3) = \mathrm{U}(2)$. Our second fibration is

$$\mathrm{SO}(4)/\,\mathrm{U}(2) \to \mathrm{G}_2 / \mathrm{U}(2) \to \mathrm{G}_2 / \mathrm{SO}(4).$$

Here $\mathfrak{p}_1$ is tangent to the fibre, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are tangent to the base: From the two fibrations we obtain the following Lie bracket relations among the $\mathfrak{p}_i$'s: $[\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{u}(2)$, $[\mathfrak{p}_1, \mathfrak{p}_2] \subset \mathfrak{p}_2 \oplus \mathfrak{p}_3$, $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{u}(2) \oplus \mathfrak{p}_1$, and $[\mathfrak{p}_3, \mathfrak{p}_3] \subset \mathfrak{u}(2)$.

Since $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$ are mutually inequivalent, any G_2-invariant metric on our space is of the form $\langle\ ,\ \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \perp x_3 Q|_{\mathfrak{p}_3}$, with $x_i > 0$, for $i = 1, 2, 3$. As in the previous cases we can write the scalar curvature on $G_2^+(\mathbb{R}^7)$ as function of x_1, x_2, and x_3 via the formula given in [W-Z]:

$$S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \binom{k}{i\,j} \frac{x_k}{x_i x_j}.$$

When we compute the non-zero Lie bracket relations between the $\mathfrak{p}_i$'s we find that $\binom{3}{1\,2} = 4$ and $\binom{1}{2\,2} = \frac{16}{3}$. Also we compute (using the basis elements for $\mathfrak{g}_2$) $b_i = 8$ for each i, and of course $d_1 = 2$, $d_2 = 4$, and $d_3 = 4$. This gives us

$$S = 8\Big(\frac{1}{x_1} + \frac{2}{x_2} + \frac{2}{x_3}\Big) - \frac{4}{3}\Big(\frac{x_1}{x_2^2} + \frac{2}{x_1}\Big) - 2\Big(\frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} + \frac{x_3}{x_1 x_2}\Big).$$

Then $\tilde{S} = S - \lambda(x_1^2 x_2^4 x_3^4 - 1)$ includes our boundary condition, volume $= 1$. Einstein metrics on $\mathrm{G}_2 / \mathrm{U}(2)$ will be critical points of $\tilde{S}$.

$$\begin{aligned}
\frac{\partial \tilde{S}}{\partial x_1} &= -\frac{16}{3x_1^2} - \frac{4}{3x_2^2} + 2\Big(\frac{x_2}{x_1^2 x_3} + \frac{x_3}{x_1^2 x_2} - \frac{1}{x_2 x_3}\Big) - 2\lambda x_1 x_2^4 x_3^4 \\
\frac{\partial \tilde{S}}{\partial x_2} &= -\frac{16}{x_2^2} + \frac{8x_1}{3x_2^3} + 2\Big(\frac{x_1}{x_2^2 x_3} + \frac{x_3}{x_1 x_2^2} - \frac{1}{x_1 x_3}\Big) - 4\lambda x_1^2 x_2^3 x_3^4 \\
\frac{\partial \tilde{S}}{\partial x_3} &= -\frac{16}{x_3^2} + 2\Big(\frac{x_1}{x_2 x_3^2} + \frac{x_2}{x_1 x_3^2} - \frac{1}{x_1 x_2}\Big) - 4\lambda x_1^2 x_2^4 x_3^3.
\end{aligned}$$

Now we look for all solutions to $\frac{\partial \tilde{S}}{\partial x_1} = \frac{\partial \tilde{S}}{\partial x_2} = \frac{\partial \tilde{S}}{\partial x_3} = 0$. We solve these using Maple, and we find the following:

$$\begin{aligned}
&\text{either } x_2 = \frac{1}{2}x_1 \text{ and } x_3 = \frac{3}{2}x_1 \\
&\text{or } x_2 = \zeta x_1 \text{ and } x_3 = \left(\frac{7}{120}\zeta^4 - \frac{7}{60}\zeta^3 - \frac{151}{96}\zeta^2 + \frac{39}{10}\zeta - \frac{21}{40}\right) x_1, \\
&\text{where } \zeta \text{ is a root of } 56\zeta^5 - 532\zeta^4 + 1570\zeta^3 - 1891\zeta^2 + 776\zeta - 60 = 0.
\end{aligned}$$

We need both a positive solution to this polynomial in order for $x_2 > 0$, and also

$$\left(\frac{7}{120}\zeta^4 - \frac{7}{60}\zeta^3 - \frac{151}{96}\zeta^2 + \frac{39}{10}\zeta - \frac{21}{40}\right) > 0, \text{ so that } x_3 > 0.$$

