HOMOGENEOUS EINSTEIN METRICS

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We consider homogeneous Einstein metrics on symmetric spaces and we describe their geometry. For compact irreducible symmetric spaces with rank$(M) > 1$, not isometric to a Lie group, we classify all homogeneous Einstein metrics. Whenever there exists a closed proper subgroup $G$ of $\text{Isom}(M)$ acting transitively on $M$ we find all $G$-homogeneous (non-symmetric) Einstein metrics on $M$. We next examine three reducible symmetric spaces on which a simple Lie group $G$ acts transitively, and we describe all $G$-invariant Einstein metrics. Finally, we discuss the $\text{SO}(n+1)$-invariant Einstein metrics on the Stiefel manifold $\text{SO}(n+1)/\text{SO}(n-1) \cong V_2(\mathbb{R}^{n+1})$. 
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We study homogeneous Einstein metrics on compact symmetric spaces and Stiefel manifolds. In chapter 1 we look at compact symmetric spaces presented homogeneously, i.e. as $M = G/H$, where $G = \text{Isom}_0(M)$ is simple, and we consider the cases where there exists a closed subgroup $G' \subset G$ which acts transitively on $M$. Denote by $H'$ the isotropy subgroup in $G'$, then $M = G'/H'$. Since $G'$ is smaller than $G$, we expect more $G'$-invariant metrics on $M$ than $G$-invariant metrics, and thus we can hope for non-symmetric $G'$-invariant Einstein metrics on our symmetric space $M$. We find the following.

**Lemma 1.1.** Let $M$ be a compact irreducible symmetric space of rank $> 1$, $M$ not isometric to a compact Lie group with biinvariant metric. Let $G = \text{Isom}_0(M)$, and $M = G/H$. Then there exists a proper subgroup $G' \subset G$ acting transitively on $M$ $\iff$

1. $G = \text{SO}(2n)$, $H = \text{U}(n)$, $G' = \text{SO}(2n - 1)$, $H' = \text{U}(n - 1)$ ($n \geq 4$);
2. $G = \text{SU}(2n)$, $H = \text{Sp}(n)$, $G' = \text{SU}(2n - 1)$, $H' = \text{Sp}(n - 1)$ ($n \geq 3$);
3. $G = \text{SO}(7)$, $H = \text{SO}(2) \text{SO}(5)$, $G' = \text{G}_2$, $H' = \text{U}(2)$ ($\text{U}(2) \subset \text{SU}(3)$);
4. $G = \text{SO}(8)$, $H = \text{SO}(3) \text{SO}(5)$, $G' = \text{Spin}(7)$, $H' = \text{SO}(4)$ ($\text{SO}(4) \subset \text{G}_2$).

**Theorem 1.2.** Among the compact irreducible symmetric spaces of rank $> 1$, not isometric to a Lie group with a biinvariant metric, only $\text{G}_2^+(\mathbb{R}^7)$, $\text{G}_3^+(\mathbb{R}^8)$, and $\text{SO}(2n)/\text{U}(n)$, for $n \geq 4$, carry non-symmetric homogeneous Einstein metrics (cases (3), (4), and (1), respectively). The Grassmannians $\text{G}_2^+(\mathbb{R}^7)$ and $\text{G}_3^+(\mathbb{R}^8)$ each carry two and $\text{SO}(2n)/\text{U}(n)$ carries one; the only homogeneous Einstein metric on $\text{SU}(2n)/\text{Sp}(n)$ is the symmetric metric.

The analogous results for symmetric spaces of rank 1 were studied in [Z] and for compact Lie groups with biinvariant metrics in [DA-Z].
Remark. The following result should be true for all compact irreducible symmetric spaces $M$, up to diffeomorphism: If $G$ is a compact connected Lie group acting transitively and effectively on $M$, then $G$ is conjugate to a subgroup of $\text{Isom}_0(M)$. This would imply that Theorem 1.2 classifies all homogeneous Einstein metrics on compact irreducible symmetric spaces of rank $> 1$, not isometric to a Lie group. Such a result is well known for rank 1 symmetric spaces, but does not seem to be known for all symmetric spaces of rank $> 1$. Partial results can be found in [O2], [O3], [S], [T].

For example, in [O3] Oniščik showed that if $M$ is diffeomorphic to $G_{2k}(\mathbb{R}^n)$ for $n$ even, $n > 5$, $1 < k < \frac{n-2}{2}$, or for $n$ odd, $2 < k < \frac{n-3}{2}$, and if a compact connected Lie group $G$ acts transitively and effectively on $M$, then $G$ is conjugate to $\text{SO}(n)$ with the standard action. Tsukada proved in [T] that if $M$ is diffeomorphic to $G_{2k+1}(\mathbb{R}^{2n})$, and $G$ is compact, connected, and simple, then if $G$ acts transitively and effectively, the action of $G$ is conjugate to the standard action.

In chapter 2 we consider three products of irreducible symmetric spaces on which simple compact Lie groups act transitively. We consider their homogeneous metrics to find the Einstein metrics among them.

**Proposition 2.1.** (1) $S^7 \times S^7 = \text{Spin}(8)/G_2$ carries two distinct Spin(8)-invariant Einstein metrics: the product metric and the metric induced by the Killing form.  
(2) $S^7 \times S^6 = \text{Spin}(7)/\text{SU}(3)$ carries three distinct Spin(7)-invariant Einstein metrics: the product metric and two others.  
(3) $S^7 \times G_2^+(\mathbb{R}^8) = \text{Spin}(8)/\text{U}(3)$ carries two distinct Spin(8)-invariant Einstein metrics: the product metric and one other.
For these examples, we could not consider diagonal metrics only; it was necessary to develop a scalar curvature formula which does not assume one has an orthonormal, or even orthogonal, basis. Few examples of this type, where diagonal metrics do not represent all homogeneous metrics, even up to isometry, have been previously examined.

In chapter 3 we look at the Stiefel manifold $V_2(\mathbb{R}^{n+1})$ of two-flags in $(n+1)$-dimensional Euclidean space. (One can also think of this space as the unit tangent bundle of the $n$-sphere $S^n$.) This can be presented homogeneously as $\text{SO}(n+1)/\text{SO}(n-1)$, and we classify the $\text{SO}(n+1)$-invariant Einstein metrics.

Chapter 0: Preliminaries

A Riemannian manifold $(M, g)$ is Einstein if $\text{Ric}(X, Y) = \lambda g(X, Y)$ for some constant $\lambda$, for all vector fields $X, Y$. We say a Riemannian manifold $(M, g)$ is a symmetric space if for all $p \in M$ there exists an isometry $\sigma_p : M \to M$ such that $\sigma_p(p) = p$ and $(d\sigma_p)_p = -\text{Id}$. Symmetric spaces make up a class of manifolds which includes spheres, projective spaces, and Grassmannians; their geometry is well understood. In fact, every symmetric space is homogeneous.

A manifold $M$ is defined to be $G$-homogeneous if we have a Riemannian metric $g$ and a closed subgroup $G \subset \text{Isom}(M, g)$ such that for any $p$ and $q \in M$, there exists a $g \in G$ with $g(p) = q$. We write $H_p = \{g \in G \mid g(p) = p\}$, called the isotropy subgroup corresponding to $p$. Notice $H_p$ is compact, since $H_p \subset \text{O}(T_pM)$. Via the map $g \mapsto g(p)$ we identify the two manifolds $G/H_p$ and $M$. Any two isotropy subgroups $H_p$ and $H_q$ are conjugate: if $q = g(p)$, then $g^{-1}H_qg = H_p$, hence we will usually suppress the point $p$.

Given a homogeneous manifold $G/H$, where $G$ is compact and $H$ is closed, what metrics can we put on $G/H$ so that $G$ acts by isometries?
Just as a left-invariant metric on a Lie group is determined by any inner product on its Lie algebra, a $G$-invariant metric on $G/H$ is determined by an inner product on $\mathfrak{g}/\mathfrak{h} \cong T_{[H]}(G/H)$, with the additional requirement that the inner product be $\text{Ad}(H)$-invariant. We identify the quotient $\mathfrak{g}/\mathfrak{h}$ with an $\text{Ad}(H)$-invariant complement $\mathfrak{p}$ to $\mathfrak{h}$ in $\mathfrak{g}$; compactness of $H$ guarantees such a $\mathfrak{p}$ exists. If $\mathfrak{g}$ is semisimple then the Killing form is $\text{Ad}(H)$-invariant and we can use it to define $\mathfrak{p} = \mathfrak{h}^\perp$. We want to consider all $\text{Ad}(H)$-invariant inner products on $\mathfrak{p}$.