There are exactly three real solutions to the quintic polynomial. We give the approximate values for x_2 and x_3, setting $x_1 = 1$:

$$\begin{aligned}
x_2 &= 0.09953 \quad x_3 = -.15252 \\
x_2 &= 0.59713 \quad x_3 = 1.22554 \\
x_2 &= 5.35063 \quad x_3 = 5.25153
\end{aligned}$$

We must eliminate the first of the solutions from the quintic, since it gives a negative value for x_3. The solution $x_2 = \frac{1}{2}x_1$ and $x_3 = \frac{3}{2}x_1$ is the symmetric metric; that is, we will see that it is SO(7)-invariant, after projection. If we denote by $\tilde{\mathfrak{p}}$ the Q-orthogonal complement to $\mathfrak{so}(2) \oplus \mathfrak{so}(5)$ in $\mathfrak{so}(7)$, then $\tilde{\mathfrak{p}} = \{E_{ij} \mid 1 \le i \le 2,\ 3 \le j \le 7\}$. When we project $\mathfrak{p}$ to $\tilde{\mathfrak{p}}$, we see that in $\mathfrak{p}_1$, the basis element $\frac{1}{\sqrt{6}}(2E_{13} + E_{46} + E_{57})$ projects to $\frac{2}{\sqrt{6}}E_{13}$, i.e., an element of norm $= \sqrt{x_1}$ projects to an element of norm $= \sqrt{\frac{2}{3}}$. In $\mathfrak{p}_2$, the basis element $\frac{1}{\sqrt{6}}(2E_{36} - E_{14} + E_{27})$ projects to $\frac{1}{\sqrt{6}}(-E_{13} + E_{27})$, so norm $= \sqrt{x_2}$ is projected to norm

$= \sqrt{\frac{1}{3}}$. In $\mathfrak{p}_3$, basis element $\frac{1}{\sqrt{2}}(E_{14} + E_{27})$ is already an element of $\tilde{\mathfrak{p}}$, so norm $= \sqrt{x_3}$ is projected to norm $= 1$. This shows that the symmetric metric in our basis satisfies $\frac{x_2}{x_1} = \frac{1}{2}$ and $\frac{x_3}{x_1} = \frac{3}{2}$. Thus we end up with two new G_2-homogeneous Einstein metrics on the Grassmannian $G_2^+(\mathbb{R}^7)$.

Remark 5.4. Notice none of these is a fibration metric, since a metric of the first fibration would require $x_1 = x_2$ and a metric of the second fibration would require $x_2 = x_3$. Many examples of Einstein metrics have been obtained by using fibrations over Einstein spaces and with Einstein fibres [Bes, ch.9]. In our example we have two fibrations; in each the fibre and base space are isotropy irreducible, so they must be Einstein. However, in each fibration the isotropy representation of the base, when restricted to U(2), decomposes into the sum of two irreducible subrepresentations. Recall $Q(X,Y) = -\frac{1}{2}\operatorname{tr} XY$ is our comparison metric. The Casimir constant corresponding to Q and to the restriction to U(2) of the isotropy representations for these homogeneous spaces differs on these two subpresentations. This implies that the O'Neill tensor restricted to horizontal vectors is not a scalar multiple of Q. Using Besse's Proposition 9.70 [Bes, p.253] we see that therefore our fibrations cannot give rise to Einstein metrics.

To see that we have three non-isometric solutions we compare the scale-invariant product: $(S)^{\frac{d}{2}}(V)^{\frac{1}{2}}$, where S is the scalar curvature and V is the volume, and d is the dimension of our homogeneous space, for our three metrics. If we let $x_2 = \frac{1}{2}x_1$ and $x_3 = \frac{3}{2}x_1$, we obtain $S = \frac{100}{3x_1}$, and $V = \frac{3^4}{2^8}x_1^{10}$, so $(S)^5(V)^{\frac{1}{2}} = \frac{2^6 5^{10}}{3^3}$. This is approximately 2.3148×10^7. The second solution gives $(S)^5(V)^{\frac{1}{2}} \cong 2.3044 \times 10^7$, and the third solution gives $(S)^5(V)^{\frac{1}{2}} \cong 1.5836 \times 10^7$.

We note that these have been found previously in [A] and [K]. They observe that one of the three metrics is Kähler and the other two are not Kähler for any complex structure on M. Neither author observes that the Kähler Einstein metric is the symmetric metric.

6. $G_3^+(\mathbb{R}^8)$

We can write the Grassmannian manifold of oriented three-planes in $\mathbb{R}^8$ homogeneously $G_3^+(\mathbb{R}^8) \cong \mathrm{SO}(8)/\mathrm{SO}(3)\,\mathrm{SO}(5)$, but we can also write it as $G_3^+(\mathbb{R}^8) \cong \mathrm{Spin}(7)/\mathrm{SO}(4)$. With respect to SO(8), $G_3^+(\mathbb{R}^8)$ is irreducible, and therefore the symmetric metric is Einstein and it is the unique SO(8)-invariant metric, up to scaling. However, we find there are two more Spin(7)-invariant Einstein metrics which are not symmetric.

We first describe how Spin(7) sits inside SO(8). We again identify $\mathbb{R}^8$ with the Cayley numbers $\mathbb{O}$; from Murakami [M] we know

$$\mathrm{Spin}(7) = \{A \in \mathrm{SO}(8) \mid \exists B \in \mathrm{SO}(8) \text{ such that } B(x)A(y) = A(xy)\ \forall x, y \in \mathbb{O}\}.$$

In this definition notice $B(1) = 1$, so $B \in \mathrm{SO}(7)$ and if A corresponds to B, $-A$ corresponds to B as well, which shows Spin(7) is indeed a double cover of SO(7). We also remark that that $\{L_a \mid a \in \mathrm{Im}(\mathbb{O}), |a| = 1\} \subset \mathrm{Spin}(7)$. (The corresponding B is conjugation by a.) We need to show that $C_a(x)L_a(y) = L_a(xy)$ for any $x, y \in \mathbb{O}$. Because a is a unit imaginary octonian, $a^{-1} = -a$, thus $C_a(x)L_a(y) = (axa^{-1})(ay) = -(axa)(ay)$. Then the first Moufang identity tells us that $-(axa)(ay) = -a(x(aay))$, and using that $aa = -1$, we have $-a(x(aay)) = a(xy) = L_a(xy)$.

It is also convenient to identify two subgroups of Spin(7), they are G_2, the automorphisms of the Cayley numbers, and SU(4), complex linear maps with respect to L_i. We see $\mathrm{G}_2 \subset \mathrm{Spin}(7)$ by letting $B = A$ in the definition of Spin(7). Murakami shows how to see that $\mathrm{SU}(4) \subset \mathrm{Spin}(7)$ with the following lemma [M].

Lemma 6.1. *In* SO(8), $\mathrm{U}(4) = \{A \in \mathrm{SO}(8) \mid iA(x) = A(ix)\ \forall x, y \in \mathbb{O}\}$ *and* $\mathrm{U}(4) \cap \mathrm{Spin}(7) = \mathrm{SU}(4)$.

Proof. Let $p^+ : \mathrm{Spin}(7) \to \mathrm{SO}(7)$ be the homomorphism sending $A \mapsto B$, for A and B in the definition of Spin(7). For every $A \in \mathrm{U}(4) \cap \mathrm{Spin}(7)$ we have $B(i) = i$, so $p^+(\mathrm{U}(4) \cap \mathrm{Spin}(7)) \subset \mathrm{SO}(6)$. And for every $B \in \mathrm{SO}(6)$ ($B(i) = i$), the corresponding A must be in $\mathrm{U}(4) \cap \mathrm{Spin}(7)$, hence $\mathrm{SO}(6) \subset p^+(\mathrm{U}(4) \cap \mathrm{Spin}(7))$. Furthermore, p^+ is a local isomorphism, thus $\mathrm{U}(4) \cap \mathrm{Spin}(7)$ is a 15-dimensional connected Lie group with a simple Lie algebra. Observe that SU(4) is the commutator subgroup of U(4). Since its Lie algebra is simple, $\mathrm{U}(4) \cap \mathrm{Spin}(7)$ is its own commutator subgroup, thus $\mathrm{U}(4) \cap \mathrm{Spin}(7) \subset \mathrm{SU}(4)$, and a dimension count tells us that these subgroups are equal. □

We embed $\mathrm{SU}(4) \subset \mathrm{SO}(8)$ via $A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. To embed SU(4) into SO(8) in this way, we want $\mathbb{R}^4 \oplus \mathbb{R}^4 \cong \mathbb{C}^4$ with $(u, v) \mapsto u + iv$; this restricts our choice of ordered bases for $\mathbb{O}$: we choose $\{1, j, \varepsilon, j\varepsilon, i, k, i\varepsilon, -k\varepsilon\}$. The intersection of our two subgroups is $\mathrm{G}_2 \cap \mathrm{SU}(4) = \mathrm{SU}(3)$, and this time SU(3) in G_2 fixes i instead of k.

We check that Spin(7) acts transitively on $G_3^+(\mathbb{R}^8)$: Let $P_0 = \mathrm{span}\{i, j, k\}$, an oriented three-plane through the origin. Let P be given by $\mathrm{span}\{v_1, v_2, v_3\}$, where v_1, v_2 and v_3 are ordered orthonormal vectors. Without loss of generality we may assume $v_1, v_2 \in \mathrm{Im}(\mathbb{O})$, since P is a three-plane, so $\dim(P \cap \mathrm{Im}(\mathbb{O})) \geq 2$. We know from the previous example that we can find an element $A \in \mathrm{G}_2$ such that $A(i) = v_1$, $A(j) = v_2$. Let $x = A^{-1}(v_3)$. Observe that

$$\langle x, i \rangle = \langle A^{-1}(v_3), i \rangle = \langle v_3, v_1 \rangle = 0$$
$$\langle x, j \rangle = \langle A^{-1}(v_3), j \rangle = \langle v_3, v_2 \rangle = 0.$$