A homogeneous space $M = G/H$ is said to be isotropy irreducible if the isotropy action, denoted $\chi : H \to \text{GL}(T_pM)$, or equivalently $\text{Ad} : H \to \text{GL}(\mathfrak{p})$, is an irreducible representation of $H$. When this is the case, the $G$-invariant metric on $G/H$ is unique, up to scaling, and it is Einstein. When $G/H$ is a symmetric space with $G$ simple and $G = \text{Isom}_0(G/H)$, then $G/H$ is an irreducible symmetric space. (In fact, the only irreducible symmetric space with $G$ not simple is $(K \times K)/\Delta K$, for $K$ a compact simple Lie group, and $(K \times K)/\Delta K$ with the symmetric metric is isometric to $K$ with a biinvariant metric.)

In 1962 A.L. Oniščik classified all simple compact Lie algebras $\mathfrak{g}$ with Lie subalgebras $\mathfrak{g}'$ and $\mathfrak{g}''$, such that $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}''$. In terms of transitive group actions, let $G$ be the simply connected compact Lie group corresponding to $\mathfrak{g}$ and let $G'$, $G''$ be Lie subgroups corresponding to $\mathfrak{g}'$, $\mathfrak{g}''$, respectively, then $G/G' = G''/(G' \cap G'')$ and $G/G'' = G'/((G' \cap G'')$. When $G/G'$ or $G/G''$ is a symmetric space, Oniščik’s list tells us when a subgroup of $G$ still acts transitively. Comparing this with the classification of compact irreducible symmetric spaces one obtains Lemma 1.1. Here are the symmetric spaces on his list [O1]: (See the
appendix of this paper for the non-symmetric homogeneous spaces on his list.

\[
\begin{align*}
\text{SO}(2n)/\text{SO}(2n-1) &= U(n)/U(n-1) = S^{2n-1} \\
\text{SO}(2n)/\text{SO}(2n-1) &= \text{SU}(n)/\text{SU}(n-1) = S^{2n-1} \\
\text{SO}(4n)/\text{SO}(4n-1) &= \text{Sp}(n)/\text{Sp}(n-1) = S^{4n-1} \\
\text{SO}(4n)/\text{SO}(4n-1) &= \text{Sp}(n)\text{U}(1)/\text{Sp}(n-1)\text{U}(1) = S^{4n-1} \\
\text{SO}(4n)/\text{SO}(4n-1) &= \text{Sp}(n)\text{Sp}(1)/\text{Sp}(n-1)\text{Sp}(1) = S^{4n-1} \\
\text{SO}(7)/\text{SO}(6) &= G_2/\text{SU}(3) = S^6 \\
\text{SO}(8)/\text{SO}(7) &= \text{Spin}(7)/G_2 = S^7 \\
\text{SO}(16)/\text{SO}(15) &= \text{Spin}(9)/\text{Spin}(7) = S^{15} \\
\text{SU}(2n)/\text{U}(2n-1) &= \text{Sp}(n)/\text{Sp}(n-1)\text{U}(1) = \mathbb{C}P^{2n} \\
\text{SO}(2n)/\text{U}(n) &= \text{SO}(2n-1)/\text{U}(n-1) = \text{spec. orth. ex. str. on } \mathbb{R}^{2n} \\
\text{SU}(2n)/\text{Sp}(n) &= \text{SU}(2n-1)/\text{Sp}(n-1) = \text{spec. orth. quat. str. on } \mathbb{C}^{2n} \\
\text{SO}(7)/\text{SO}(2)\text{SO}(5) &= G_2/U(2) = G_2^{+}(\mathbb{R}^7) \\
\text{SO}(8)/\text{SO}(3)\text{SO}(5) &= \text{Spin}(7)/\text{SO}(4) = G_3^{+}(\mathbb{R}^8).
\end{align*}
\]

Each of these symmetric spaces in the left-hand presentation is an irreducible symmetric space. Up to scaling, each has exactly one Einstein metric, the symmetric metric, homogeneous with respect to the left-hand presentation. However, with respect to the right-hand presentation, only the sixth and seventh symmetric spaces are isotropy irreducible.

The first nine examples are discussed in [Z]. In this paper we consider the last four examples in the table. Of these, the first is originally described in [W-Z2]. On SU(2n)/Sp(n),
the only homogeneous Einstein metric is the original one. However, the last two spaces each carry two new Einstein metrics, homogeneous with respect to the smaller group.

For any homogeneous space \( M = G/H \), with \( g = h \oplus p \) on the Lie algebra level, we parametrize the space of \( G \)-invariant metrics on \( M \) by decomposing \( p \) into its \( \text{Ad}(H) \)-irreducible subspaces, \( p = p_1 \oplus p_2 \oplus \cdots \oplus p_k \). If the \( p_i \)'s are pairwise inequivalent representations, a \( G \)-homogeneous metric is determined by an inner product on \( p \) of the form \( \langle \cdot, \cdot \rangle = x_1 Q|_{p_1} \perp x_2 Q|_{p_2} \perp \cdots \perp x_k Q|_{p_k} \), for \( Q \) an \( \text{Ad}(H) \)-invariant inner product and \( x_i > 0 \) for all \( i \). If \( p_i \) and \( p_j \) are equivalent for some \( i \) and \( j \), then \( \langle p_i, p_j \rangle \) does not necessarily vanish. We discuss examples of this type in chapters 2 and 3.

Assume \( G/H \) is compact, and let \( S(g) \) denote the scalar curvature of \( g \). Einstein metrics are the critical points of the total scalar curvature functional

\[
T(g) = \int_M S(g) d\text{vol}_g
\]

on the space \( \mathcal{M} \) of Riemannian metrics of volume one [Ber], [H]. Let \( \mathcal{M}_G \) denote the set of all \( G \)-invariant metrics of volume one on \( M \). Notice that on \( \mathcal{M}_G \), \( T(g) \simeq S(g) \). Furthermore, critical points of \( T|_{\mathcal{M}_G} \) are precisely \( G \)-invariant Einstein metrics of volume one [Bes, p.121].

If for our homogeneous space \( G/H \), every homogeneous metric is diagonal, i.e., \( \langle \cdot, \cdot \rangle = x_1 Q|_{p_1} \perp x_2 Q|_{p_2} \perp \cdots \perp x_k Q|_{p_k} \), with \( x_i > 0 \) for all \( i \), then we use equation (1.3) for the scalar curvature given in [W-Z2].

\[
S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \binom{k}{i,j} \frac{x_k}{x_i x_j}.
\]
In their formula, for each $i$, $-B|_{p_i} = b_i Q|_{p_i}$, where $B$ denotes the Killing form, and $d_i = \text{dim}(p_i)$; the triple $\left( k_{ij} \right) = \sum Q([X_\alpha, X_\beta], X_\gamma)^2$, summed over $\{X_\alpha\}$, $\{X_\beta\}$, and $\{X_\gamma\}$: $Q$-orthonormal bases for $p_i$, $p_j$, and $p_k$, respectively. Notice $\left( k_{ij} \right)$ is symmetric in all three entries.
Let $\mathcal{M}_{G'}$ denote the space of $G'$-invariant metrics of volume one. Our results are summarized in the following table. In the next four sections we discuss each of the spaces in turn, proving Theorem 1.2.

<table>
<thead>
<tr>
<th>$G/H$</th>
<th>$G'/H'$</th>
<th>$\dim \mathcal{M}_{G'}$</th>
<th>no. Einstein</th>
</tr>
</thead>
<tbody>
<tr>
<td>SO($2n$)/U($n$)</td>
<td>SO($2n-1$)/U($n-1$)</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>SU($2n$)/Sp($n$)</td>
<td>SU($2n-1$)/Sp($n-1$)</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>SO($7$)/SO($2$)SO($5$)</td>
<td>G$_2$/U($2$)</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>SO($8$)/SO($3$)SO($5$)</td>
<td>Spin($7$)/SO($4$)</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

1.1. SO($2n$)/U($n$)

We begin with the symmetric space SO($2n$)/U($n$). Consider the space of orthogonal complex structures on $\mathbb{R}^{2n}$, and let $M_0$ be the connected component containing $J_0$, the complex structure represented by \[
\begin{pmatrix}
0 & \text{Id}_n \\
-\text{Id}_n & 0
\end{pmatrix}
\] with respect to the standard basis on $\mathbb{R}^{2n}$.