We claim there exists $A' \in \mathrm{Spin}(7)$ such that $A'(i) = i$, $A'(j) = j$, and $A'(k) = x$. This is because the subgroup of Spin(7) fixing one unit octonian is conjugate to G_2, and the subgroup of Spin(7) fixing two orthonormal octonians is conjugate to SU(3), which acts transitively on $S^5(1) \subset \operatorname{span}\{i,j\}^\perp$. The composition $A \circ A' \in \mathrm{Spin}(7)$ is our map taking $i \mapsto v_1$, $j \mapsto v_2$, and $k \mapsto v_3$, so that P_0 goes to P, and this shows Spin(7) acts transitively on $G_3^+(\mathbb{R}^8)$.

Next we must determine the isotropy subgroup H of P_0. We claim that $H \subset \mathrm{G}_2$. To see this, we note that if $A(P_0) = P_0$, then if B is the element of SO(7) in the definition of Spin(7) corresponding to A, since $B(i)A(j) = A(k)$ we know that $B(i) \in \operatorname{span}\{i,j,k\}$. Thus $A(1) = -B(i)A(i) \in \operatorname{span}\{1,i,j,k\}$, and furthermore $A(1) \perp P_0$, so $A(1) = \pm 1$. Since Spin(7) and $\mathrm{G}_3^+(\mathbb{R}^8)$ are connected and simply connected, we know H is connected, hence $A(1) = 1$. From the definition of Spin(7) it follows that $A = B$ and this implies $H \subset \mathrm{G}_2$. Furthermore, any element of H takes 1 to itself and preserves the standard quaternionic subalgebra $\operatorname{span}\{1,i,j,k\}$. Thus $H \subset \mathrm{SO}(4) \subset \mathrm{G}_2$, and by a dimension count $H = \mathrm{SO}(4)$.

Now we are ready to find the isotropy representation. On the Lie algebra level we have

$$\begin{aligned}\mathfrak{spin}(7) = &\operatorname{span}\{E_{ij} + E_{4+i,4+j}, E_{i,4+j} + E_{j,4+i} \mid 1 \le i < j \le 4\}\\ &\oplus \operatorname{span}\{E_{i,4+i} - E_{48} \mid 1 \le i \le 3\}\\ &\oplus \operatorname{span}\{E_{27} - E_{45}, E_{23} + E_{58}, E_{24} - E_{57}, E_{28} + E_{35}, E_{56} - E_{78}\\ &\qquad 2E_{25} - E_{38} + E_{47}\}.\end{aligned}$$

The subalgebra corresponding to the isotropy subgroup is $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$:

$$\begin{aligned}\mathfrak{h} = &\operatorname{span}\{E_{37} - E_{48}, E_{34} + E_{78}, E_{38} + E_{47}\}\\ &\oplus \operatorname{span}\{2E_{56} + E_{34} - E_{78}, 2E_{26} - E_{48} - E_{37}, 2E_{25} - E_{38} + E_{47}\}.\end{aligned}$$

Notice each copy of $\mathfrak{su}(2)$ is an ideal in $\mathfrak{h}$ and its basis vectors above are orthogonal with respect to the inner product on $\mathfrak{spin}(7)$ given by $Q(X,Y) = -\frac{1}{2}\operatorname{tr}(XY)$. As usual we denote by $\mathfrak{p}$ the Q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{spin}(7)$.

There are two fibrations of our symmetric space $\mathrm{Spin}(7)/\mathrm{SO}(4)$. The first is

$$\mathrm{G}_2/\mathrm{SO}(4) \to \mathrm{Spin}(7)/\mathrm{SO}(4) \to \mathrm{Spin}(7)/\mathrm{G}_2 \cong S^7.$$

Let $\mathfrak{p}'$ be the subspace tangent to the fibre; let $\mathfrak{p}''$ be the subspace tangent to the base. Let θ_k denote the unique irreducible complex representation of $\mathfrak{su}(2)$ in k dimensions; the first fibration tells us that $\mathfrak{p}' = [\theta_2 \hat{\otimes} \theta_4]_{\mathbb{R}}$, since this is the representation of the symmetric space $\mathrm{G}_2/\mathrm{SO}(4)$ [W, p.287]. The isotropy representation of $\mathrm{Spin}(7)/\mathrm{G}_2$ is the seven-dimensional representation of $\mathrm{G}_2 \subset \mathrm{SO}(7)$. We restrict this representation to SO(4), to see that $\mathfrak{p}'' = [\rho_3 \hat{\otimes} \mathrm{Id}] \oplus [\theta_2 \hat{\otimes} \theta_2]_{\mathbb{R}}$, where ρ_3 denotes the standard representation of $\mathfrak{so}(3)$ on $\mathbb{R}^3$. We let $\mathfrak{p}_1 = [\rho_3 \hat{\otimes} \mathrm{Id}]$, and $\mathfrak{p}_2 = [\theta_2 \hat{\otimes} \theta_2]_{\mathbb{R}}$, and $\mathfrak{p}_3 = [\theta_2 \hat{\otimes} \theta_4]_{\mathbb{R}}$. We have $\dim(\mathfrak{p}_1) = 3$, $\dim(\mathfrak{p}_2) = 4$, and $\dim(\mathfrak{p}_3) = 8$.

For the second fibration, we first need some explanation.

Lemma 6.2. *The compact group* $(\mathrm{Spin}(4)\times\mathrm{Spin}(3))/\{\pm(\mathrm{Id},\mathrm{Id})\}$ *satisfies* $\mathrm{SO}(4)\subset(\mathrm{Spin}(4)\times\mathrm{Spin}(3))/\{\pm(\mathrm{Id},\mathrm{Id})\}\subset\mathrm{Spin}(7)$.

Proof. Recall, for any n, $\mathrm{Spin}(n)$ is the simply connected double cover of $\mathrm{SO}(n)$; let $\pi:\mathrm{Spin}(n)\to\mathrm{SO}(n)$ denote the two-fold homomorphism. Because $\pi_1(\mathrm{SO}(k))\to\pi_1(\mathrm{SO}(n))$ is a surjection, $\pi(\mathrm{Spin}(k))=\mathrm{SO}(k)$ for all $k\le n$. Observe that $ker(\pi)=\{\pm(\mathrm{Id},\mathrm{Id})\}=\mathrm{Spin}(k)\cap\mathrm{Spin}(n-k)$, thus it is $(\mathrm{Spin}(k)\times\mathrm{Spin}(n-k))/\{\pm(\mathrm{Id},\mathrm{Id})\}\subset\mathrm{Spin}(n)$. In $\mathrm{Spin}(7)$ we can consider the subgroup $(\mathrm{Spin}(3)\times\mathrm{Spin}(4))/\{\pm(\mathrm{Id},\mathrm{Id})\}$. We know $\mathrm{Spin}(4)\cong\mathrm{Spin}(3)\times\mathrm{Spin}(3)$ (and $\mathrm{Spin}(3)\cong\mathrm{SU}(2)$). Thus $\mathrm{Spin}(7)$ has a subgroup $(\mathrm{Spin}(3)\times\mathrm{Spin}(3)\times\mathrm{Spin}(3))/\{\pm(\mathrm{Id},\mathrm{Id},\mathrm{Id})\}$. Our isotropy subgroup $\mathrm{SO}(4)\subset\mathrm{Spin}(7)$ is exactly $(\Delta\,\mathrm{Spin}(3)\times\mathrm{Spin}(3))/\{\pm(\mathrm{Id},\mathrm{Id})\}$, where $\Delta\,\mathrm{Spin}(3)$ is the diagonal subgroup of $\mathrm{Spin}(3)\times\mathrm{Spin}(3)$ isomorphic to $\mathrm{Spin}(3)$. This can be seen via the restriction to $\mathrm{SO}(4)$ of the homomorphism from $\mathrm{Spin}(7)$ to $\mathrm{SO}(7)$ taking A to B (A and B in the definition of $\mathrm{Spin}(7)$). □

We obtain the following fibration:

$$\begin{array}{rcl} S^3\cong \frac{\mathrm{Spin}(4)\times\mathrm{Spin}(3)}{\{\pm(\mathrm{Id},\mathrm{Id})\}}\Big/\frac{\mathrm{Spin}(3)\times\Delta\,\mathrm{Spin}(3)}{\{\pm(\mathrm{Id},\mathrm{Id})\}} & \longrightarrow \mathrm{Spin}(7)/\,\mathrm{SO}(4) & \cong G_3^+(\mathbb{R}^8) \\ & \downarrow & \\ & \mathrm{Spin}(7)\Big/\frac{\mathrm{Spin}(4)\times\mathrm{Spin}(3)}{\{\pm(\mathrm{Id},\mathrm{Id})\}} & \cong G_3^+(\mathbb{R}^7). \end{array}$$

In the second fibration, it is $\mathfrak{p}_1$ which is the subspace tangent to the fibre, while $\mathfrak{p}_2\oplus\mathfrak{p}_3$ is the subspace tangent to the base.