We will show that $M_0 \cong \text{SO}(2n)/\text{U}(n)$. Let $J \in M_0$. Since $J$ is an orthogonal complex structure $\|Jv\| = \|v\|$ and $J^2 = -\text{Id}$. We construct an orthonormal basis $\{v_\alpha\}$ for $\mathbb{R}^{2n}$ such that for $1 \leq i \leq n$, $Jv_i = v_{n+i}$ and $Jv_{n+i} = -Jv_i$. Let $v_1 = e_1$, and let $v_{n+1} = Jv_1$. We have $\langle v_1, Jv_1 \rangle = \langle Jv_1, J^2v_1 \rangle = -\langle v_1, Jv_1 \rangle$, hence $\{v_1, v_{n+1}\}$ is an orthonormal basis for a $J$-invariant subspace. Let $v_2$ be any unit vector in $\text{span}\{v_1, v_{n+1}\}^\perp$, and let $v_{n+2} = Jv_2$. Continue up to $v_n, v_{2n}$. With respect to the basis $\{v_\alpha\}$, $J = \begin{pmatrix} 0 & \text{Id}_n \\ -\text{Id}_n & 0 \end{pmatrix}$. Let $P$ be the change of basis transformation from the standard basis to $\{v_\alpha\}$, then $J = PJ_0P^{-1}$. The hypothesis that $J$ be in the connected component containing $J_0$ corresponds exactly to the fact that $P$ must be in SO($2n$). Via conjugation, SO($2n$) acts transitively on $M_0$. 


The isotropy subgroup of $J_0$ is the set of all $P \in \text{SO}(2n)$ such that $PJ_0 = J_0P$. If we identify $\mathbb{R}^n \oplus \mathbb{R}^n \cong \mathbb{C}^n$ via $(u, v) \mapsto u + iv$ then $J_0$ is multiplication by $i$ and hence $PJ_0 = J_0P$ implies $P \in \text{SO}(2n) \cap \text{GL}(n, \mathbb{C}) = \text{U}(n)$. Thus $M_0 \cong \text{SO}(2n)/\text{U}(n)$, where we have $\text{U}(n)$ embedded in $\text{SO}(2n)$ in the following way: $A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. It is well known that $\text{SO}(2n)/\text{U}(n)$ is an irreducible symmetric space [W, p.287]. Let $\mu_n$ denote the standard complex $n$-dimensional representation of $\text{U}(n)$; the isotropy representation of $\text{U}(n)$ is $[\wedge^2 \mu_n]_{\mathbb{R}}$. Since $\wedge^2 \mu_n$ is unitary, $[\wedge^2 \mu_n]_{\mathbb{R}}$ is the irreducible real representation whose complexification is isomorphic to the direct sum of $\wedge^2 \mu_n$ and its dual.

If we look at the low dimensional examples, for $n \leq 4$, we find $\text{SO}(4)/\text{U}(2) = S^2$, $\text{SO}(6)/\text{U}(3) = \mathbb{C}P^3$, and $\text{SO}(8)/\text{U}(4) = G_2^+ (\mathbb{R}^8)$. For $n \geq 4$ the rank of the symmetric space is greater than one.

Notice that in our description above we let $v_1 = e_1$, so the subgroup $\text{SO}(2n - 1) \subset \text{SO}(2n)$ fixing span$\{e_1\}$ also acts transitively on $M_0$. The isotropy subgroup of $\text{SO}(2n - 1)$ corresponding to $J_0$ is $\text{U}(n - 1) \subset \text{SO}(2n - 2)$, where $\text{SO}(2n - 2)$ is the subgroup fixing $e_1$ and $e_{n+1}$.

On the Lie algebra we have, for $X, Y \in \mathfrak{gl}(n - 1, \mathbb{R})$,

\[ u(n - 1) \cong \left\{ \begin{pmatrix} X & 0 & -Y \\ 0 & 0 & 0 \\ Y & 0 & X \end{pmatrix} \middle| X = -X^t, \ Y = Y^t \right\} \subset \mathfrak{so}(2n - 1); \]

therefore, $\mathfrak{p} \cong \left\{ \begin{pmatrix} X & v & Y \\ -v^t & 0 & -w^t \\ Y & w & -X \end{pmatrix} \middle| X = -X^t, \ Y = -Y^t, \ v, w \in \mathbb{R}^{n-1} \right\}$.

We find that $\mathfrak{p}$ decomposes into the sum of two irreducible representations. In fact $\text{U}(n-1) \subset \text{SO}(2n - 2) \subset \text{SO}(2n - 1)$ gives rise to the following fibration:

\[ \text{SO}(2n - 2)/\text{U}(n - 1) \to \text{SO}(2n - 1)/\text{U}(n - 1) \to \text{SO}(2n - 1)/\text{SO}(2n - 2) \cong S^{2n-2}. \]
Both base and fibre are irreducible symmetric spaces. Let $p_1$ denote the Ad $U(n-1)$-invariant complement to $u(n-1)$ in $\mathfrak{so}(2n-2)$; $p_1$ corresponds to the tangent space of the fibre. This representation of $U(n-1)$ on $p_1$ is the irreducible isotropy representation of the fibre symmetric space, $[\wedge^2 \mu_{n-1}]_{\mathbb{R}}$ [W, p.287]. Let $p_2$ denote the Ad $SO(2n-2)$-invariant complement to $\mathfrak{so}(2n-2)$ in $\mathfrak{so}(2n-1)$; in our fibration $p_2$ corresponds to the tangent space of the base. The representation of $U(n-1)$ on $p_2$ is the restriction of the standard representation of $SO(2n-2)$ on $p_2 \cong \mathbb{R}^{2n-2}$ to $U(n-1)$, which is $[\mu_n]_{\mathbb{R}}$, again irreducible.

We have $p = p_1 \oplus p_2$.

The dimensions of $p_1$ and $p_2$ are $(n-1)(n-2)$ and $2(n-1)$, respectively. We see that $p_1$ and $p_2$ are clearly inequivalent representations of $U(n-1)$ for $n \neq 4$, and for $n = 4$, while $\mu_3$ and $\wedge^2 \mu_3$ are equivalent representations on SU(3), they are inequivalent on the center of U(3). We apply Schur’s lemma to know that $\langle p_1, p_2 \rangle$ and $\text{Ric}(p_1, p_2)$ must vanish. Thus any $SO(2n-1)$-homogeneous metric on $M_0$ must be of the form $\langle \cdot, \cdot \rangle = x_1 Q|_{p_1} \perp x_2 Q|_{p_2}$, where $Q(X,Y) = -\frac{1}{2} \text{tr}(XY)$ and $x_1, x_2 > 0$. We express the scalar curvature in terms of $x_1$ and $x_2$ using [W-Z2, (1.3)]. The equation is

$$S = \frac{1}{2} \sum_i d_i b_i - \frac{1}{4} \sum_{i,j,k} \binom{k}{ij} \frac{x_k}{x_i x_j}.$$ 

Since for $\mathfrak{so}(k)$, $-B(X,Y) = (k-2) \text{tr}(XY)$, we have $b_1 = b_2 = 2(2n-3)$. From the fibration it follows that $[p_1, p_1] \subset u(n-1)$, $[p_2, p_2] \subset u(n-1) \oplus p_1$, hence the only nonzero triple (up to rearrangements) is $\binom{1}{22}$.