$$\begin{aligned} \mathfrak{p}_1 &= \mathrm{span}\{3E_{12}-E_{34}+E_{56}+E_{78}, 3E_{15}-E_{26}-E_{37}-E_{48}, 3E_{16}+E_{25}+E_{38}-E_{47}\} \\ \mathfrak{p}_2 &= \mathrm{span}\{3E_{13}+E_{24}+E_{57}-E_{68}, 3E_{14}-E_{23}+E_{58}+E_{67}, \\ &\qquad 3E_{17}-E_{28}+E_{35}+E_{46}, 3E_{18}+E_{27}-E_{36}+E_{45}\} \\ \mathfrak{p}_3 &= \mathrm{span}\{E_{23}+E_{67}, E_{24}+E_{68}, E_{27}+E_{36}, E_{28}+E_{46}, 2E_{57}-E_{24}+E_{68}, \\ &\qquad 2E_{58}+E_{23}-E_{67}, 2E_{35}+E_{28}-E_{46}, 2E_{45}-E_{27}+E_{36}\}. \end{aligned}$$

Since each of the $\mathfrak{p}_i$'s has a different dimension, they are inequivalent representations. This means any $\mathrm{Spin}(7)$-invariant metric on $G_3^+(\mathbb{R}^8)$ is determined by an inner product on $\mathfrak{p}$ satisfying

$$\langle\ ,\ \rangle = x_1 Q|_{\mathfrak{p}_1}\perp x_2 Q|_{\mathfrak{p}_2}\perp x_3 Q|_{\mathfrak{p}_3},\ \text{for } x_1,x_2,x_3>0.$$

From the fibrations we obtain the following Lie bracket relations:

$$\begin{aligned} &[\mathfrak{p}_1,\mathfrak{p}_1]\subset\mathfrak{h}\oplus\mathfrak{p}_1, \quad [\mathfrak{p}_1,\mathfrak{p}_2]\subset\mathfrak{p}_2\oplus\mathfrak{p}_3, \\ &[\mathfrak{p}_2,\mathfrak{p}_2]\subset\mathfrak{h}\oplus\mathfrak{p}_1, \quad [\mathfrak{p}_3,\mathfrak{p}_3]\subset\mathfrak{h}. \end{aligned}$$

Recall the scalar curvature formula from [W-Z]:

$$S = \frac{1}{2}\sum_i \frac{d_i b_i}{x_i} - \frac{1}{4}\sum_{i,j,k} \binom{k}{i\,j} \frac{x_k}{x_i x_j}.$$

On Spin(7) we find that $b_i = 10$ for $i = 1, 2, 3$, and we know $d_1 = 3$, $d_2 = 4$, and $d_3 = 8$. From the Lie bracket relations we know $\binom{1}{1\,1}$, $\binom{1}{2\,2}$, $\binom{1}{2\,3} \neq 0$; all other triples (except rearrangements) are zero. We find that $\binom{1}{1\,1} = 2$, $\binom{1}{2\,2} = 4$, and $\binom{3}{1\,2} = 8$.

We now have the scalar curvature function in x_1, x_2, x_3:

$$S = \frac{25}{2x_1} + \frac{20}{x_2} + \frac{40}{x_3} - \frac{x_1}{x_2^2} - 4\left(\frac{x_1}{x_2x_3} + \frac{x_2}{x_1x_3} + \frac{x_3}{x_1x_2}\right).$$

We want the critical points of the scalar curvature function with the constraint equation of volume 1: $\tilde{S} = S - \lambda(x_1^3x_2^4x_3^8 - 1)$.

$$\frac{\partial \tilde{S}}{\partial x_1} = -\frac{25}{2x_1^2} - \frac{1}{x_2^2} - \frac{4}{x_2x_3} + \frac{4x_2}{x_1^2x_3} + \frac{4x_3}{x_1^2x_2} - 3\lambda x_1^2x_2^4x_3^8$$
$$\frac{\partial \tilde{S}}{\partial x_2} = -\frac{20}{x_2^2} + \frac{2x_1}{x_2^3} + \frac{4x_1}{x_2^2x_3} - \frac{4}{x_1x_3} + \frac{x_3}{x_1x_2^2} - 4\lambda x_1^3x_2^3x_3^8$$
$$\frac{\partial \tilde{S}}{\partial x_3} = -\frac{40}{x_3^2} + \frac{4x_1}{x_2x_3^2} + \frac{4x_2}{x_1x_3^2} - \frac{4}{x_1x_2} - 8\lambda x_1^3x_2^4x_3^7.$$

A solution to $\frac{\partial \tilde{S}}{\partial x_1} = \frac{\partial \tilde{S}}{\partial x_2} = \frac{\partial \tilde{S}}{\partial x_3} = 0$ is equivalent to the simultaneous solution of the following two polynomials:

$$10x_1x_2^2 - 10x_1x_2x_3 + x_1^2x_2 + x_1^2x_3 + 3x_2x_3^2 - 3x_2^3 = 0$$
$$-11x_1^2x_2 + 11x_2x_3^2 + 5x_2^3 - 25x_2^2x_3 + 30x_1x_2^2 - 2x_1^2x_3 = 0.$$