Let $E_{ij}$ denote the skew-symmetric matrix in $\mathfrak{so}(2n-1)$ with 1 in the $ij^{th}$ entry and $-1$ in the $ji^{th}$ entry, and zeros everywhere else.

$$p_1 = \text{span}\{E_{ij} - E_{n+i,n+j}, E_{i,n+j} - E_{j,n+i} \mid 1 \leq i < j \leq n-1\}$$

$$p_2 = \text{span}\{-E_{in}, E_{n,n+i} \mid 1 \leq i \leq n-1\}.$$
We find \( \binom{1}{2} = 2(n-1)(n-2) \), and then we substitute into the scalar curvature equation to find \( S \). Let \( \tilde{S} \) be the equation for \( S \) with the boundary constraint that volume = 1:

\[
S = \frac{2(n-1)(n-2)^2}{x_1} + \frac{2(n-1)(2n-3)}{x_2} - \frac{(n-1)(n-2)x_1}{2x_2^2},
\]

\[
\tilde{S} = S - \lambda x_1^{(n-1)(n-2)} x_2^{2(n-1)} - 1).
\]

We find the partial derivatives of \( \tilde{S} \):

\[
\frac{\partial \tilde{S}}{\partial x_1} = -2(n-1)(n-2)^2 x_1^2 - \frac{(n-1)(n-2)}{2x_2^3} - (n-1)(n-2)\lambda x_1^{(n-1)(n-2)-1} x_2^{2(n-1)}
\]

\[
\frac{\partial \tilde{S}}{\partial x_2} = -2(n-1)(2n-3) x_2^2 + \frac{(n-1)(n-2)x_1}{x_2^3} - 2(n-1)\lambda x_1^{(n-1)(n-2)} x_2^{2n-3}.
\]

Setting both equations equal to zero is equivalent to the following equation:

\[
\frac{n-1}{2} x_1 + 2(n-2) x_2^2 x_1 = (2n-3)x_2.
\]

We find that the solutions are \( x_2 = \frac{1}{2} x_1 \), and \( x_2 = \frac{2(n-1)}{2(n-3)} x_1 \). The second solution is a (non-symmetric) SO(\( 2n-1 \))-invariant Einstein metric, discovered earlier in [W-Z2, Ex. 3.6]. The first solution is the SO(\( 2n \))-invariant symmetric metric, but this is not obvious until we see how to compare them.

Let \( \tilde{p} \) denote the \( Q \)-orthogonal complement to \( u(n) \) in \( \mathfrak{so}(2n) \):

\[
\tilde{p} = \text{span}\{E_{ij} - E_{n+i,n+j}, E_{i,n+j} - E_{j,n+i} \mid 1 \leq i < j \leq n\}.
\]

We must project \( p \) to \( \tilde{p} \). We take a basis element of \( p_1 \): \( \frac{1}{\sqrt{2}} (E_{ij} - E_{n+i,n+j}) \). Under the embedding of \( \mathfrak{so}(2n-1) \) in \( \mathfrak{so}(2n) \), \( \frac{1}{\sqrt{2}} (E_{ij} - E_{n+i,n+j}) \mapsto \frac{1}{\sqrt{2}} (E_{i+1,j+1} - E_{n+i+1,n+j+1}) \), already in \( \tilde{p} \); hence an element of norm = \( \sqrt{x_1} \) is sent to an element of norm = 1. A basis element of \( p_2 \) is \( -E_{in} \), which the embedding sends to \( -E_{i+1,n+1} \). Next we write \( -E_{i+1,n+1} \) as the sum of an element in \( u(n) \) and an element in \( \tilde{p} \):

\[
-E_{i+1,n+1} = -\frac{1}{2} (E_{i+1,n+1} + E_{1,n+i+1}) - \frac{1}{2} (E_{i+1,n+1} - E_{1,n+i+1}).
\]
This shows that an element of norm $= \sqrt{x_2}$ is sent to an element of norm $= \frac{1}{\sqrt{2}}$. The symmetric metric on $\text{SO}(2n) / \text{U}(n)$ is given by the restriction of $Q$ to $\tilde{p}$. It corresponds to $\frac{1}{2} = \frac{x_1}{x_2}$, i.e., $x_2 = \frac{1}{2} x_1$.

To see that these metrics are distinct, we can compare the scale-invariant product $(S)^\frac{n}{2} (V)^\frac{1}{2}$, where $S$ is the scalar curvature, $V$ is the volume, and $d$ is the dimension of $M$. The first metric has $S = \frac{2n(n-1)}{x_1}$ and $V = x_1^{(n-1)(n-2)} x_2^{2(n-1)}$, so that

$$(S)^{\frac{n(n-1)}{2}} (V)^{\frac{1}{2}} = 2^{\frac{(n-1)(n-2)}{2}} (n(n-1)^2)^{\frac{n(n-1)}{2}}.$$ 

The second metric has $S = \frac{2n(n-2)(n^2-n-1)}{(n-1)x_1}$, and $V = (\frac{n-1}{2(n-2)})^{2(n-1)} x_1^n x_2^{(n-1)}$, hence

$$(S)^{\frac{n(n-1)}{2}} (V)^{\frac{1}{2}} = \left(\frac{2n(n-2)(n^2-n-1)}{n-1}\right)^{\frac{n(n-1)}{2}} \left(\frac{n-1}{2(n-2)}\right)^{n-1}.$$ 

1.2. $\text{SU}(2n)/\text{Sp}(n)$

Our next example is the symmetric space $M = \text{SU}(2n)/\text{Sp}(n)$, an analogue of the previous example. This is the set of special orthogonal quaternionic structures on $\mathbb{C}^{2n}$. We identify $\mathbb{R}^{4n} \cong \mathbb{C}^{2n}$ via a fixed orthogonal complex structure $I$ on $\mathbb{R}^{4n}$. An orthogonal quaternionic structure on $\mathbb{C}^{2n}$ is given by $J \in \text{SO}(4n)$ such that $J^2 = -\text{Id}$ and $IJ = -JI$.

As a map from $\mathbb{C}^{2n}$ to itself, $J$ is complex anti-linear, i.e., $J(\lambda v) = \bar{\lambda} J(v)$. We show that the set of all orthogonal quaternionic structures can be written homogeneously as $\text{U}(2n)/\text{Sp}(n)$, and we call the submanifold $M = \text{SU}(2n)/\text{Sp}(n)$ the set of special orthogonal quaternionic structures. We first observe that if $I = \begin{pmatrix} 0 & \text{Id}_{2n} \\ -\text{Id}_{2n} & 0 \end{pmatrix}$ and if we identify $\mathbb{C}^n \oplus \mathbb{C}^n = \mathbb{H}$ via $(u, v) \mapsto u + jv$, then multiplication by $j$ on $\mathbb{C}^{2n} = \mathbb{R}^{4n}$ becomes

$$J_0 = \begin{pmatrix} 0 & \text{Id}_n & 0 & 0 \\ -\text{Id}_n & 0 & 0 & 0 \\ 0 & 0 & \text{Id}_n & 0 \\ 0 & 0 & 0 & -\text{Id}_n \end{pmatrix},$$
which is the standard orthogonal quaternionic structure. We want to show that $U(2n)$ acts transitively on \( \{ J \in SO(4n) \mid J^2 = -\text{Id} \text{ and } IJ = -JI \} \), and the isotropy subgroup is $\text{Sp}(n)$. Since $U(2n) = \text{GL}(2n, \mathbb{C}) \cap SO(4n)$, $A$ is unitary when $A \in SO(4n)$ and $AI = IA$, and for $A$ unitary, $AJA^{-1}$ is a quaternionic structure if $J$ is: $(AJA^{-1})(AJA^{-1}) = -\text{Id}$ and $(AJA^{-1})I = AJA^{-1} = -AIJA^{-1} = -I(AJA^{-1})$. Furthermore, $IJ$ is also a quaternionic structure:

\[
(IJ)(IJ) = -I(IJ)J = -(-\text{Id})^2, \text{ and } (IJ)I = -I(IJ).
\]

Given an orthogonal quaternionic structure $J$, we construct a unitary basis of $\mathbb{C}^{2n}$ in which $J$ is represented by the matrix $J_0$. Let $v_1 = e_1$, the first element of the standard basis. Let $v_{n+1} = Jv_1$, $v_{2n+1} = Iv_1$, and $v_{3n+1} = IJv_1$. Clearly, \( \{ v_1, Jv_1, Iv_1, IJv_1 \} \) is an orthonormal basis for a 4-plane invariant under $I$, $J$, and $IJ$. Choose $v_2$ to be any unit vector in the orthogonal complement, and repeat the process above, continuing up to $v_n$, $v_{2n}$, $v_{3n}$, $v_{4n}$. Notice this is a unitary basis for $\mathbb{R}^{4n}$, since $v_{2n+i} = Iv_i$ for all $1 \leq i \leq 2n$.