We obtain three solutions, using Maple. The first is the symmetric solution: $x_1 = \frac{3}{4}x_3$, $x_2 = \frac{1}{4}x_3$. Let $\tilde{\mathfrak{p}}$ denote the Q-orthogonal complement to $\mathfrak{so}(3)\oplus\mathfrak{so}(5)$ in $\mathfrak{so}(8)$; $\tilde{\mathfrak{p}} = \mathrm{span}\{E_{ij} \mid i = 2, 5, 6,\ j = 1, 3, 4, 7, 8\}$. (Of course we take $\mathfrak{so}(3) \oplus \mathfrak{so}(5)$ corresponding to P_0.) We must project $\mathfrak{p}_1, \mathfrak{p}_2$, and $\mathfrak{p}_3$ to $\tilde{\mathfrak{p}}$. In $\mathfrak{p}_1$, we take the basis element $\frac{1}{2\sqrt{3}}(3E_{12} - E_{34} + E_{56} + E_{78})$ of norm $= \sqrt{x_1}$. It is projected to $-\frac{\sqrt{3}}{2}E_{12}$ in $\tilde{\mathfrak{p}}$, an element of norm $= \frac{\sqrt{3}}{2}$. In $\mathfrak{p}_2$ we take the basis element $\frac{1}{2\sqrt{3}}(3E_{13} + E_{24} + E_{57} - E_{68})$ of norm $= \sqrt{x_2}$, which projects to $\frac{1}{2\sqrt{3}}(E_{24} + E_{57} - E_{68})$ in $\tilde{\mathfrak{p}}$, an element of norm $= \frac{1}{2}$. Finally, the element $\frac{1}{\sqrt{2}}(E_{23} + E_{67})$ is already in $\tilde{\mathfrak{p}}$, so norm $= \sqrt{x_3}$ corresponds to norm $= 1$. Hence $x_1 = \frac{3}{4}x_3$, $x_2 = \frac{1}{4}x_3$ is indeed the symmetric metric.

The second and third solutions are $x_2 = \eta x_3$, and

$$x_1 = \left(-\frac{629}{1980} + \frac{5689}{660}\eta - \frac{3799}{165}\eta^2 + \frac{13559}{495}\eta^3 - \frac{392}{33}\eta^4\right)x_3,$$

for η a positive root of the polynomial

$$4704t^5 - 11788t^4 + 10400t^3 - 3315t^2 - 398t + 289.$$

This polynomial has three real roots, of which two are positive and yield two positive values for x_1 and x_2 in terms of x_3. We give the approximate values, setting $x_3 = 1$:

$$\begin{aligned} x_1 &= -.241854 & x_2 &= -4.177304 \\ x_1 &= .425179 & x_2 &= .902192 \\ x_1 &= 1.100300 & x_2 &= .369813 \end{aligned}$$

These are two new Einstein metrics on $G_3^+(\mathbb{R}^8)$.

Remark 6.3. None of these is a fibration metric, since the first fibration required $x_1 = x_2$, and the second required $x_2 = x_3$. Just as in the previous example in both fibrations the fibre and base space are isotropy irreducible, therefore Einstein. However, the isotropy representation of each base, when restricted to SO(4), again decomposes into the sum of two irreducible subrepresentations, where the Casimir constant corresponding to Q and to the restriction to SO(4) of the isotropy representations for these homogeneous spaces differs. Again this implies the O'Neill tensor restricted to horizontal vectors is not a scalar multiple of Q. Using Besse's Proposition 9.70 [Bes, p.253] we know our fibrations cannot give rise to Einstein metrics.

We verify that they are all distinct by estimating the (scale-invariant) product: $(S)^{\frac{15}{2}}(V)^{\frac{1}{2}}$, where S is the scalar curvature of the metric and V is the volume of the metric. For the symmetric metric, $S = \frac{90}{x_3}$ and $V = \frac{3^3}{2^{14}} x_3^{15}$, and so $(S)^{\frac{15}{2}}(V)^{\frac{1}{2}} \cong 1.84200 \times 10^{13}$. For the two new metrics, we find that $(S)^{\frac{15}{2}}(V)^{\frac{1}{2}} \cong 1.80936 \times 10^{13}$, and $(S)^{\frac{15}{2}}(V)^{\frac{1}{2}} \cong 1.61159 \times 10^{13}$, respectively. This shows that they are non-isometric.

7. Appendix

Oniščik in fact lists more triples of Lie algebras $(\mathfrak{g}, \mathfrak{g}', \mathfrak{g}'')$, but the extra triples can be obtained by combining the information given on page 4. For example, in addition to $\mathrm{SO}(2n)/\mathrm{SO}(2n-1) = \mathrm{SU}(n)/\mathrm{SU}(n-1)$ and $\mathrm{SO}(4n)/\mathrm{SO}(4n-1) = \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$, he lists $\mathrm{SU}(2n)/\mathrm{SU}(2n-1) = \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$, which follows from the inclusions $\mathrm{Sp}(n) \subset \mathrm{SU}(2n) \subset \mathrm{SO}(4n)$. Many of the triples on his list come from the subgroups of SO(8): In addition to $\mathrm{SO}(7) \subset \mathrm{SO}(8)$, we have

$$\begin{aligned} \mathrm{Sp}(2) \subset \mathrm{Sp}(2)\,\mathrm{U}(1) \subset \mathrm{Sp}(2)\,\mathrm{Sp}(1) \subset \mathrm{SO}(8), \\ \mathrm{U}(2) \subset \mathrm{SU}(3) \subset \mathrm{SU}(4) \subset \mathrm{U}(4) \subset \mathrm{SO}(8), \\ \text{and } \mathrm{SO}(4) \subset \mathrm{G}_2 \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8). \end{aligned}$$

We note that SO(8) contains two copies of Spin(7) and that there is an outer automorphism of SO(8) of order three, called the triality automorphism, which interchanges the Spin(7)'s, and on the Lie algebra level, it interchanges the $\mathfrak{spin}(7)'\mathfrak{s}$ and the standard embedding of $\mathfrak{so}(7)$. This yields equalities like the following:

$$\mathrm{SO}(8)/\,\mathrm{Spin}(7) = \mathrm{SO}(6)/\,\mathrm{SU}(3) \text{ (with double cover } \mathrm{SO}(8)/\,\mathrm{SO}(7) = \mathrm{SU}(4)/\,\mathrm{SU}(3))$$
$$\mathrm{SO}(8)/\,\mathrm{Spin}(7) = \mathrm{SO}(5)/\,\mathrm{SU}(2) \text{ (with double cover } \mathrm{SO}(8)/\,\mathrm{SO}(7) = \mathrm{Sp}(2)/\,\mathrm{Sp}(1)).$$

We include some intersections of subgroups of SO(8) and related equalities:

$$\mathrm{G}_2 = \mathrm{SO}(7) \cap \mathrm{Spin}(7) \text{ implies } \mathrm{SO}(8)/\,\mathrm{SO}(7) = \mathrm{Spin}(7)/\,\mathrm{G}_2;$$
$$\mathrm{Sp}(1)\,\mathrm{Sp}(1) = \mathrm{SO}(7) \cap \mathrm{Sp}(2)\,\mathrm{Sp}(1) \text{ implies } \mathrm{SO}(8)/\,\mathrm{SO}(7) = \mathrm{Sp}(2)\,\mathrm{Sp}(1)/\,\mathrm{Sp}(1)\,\mathrm{Sp}(1);$$
$$\mathrm{SU}(3) = \mathrm{SO}(7) \cap \mathrm{SU}(4) \text{ implies } \mathrm{SO}(8)/\,\mathrm{SO}(7) = \mathrm{SU}(4)/\,\mathrm{SU}(3).$$

Here are the non-symmetric homogeneous spaces on Oniščik's list [O1]:

$$\begin{aligned}
\mathrm{SO}(7)/\,\mathrm{SO}(5) &= \mathrm{G}_2\,/\,\mathrm{SU}(2) && = V_2(\mathbb{R}^7)\\
\mathrm{SO}(8)/\,\mathrm{SO}(6) &= \mathrm{Spin}(7)/\,\mathrm{SU}(3) && = V_2(\mathbb{R}^8)\\
\mathrm{SO}(8)/\,\mathrm{SO}(5) &= \mathrm{Spin}(7)/\,\mathrm{SU}(2) && = V_3(\mathbb{R}^8)\\
\mathrm{SO}(8)/\,\mathrm{SO}(2)\,\mathrm{SO}(5) &= \mathrm{Spin}(7)/\,\mathrm{SO}(2)\,\mathrm{SU}(2)\\
\mathrm{SO}(16)/\,\mathrm{Spin}(9) &= \mathrm{SO}(15)/\,\mathrm{Spin}(7)\\
\mathrm{SO}(2n)/\,\mathrm{SU}(n) &= \mathrm{SO}(2n-1)/\,\mathrm{SU}(n-1)\\
\mathrm{SO}(4n)/\,\mathrm{Sp}(n) &= \mathrm{SO}(4n-1)/\,\mathrm{Sp}(n-1)\\
\mathrm{SO}(4n)/\,\mathrm{Sp}(n)\,\mathrm{U}(1) &= \mathrm{SO}(4n-1)/\,\mathrm{Sp}(n-1)\,\mathrm{U}(1)\\
\mathrm{SO}(4n)/\,\mathrm{Sp}(n)\,\mathrm{Sp}(1) &= \mathrm{SO}(4n-1)/\,\mathrm{Sp}(n-1)\,\mathrm{Sp}(1).
\end{aligned}$$

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