The isotropy subgroup of $U(2n)$ corresponding to $J_0$ is all $A \in U(2n)$ such that $AJ_0 = J_0A$. i.e., $A$ commutes with $I$, $J$, and $IJ$: $A$ is quaternionic linear. We embed $\text{Sp}(n) \subset U(2n)$ via $A + jB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. The image of this embedding is contained in $SU(2n)$, and we now restrict ourselves to the orbit of $SU(2n)$, which is the symmetric space of special orthogonal quaternionic structures on $\mathbb{R}^{4n}$, or $SU(2n)/\text{Sp}(n)$. This symmetric space is irreducible; up to scaling the symmetric metric is the unique $SU(2n)$-invariant metric, and it is Einstein. Notice that when $n = 2$, $SU(4)/\text{Sp}(2) = S^5$; for $n \geq 3$ the rank of the symmetric space is greater than one.
Let $\tilde{p}$ be the orthogonal complement to $\mathfrak{sp}(n)$ in $\mathfrak{su}(2n)$ with respect to the inner product $Q(X,Y) = -\frac{1}{2} \text{tr}(XY)$.

We have $\mathfrak{su}(2n) = \left\{ \begin{pmatrix} X & Y \\ -Y^t & Z \end{pmatrix} \mid X, Z \in \mathfrak{u}(n), \ Y \in \mathfrak{gl}(n, \mathbb{C}), \ \text{tr} \ Z = - \text{tr} \ X \right\}$

and $\mathfrak{sp}(n) \cong \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \mid X \in \mathfrak{u}(n), \ Y = Y^t \in \mathfrak{gl}(n, \mathbb{C}) \right\}$,

hence $\tilde{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \mid X \in \mathfrak{su}(n), \ Y = Y^t \in \mathfrak{gl}(n, \mathbb{C}), \ \text{and} \ \text{tr} \ X = 0 \right\}$.

The isotropy representation of $\text{Sp}(n)$ on $\tilde{p}$ is $[\wedge^2 \nu_n - \text{Id}]_{\mathbb{R}}$, where $\nu_n$ is the standard representation of $\text{Sp}(n)$ on $\mathbb{H}^n \cong \mathbb{C}^{2n}$. (The representation $\wedge^2 \nu_n$ is the sum of a complex $(2n+1)(n-1)$-dimensional irreducible representation and a one-dimensional trivial representation. We denote by $\wedge^2 \nu_n - \text{Id}$ the non-trivial summand.) We write $[\wedge^2 \nu_n - \text{Id}]_{\mathbb{R}}$ for the real representation whose complexification is $\wedge^2 \nu_n - \text{Id}$.

The subgroup $\text{SU}(2n-1) \subset \text{SU}(2n)$ fixing $e_1$ acts transitively on $M$, just as in the previous example. The isotropy subgroup of $\text{SU}(2n-1)$ corresponding to $J_0$ is

$$H = \left\{ \begin{pmatrix} A & 0 & -B \\ 0 & 1 & 0 \\ B & 0 & A \end{pmatrix} \in \text{SU}(2n-1) \mid A + jB \in \text{Sp}(n-1, \mathbb{C}) \right\}$$

$= \text{Sp}(n-1) \subset \text{SU}(2n-2)$ fixing $e_{2n+1}$. In $\mathfrak{su}(2n-1)$, for $X,Y \in \mathfrak{gl}(n-1, \mathbb{C})$,

$$\mathfrak{h} = \mathfrak{sp}(n-1) = \left\{ \begin{pmatrix} X & 0 & -Y \\ 0 & 0 & 0 \\ Y & 0 & X \end{pmatrix} \mid X \in \mathfrak{u}(n-1), \ Y = Y^t \right\}.$$

We denote by $\mathfrak{p}$ the $Q$-orthogonal complement to $\mathfrak{sp}(n-1)$ in $\mathfrak{su}(2n-1)$:

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & -\bar{u}^t \\ u & z & v^t \\ Y & -\bar{v} & -\bar{X} \end{pmatrix} \mid X \in \mathfrak{u}(n-1), \ Y = -Y^t, \ u,v \in \mathbb{C}^{n-1}, \ z = -2 \text{tr} \ X \right\}.$$

We have the following fibration of our symmetric space, which tells us how to decompose $\mathfrak{p}$ into irreducible $\text{Ad} \text{Sp}(n-1)$-invariant subrepresentations:

$$\text{SU}(2n-2) / \text{Sp}(n-1) \to \text{SU}(2n-1) / \text{Sp}(n-1) \to \text{SU}(2n-1) / \text{SU}(2n-2) = S^{4n-3}.$$
From the fibration we see that \( \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}' \), where \( \mathfrak{p}_1 \) is tangent to the fibre and the \( \text{Ad} \, \text{Sp}(n-1) \) action on \( \mathfrak{p}_1 \) is \( [\wedge^2 \nu_{n-1} - \text{Id}]_\mathbb{R} \), and \( \dim(\mathfrak{p}_1) = (2n-1)(n-2) \). The subspace \( \mathfrak{p}' \) is tangent to the base, and \( \text{Ad} \, \text{SU}(2n-2) \) acts on \( \mathfrak{p}' \) by \( [\mu_{2n-2}]_\mathbb{R} \oplus \text{Id} \), which when restricted to \( \text{Sp}(n-1) \) is \( [\nu_{n-1}]_\mathbb{R} \oplus \text{Id} \). That is, \( \mathfrak{p}' = \mathfrak{p}_2 \oplus \mathfrak{p}_3 \); \( \dim(\mathfrak{p}_2) = 4(n-1) \) and \( \dim(\mathfrak{p}_3) = 1 \) (the \( \text{Ad} \, \text{Sp}(n-1) \) action on \( \mathfrak{p}_3 \) is trivial). The set of elements listed below gives a \( \mathbb{Q} \)-orthogonal basis for \( \mathfrak{p} \). We write \( E_{ij} \) for the skew-symmetric \((2n-1) \times (2n-1)\) matrix with 1 in the \( ij \)th entry and \(-1\) in the \( ji \)th entry, and zeros elsewhere. We denote by \( F_{ij} \) the symmetric \((2n-1) \times (2n-1)\) matrix with 1 in both the \( ij \)th and \( ji \)th entries.

\[
\mathfrak{p}_1 = \text{span}\{ (E_{kl} - E_{n+k,n+l}), i(F_{kl} + F_{n+k,n+l}), (E_{k,n+l} - E_{l,n+k}), i(F_{k,n+l} - F_{l,n+k}) \mid 1 \leq k < l \leq n-1 \} \\
\oplus \text{span}\{ i(F_{kk} - F_{n-1,n-1} + F_{n+k,n+k} - F_{2n-1,2n-1}) \mid 1 \leq k < n-1 \}
\]

\[
\mathfrak{p}_2 = \text{span}\{ E_{nk}, iF_{nk}, E_{n+k}, iF_{n,n+k} \mid 1 \leq k \leq n-1 \}
\]

\[
\mathfrak{p}_3 = \text{span}\{ \text{diag}(\eta, \ldots, \eta, -2(n-1)\eta, \eta, \ldots, \eta) \}, \text{where } \eta = \frac{i}{\sqrt{(2n-1)(n-1)}}.
\]

Since \( \mathfrak{p}_1 \), \( \mathfrak{p}_2 \), and \( \mathfrak{p}_3 \) are inequivalent irreducible representations of \( \text{Sp}(n-1) \), any \( \text{SU}(2n-1) \)-invariant metric on \( M \) must take the form

\[
\langle \cdot, \cdot \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \perp x_3 Q|_{\mathfrak{p}_3}, \text{ with } x_i > 0 \text{ for } i = 1, 2, 3.
\]

To find all \( \text{SU}(2n-1) \)-invariant Einstein metrics on \( M \), we solve for the critical points of the scalar curvature equation in terms of \( x_1, x_2, \) and \( x_3 \) (restricting to unit volume). As in the previous case we use the formula given in [W-Z2, (1.3)]:

\[
S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \binom{k}{ij} \frac{x_k}{x_i x_j}.
\]
In our example we have $b_i = 4(2n - 1)$ for all $i$, since for $\mathfrak{su}(k)$, $-B(X, Y) = 2k \operatorname{tr}(XY)$.

And $d_1 = (2n - 1)(n - 1)$, $d_2 = 4(n - 1)$, $d_3 = 1$. We find that

$$[p_1, p_1] \subset \mathfrak{sp}(n - 1), \quad [p_1, p_2] \subset p_2,$$

$$[p_2, p_2] \subset \mathfrak{sp}(n - 1) + p_1 + p_3, \quad [p_1, p_3] = 0,$$

$$[p_3, p_3] = 0, \quad [p_2, p_3] \subset p_2.$$

Therefore $(\frac{1}{2}, 2)$, $(\frac{3}{2}, 2)$ (and rearrangements) are the only nonzero triples. We compute $(\frac{1}{2}, 2) = 4(2n - 1)(n - 2)$ and $(\frac{3}{2}, 2) = 4(2n - 1)$. We now have the equation for the scalar curvature of $M$ in terms of $x_1$, $x_2$, and $x_3$.

$$S = (2n - 1) \left( \frac{4(n - 1)(n - 2)}{x_1} + \frac{8(n - 1)}{x_2} - \frac{x_3}{x_2^2} - \frac{(n - 2)x_1}{x_1^2} \right).$$

We normalize for volume 1 metrics: $\tilde{S} = S - \lambda(x_1^{d_1}x_2^{d_2}x_3 - 1)$.

$$\frac{\partial \tilde{S}}{\partial x_1} = -\frac{4(2n - 1)(n - 1)(n - 2)}{x_1^2} - \frac{(2n - 1)(n - 2)}{x_2^2} - (2n - 1)(n - 2)\lambda x_1^{d_1 - 1}x_2^{d_2}x_3$$

$$\frac{\partial \tilde{S}}{\partial x_2} = \frac{8(2n - 1)(n - 1)}{x_2^2} + \frac{2(2n - 1)((n - 2)x_1 + x_3)}{x_3^2} - 4(n - 1)\lambda x_1^{d_1}x_2^{d_2 - 1}x_3$$

$$\frac{\partial \tilde{S}}{\partial x_3} = -\frac{(2n - 1)}{x_2^2} - \lambda x_1^{d_1}x_2^{d_2}.$$

Setting $\frac{\partial \tilde{S}}{\partial x_1} = \frac{\partial \tilde{S}}{\partial x_2} = \frac{\partial \tilde{S}}{\partial x_3} = 0$ simultaneously is equivalent to

$$4(n - 1)x_2^2 + x_1^2 = (2n - 1)x_1x_3 = 2(2n - 1)x_1x_2 - \frac{2n - 1}{2n - 2}((n - 2)x_1^2 + x_1x_3).$$

There is only one solution; it is $x_2 = \frac{1}{2}x_1$ and $x_3 = \frac{n}{2n - 1}x_1$, unique up to scaling. This is not a new metric, rather it is the symmetric metric, which we knew must solve our equations.

(It is $\text{SU}(2n)$-invariant, hence $\text{SU}(2n - 1)$-invariant.) Thus the only homogeneous Einstein metric on $M = \text{SU}(2n - 1)/\text{Sp}(n - 1) \cong \text{SU}(2n)/\text{Sp}(n)$ is the symmetric metric.
The Grassmann manifold of oriented two-planes through the origin in $\mathbb{R}^7$ is generally written homogeneously $G_2^+ (\mathbb{R}^7) \cong \text{SO}(7)/\text{SO}(2) \text{SO}(5)$. It is irreducible: the symmetric metric is not only Einstein, it is the only $\text{SO}(7)$-invariant metric. We will show that this Grassmannian manifold can also be written homogeneously as $G_2 / \text{U}(2)$ and we find it carries two non-symmetric $G_2$-invariant Einstein metrics.

First we must see how $G_2 \subset \text{SO}(7)$, following [M, p.190]. We identify $\mathbb{R}^8$ with the Cayley numbers, or Octonians, the normed division algebra $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon$. Then $G_2$ is the set of automorphisms of $\mathbb{O}$. Any automorphism of the Cayley numbers must take 1 to itself and must preserve the inner product, so elements of $G_2$ also preserve $\text{Im}(\mathbb{O})$, the space of imaginary Cayley numbers, the orthogonal complement to 1. In this way we see $G_2 \subset \text{SO}(7)$. To see that $G_2$ acts transitively on $G_2^+ (\mathbb{R}^7)$, we use the following observation [M, p.186].

**Lemma 1.1.** Given three imaginary orthogonal unit octonians: $v_1, v_2, \text{ and } v_3 \in \{v_1, v_2, v_1v_2\}^\perp$, there exists a unique automorphism $A$ of $\mathbb{O}$ with $A(i) = v_1, A(j) = v_2, \text{ and } A(\varepsilon) = v_3$.

Using the lemma we take any $P = \text{span}\{v, w\}$ to the oriented two-plane $P_0 = \text{span}\{i, j\}$, where we take $v$ and $w$ an orthonormal basis for $P$. We must find the isotropy subgroup fixing $P_0$. It will be useful to recall the following well known fact.

**Lemma 1.2.** The quotient $G_2 / \text{SU}(3) \cong S^6$.

**Proof.** By the previous lemma, $G_2$ acts transitively on $S^6(1) \subset \text{Im}(\mathbb{O})$. We need to show that the isotropy subgroup $H_v$ of $G_2$ corresponding to any $v \in S^6$ is $\text{SU}(3)$. We observe first
that $A(v) = v$ implies the map $L_v$ is a complex structure on $\mathbb{O}$ and on $V = \text{span}\{1, v\}^\perp$. This shows that $H_v \subset U(V) \cong U(3)$. Furthermore, $\dim(H_v) = 8$. Since $G_2$ and $S^6$ are connected and simply connected, $H_v$ is connected.

Consider the homomorphism $\text{det} : H_v \to S^1$; it must be either trivial or onto. If it is trivial, $H_v \cong SU(3)$. If it is onto, then let $H'$ denote the kernel. Then $H'$ is a normal subgroup of $H_v$ of dimension seven, and if $H_0'$ is the connected component of the identity, $\text{rank}(H_0') \leq \text{rank}(SU(3)) = 2$. But the only compact connected seven-dimensional Lie groups, up to finite cover, are $T^7$, $S^3 \times T^4$, and $S^3 \times S^3 \times S^1$, and each of these has rank $> 2$:

$$\text{rank}(T^7) = 7, \quad \text{rank}(S^3 \times T^4) = 5, \quad \text{rank}(S^3 \times S^3 \times S^1) = 3.$$  

This shows $H' = H_v$, and thus $H_v \cong SU(3).$ \hfill \Box

We are now ready to describe the isotropy subgroup $H$ corresponding to the oriented two-plane $P_0$. Since $G_2$ and $G_2^+(\mathbb{R}^7)$ are connected and simply connected, we know $H$ is connected. If we have $A \in G_2$ such that $A(P_0) = P_0$ (with orientation), then $A(i) = i \cos \theta - j \sin \theta$, $A(j) = i \sin \theta + j \cos \theta$, thus $A(k) = A(i)A(j) = k$. The isotropy subgroup $H \subset \{A \in G_2 \mid A(k) = k\} \cong SU(3)$, and since the $ij$-plane is a complex line with respect to our complex structure $L_k$, $H$ must preserve this complex line and the complex two-plane perpendicular to it. Hence $H \subset S(U(1)U(2)) \subset SU(3)$. A dimension count shows $H = S(U(1)U(2)) \cong U(2)$.

If we look on the Lie algebra level, and we take $\{i, j, k, \varepsilon, i\varepsilon, j\varepsilon, k\varepsilon\}$ as our basis for $\text{Im}(\mathbb{O})$, then using the automorphism property we can write $g_2 \subset \mathfrak{so}(7)$ in the following way:

$$g_2 = \text{span}\{E_{12} + E_{56}, E_{47} + E_{56}, E_{45} + E_{67}, E_{46} - E_{57}, E_{46} + E_{57} + 2E_{13},$$

$$E_{67} - E_{45} + 2E_{23}, E_{14} - E_{27} - 2E_{36}, E_{15} + E_{26} - 2E_{37}, E_{16} - E_{25} + 2E_{34},$$

$$\ldots\}.$$
We use the inner product on $g_2$ given by 

$$Q(X, Y) = -\frac{1}{2} \text{tr}(XY),$$

in which the basis for $g_2$ above is orthogonal. The subalgebra $\mathfrak{h}$ corresponds to $H \cong U(2)$:

$$u(2) \cong \text{span}\{2E_{12} + E_{56} - E_{47}, E_{47} + E_{56}, E_{45} + E_{67}, E_{46} - E_{57}\}.$$

The isotropy representation of $U(2)$ is the action of $\text{Ad} \, U(2)$ on $\mathfrak{p}$, the $Q$-orthogonal complement of $\mathfrak{u}(2)$ in $g_2$. We can use the following fibrations to decompose $\mathfrak{p}$ into its irreducible $\text{Ad} \, U(2)$ representations: First,

$$\mathbb{C}P^2 \cong SU(3) / U(2) \rightarrow G_2 / U(2) \rightarrow G_2 / SU(3) \cong S^6.$$

The tangent space to the base is isomorphic to $[\mu_3]_\mathbb{R}$; when restricted to $U(2)$ it gives $[(\mu_1 \otimes \text{Id}) \oplus (\text{Id} \otimes \mu_2)]_\mathbb{R} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. Thus, $\mathfrak{p}_1 = [\mu_1 \otimes \text{Id}]_\mathbb{R}$, $\mathfrak{p}_2 = [\text{Id} \otimes \mu_2]_\mathbb{R}$. The tangent space to the fibre is $\mathfrak{p}_3 = [\mu_1 \otimes \mu_2]_\mathbb{R}$. We have $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$.

$$\mathfrak{p}_1 = \text{span}\{E_{46} + E_{57} + 2E_{13}, E_{67} - E_{45} + 2E_{23}\},$$

$$\mathfrak{p}_2 = \text{span}\{E_{14} - E_{27} - 2E_{36}, E_{15} + E_{26} - 2E_{37}, E_{16} - E_{25} + 2E_{34}, E_{17} + E_{24} + 2E_{35}\},$$

$$\mathfrak{p}_3 = \text{span}\{E_{14} + E_{27}, E_{26} - E_{15}, E_{16} + E_{25}, E_{24} - E_{17}\}.$$

We have a second fibration of our manifold: we claim $U(2) \subset SO(4) \subset G_2$. Before showing this, we briefly discuss the embedding $SO(4) \subset G_2$ and the irreducible symmetric space $G_2 / SO(4)$.

**Lemma 1.3.** *The quotient $G_2 / SO(4)$ is the space of quaternionic subalgebras of the Cayley numbers, $\mathcal{O}$.***
Proof. We have $\text{SO}(4) \cong (\text{Sp}(1) \times \text{Sp}(1))/\{(q_1, q_2) \cong (-q_1, -q_2)\}$, and it acts on $\mathbb{H} \cong \mathbb{H} \oplus \mathbb{H} \varepsilon$ by
\[
(q_1, q_2) : a + b \varepsilon \mapsto q_1 a \bar{q}_1 + (q_2 b \bar{q}_1) \varepsilon.
\]
A calculation shows that $\text{SO}(4) \subset G_2$, and this embedding of $\text{SO}(4)$ in $G_2$ can also be described as the subgroup of $G_2$ which leaves the subalgebra $\mathbb{H} \cong \text{span}\{1, i, j, k\}$ invariant.

\[
\square
\]

Since $\text{U}(2)$ is the subgroup of $G_2$ preserving the plane spanned by $i$ and $j$, elements of $\text{U}(2)$ take 1 to itself and $k$ to itself, hence they preserve $\text{span}\{1, i, j, k\}$. This also shows $\text{U}(2) \subset \text{SO}(4) \cap \text{SU}(3)$. We also have $\text{SO}(4) \cap \text{SU}(3) \subset \text{U}(2)$: under $G_2$, 1 $\mapsto$ 1; under $\text{SO}(4)$, $\text{span}\{1, i, j, k\} \mapsto \text{span}\{1, i, j, k\}$; under $\text{SU}(3)$, $k \mapsto k$. Thus $\text{SO}(4) \cap \text{SU}(3) = \text{U}(2)$. Our second fibration is
\[
\text{SO}(4)/\text{U}(2) \to G_2/\text{U}(2) \to G_2/\text{SO}(4).
\]
Here $p_1$ is tangent to the fibre, $p_2$ and $p_3$ are tangent to the base: From the two fibrations we obtain the following Lie bracket relations among the $p_i$'s: $[p_1, p_1] \subset u(2)$, $[p_1, p_2] \subset p_2 \oplus p_3$, $[p_2, p_2] \subset u(2) \oplus p_1$, and $[p_3, p_3] \subset u(2)$.

Since $p_1$, $p_2$, and $p_3$ are mutually inequivalent, any $G_2$-invariant metric on our space is of the form $\langle \ , \ \rangle = x_1 Q|_{p_1} \perp x_2 Q|_{p_2} \perp x_3 Q|_{p_3}$, with $x_i > 0$, for $i = 1, 2, 3$. As in the previous cases we can write the scalar curvature on $G_2^+(\mathbb{R}^7)$ as function of $x_1, x_2,$ and $x_3$ via the formula given in [W-Z]:
\[
S = \frac{1}{2} \sum_i d_i b_i x_i - \frac{1}{4} \sum_{i,j,k} \binom{k}{ij} x_k x_i x_j.
\]
When we compute the non-zero Lie bracket relations between the $p_i$'s we find that $\binom{3}{12} = 4$ and $\binom{1}{22} = \frac{16}{3}$. Also we compute (using the basis elements for $g_2$) $b_i = 8$ for each $i$, and of
course $d_1 = 2$, $d_2 = 4$, and $d_3 = 4$. This gives us

$$S = 8\left(\frac{1}{x_1} + \frac{2}{x_2} + \frac{2}{x_3}\right) - \frac{4}{3}\left(\frac{x_1}{x_2^2} + \frac{2}{x_1}\right) - 2\left(\frac{x_1}{x_2x_3} + \frac{x_2}{x_1x_3} + \frac{x_3}{x_1x_2}\right).$$

Then $\tilde{S} = S - \lambda(x_1^2x_2^4x_3^4 - 1)$ includes our boundary condition, volume = 1. Einstein metrics on $G_2/U(2)$ will be critical points of $\tilde{S}$.

$$\frac{\partial \tilde{S}}{\partial x_1} = -\frac{16}{3x_1^2} - \frac{4}{3x_2^2} + 2\left(\frac{x_2}{x_1^2x_3} + \frac{x_3}{x_1^2x_2} - \frac{1}{x_2x_3}\right) - 2\lambda x_1x_2^4x_3^4,$$

$$\frac{\partial \tilde{S}}{\partial x_2} = -\frac{16}{x_2^3} + \frac{8x_1}{3x_2^3} + 2\left(\frac{x_1}{x_2^2x_3} + \frac{x_3}{x_2^2x_2} - \frac{1}{x_2x_3}\right) - 4\lambda x_1^2x_2^3x_3^4,$$

$$\frac{\partial \tilde{S}}{\partial x_3} = -\frac{16}{x_3^2} + 2\left(\frac{x_1}{x_2^2x_3} + \frac{x_2}{x_1^2x_3} - \frac{1}{x_1x_2}\right) - 4\lambda x_1^2x_2^4x_3^3.$$

Now we look for all solutions to $\frac{\partial \tilde{S}}{\partial x_1} = \frac{\partial \tilde{S}}{\partial x_2} = \frac{\partial \tilde{S}}{\partial x_3} = 0$. We solve these using Maple, and we find the following:

either $x_2 = \frac{1}{2}x_1$ and $x_3 = \frac{3}{2}x_1$

or $x_2 = \zeta x_1$ and $x_3 = \left(\frac{7}{120}\zeta^4 - \frac{7}{60}\zeta^3 - \frac{151}{96}\zeta^2 + \frac{39}{10}\zeta - \frac{21}{40}\right)x_1$,

where $\zeta$ is a root of $56\zeta^5 - 532\zeta^4 + 1570\zeta^3 - 1891\zeta^2 + 776\zeta - 60 = 0$.

We need both a positive solution to this polynomial in order for $x_2 > 0$, and also

$$\left(\frac{7}{120}\zeta^4 - \frac{7}{60}\zeta^3 - \frac{151}{96}\zeta^2 + \frac{39}{10}\zeta - \frac{21}{40}\right) > 0,$$

so that $x_3 > 0$.

There are exactly three real solutions to the quintic polynomial. We give the approximate values for $x_2$ and $x_3$, setting $x_1 = 1$:

$$x_2 = 0.09953 \quad x_3 = -0.15252$$

$$x_2 = 0.59713 \quad x_3 = 1.22554$$

$$x_2 = 5.35063 \quad x_3 = 5.25153$$

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We must eliminate the first of the solutions from the quintic, since it gives a negative value for \( x_3 \). The solution \( x_2 = \frac{1}{2}x_1 \) and \( x_3 = \frac{3}{2}x_1 \) is the symmetric metric; that is, we will see that it is \( \text{SO}(7) \)-invariant, after projection. If we denote by \( \tilde{\mathfrak{p}} \) the \( Q \)-orthogonal complement to \( \mathfrak{so}(2) \oplus \mathfrak{so}(5) \) in \( \mathfrak{so}(7) \), then \( \tilde{\mathfrak{p}} = \{ E_{ij} \mid 1 \leq i \leq 2, \ 3 \leq j \leq 7 \} \). When we project \( \mathfrak{p} \) to \( \tilde{\mathfrak{p}} \), we see that in \( \mathfrak{p}_1 \), the basis element \( \frac{1}{\sqrt{6}}(2E_{13} + E_{46} + E_{57}) \) projects to \( \frac{2}{\sqrt{6}}E_{13} \), i.e., an element of norm \( \sqrt{x_1} \) projects to an element of norm \( \sqrt{\frac{2}{3}} \). In \( \mathfrak{p}_2 \), the basis element \( \frac{1}{\sqrt{6}}(2E_{36} - E_{14} + E_{27}) \) projects to \( \frac{1}{\sqrt{6}}(-E_{13} + E_{27}) \), so norm \( \sqrt{x_2} \) is projected to norm \( \sqrt{\frac{1}{3}} \). In \( \mathfrak{p}_3 \), basis element \( \frac{1}{\sqrt{2}}(E_{14} + E_{27}) \) is already an element of \( \tilde{\mathfrak{p}} \), so norm \( \sqrt{x_3} \) is projected to norm \( 1 \). This shows that the symmetric metric in our basis satisfies \( \frac{x_2}{x_1} = \frac{1}{2} \) and \( \frac{x_3}{x_1} = \frac{3}{2} \). Thus we end up with two new \( G_2 \)-homogeneous Einstein metrics on the Grassmannian \( G_2^+(\mathbb{R}^7) \).

\emph{Remark.} Notice none of these is a fibration metric, since a metric of the first fibration would require \( x_1 = x_2 \) and a metric of the second fibration would require \( x_2 = x_3 \). Many examples of Einstein metrics have been obtained by using fibrations over Einstein spaces and with Einstein fibres [Bes, ch.9]. In our example we have two fibrations; in each the fibre and base space are isotropy irreducible, so they must be Einstein. However, in each fibration the isotropy representation of the base, when restricted to \( \text{U}(2) \), decomposes into the sum of two irreducible subrepresentations. Recall \( Q(X, Y) = -\frac{1}{2} \text{tr } XY \) is our comparison metric. The Casimir constant corresponding to \( Q \) and to the restriction to \( \text{U}(2) \) of the isotropy representations for these homogeneous spaces differs on these two subrepresentations. This implies that the O’Neill tensor restricted to horizontal vectors is not a scalar multiple of \( Q \). Using Besse’s Proposition 9.70 [Bes, p.253] we see that therefore our fibrations cannot give rise to Einstein metrics.
To see that we have three non-isometric solutions we compare the scale-invariant product:
\[(S)^{\frac{d}{2}}(V)^{\frac{1}{2}},\]
where \(S\) is the scalar curvature and \(V\) is the volume, and \(d\) is the dimension of our homogeneous space, for our three metrics. If we let \(x_2 = \frac{1}{2}x_1\) and \(x_3 = \frac{3}{2}x_1\), we obtain \(S = \frac{100}{3x_1}\), and \(V = \frac{2^4}{3^2}x_1^{10}\), so \((S)^{\frac{5}{2}}(V)^{\frac{1}{2}} = \frac{2^{10}5}{3^3}\). This is approximately \(2.3148 \times 10^7\). The second solution gives \((S)^{\frac{5}{2}}(V)^{\frac{1}{2}} \approx 2.3044 \times 10^7\), and the third solution gives \((S)^{\frac{5}{2}}(V)^{\frac{1}{2}} \approx 1.5836 \times 10^7\).

We note that these have been found previously in [A] and [K]. They observe that one of the three metrics is Kähler and the other two are not Kähler for any complex structure on \(M\). Neither author observes that the Kähler Einstein metric is the symmetric metric.

### 1.4. \(G_3^+(\mathbb{R}^8)\)

We can write the Grassmannian manifold of oriented three-planes in \(\mathbb{R}^8\) homogeneously \(G_3^+(\mathbb{R}^8) \cong SO(8)/SO(3)SO(5)\), but we can also write it as \(G_3^+(\mathbb{R}^8) \cong \text{Spin}(7)/SO(4)\). With respect to \(SO(8)\), \(G_3^+(\mathbb{R}^8)\) is irreducible, and therefore the symmetric metric is Einstein and it is the unique \(SO(8)\)-invariant metric, up to scaling. However, we find there are two more \(\text{Spin}(7)\)-invariant Einstein metrics which are not symmetric.

We first describe how \(\text{Spin}(7)\) sits inside \(SO(8)\). We again identify \(\mathbb{R}^8\) with the Cayley numbers \(\mathbb{O}\); from Murakami [M] we know

\[
\text{Spin}(7) = \{ A \in \text{SO}(8) \mid \exists B \in \text{SO}(8) \text{ such that } B(x)A(y) = A(xy) \ \forall x, y \in \mathbb{O} \}.
\]

In this definition notice \(B(1) = 1\), so \(B \in \text{SO}(7)\) and if \(A\) corresponds to \(B\), \(-A\) corresponds to \(B\) as well, which shows \(\text{Spin}(7)\) is indeed a double cover of \(\text{SO}(7)\). We also remark that that \(\{ L_a \mid a \in \text{Im}(\mathbb{O}), |D| = |\mathbb{O}| \} \subset \text{Spin}(\mathbb{O})\). (The corresponding \(B\) is conjugation by \(a\).) We need to show that \(C_a(x)L_a(y) = L_a(xy)\) for any \(x, y \in \mathbb{O}\). Because \(a\) is a unit
imaginary octonian, \( a^{-1} = -a \), thus \( C_a(x)L_a(y) = (axa^{-1})(ay) = -(axa)(ay) \). Then the first Moufang identity tells us that \(-(axa)(ay) = -a(xaay)\), and using that \( aa = -1 \), we have \(-a(xaay)) = a(xy) = L_a(xy)\).

It is also convenient to identify two subgroups of \( \text{Spin}(7) \), they are \( G_2 \), the automorphisms of the Cayley numbers, and \( SU(4) \), complex linear maps with respect to \( L_i \). We see \( G_2 \subset \text{Spin}(7) \) by letting \( B = A \) in the definition of \( \text{Spin}(7) \). Murakami shows how to see that \( SU(4) \subset \text{Spin}(7) \) with the following lemma [M].

**Lemma 1.4.** In \( \text{SO}(8) \), \( U(4) = \{ A \in \text{SO}(8) \mid iA(x) = A(ix) \ \forall x, y \in \mathbb{O} \} \) and \( U(4) \cap \text{Spin}(7) = SU(4) \).

**Proof.** Let \( p^+ : \text{Spin}(7) \rightarrow \text{SO}(7) \) be the homomorphism sending \( A \mapsto B \), for \( A \) and \( B \) in the definition of \( \text{Spin}(7) \). For every \( A \in U(4) \cap \text{Spin}(7) \) we have \( B(i) = i \), so \( p^+(U(4) \cap \text{Spin}(7)) \subset \text{SO}(6) \). And for every \( B \in \text{SO}(6) \) \((B(i) = i)\), the corresponding \( A \) must be in \( U(4) \cap \text{Spin}(7) \), hence \( \text{SO}(6) \subset p^+(U(4) \cap \text{Spin}(7)) \). Furthermore, \( p^+ \) is a local isomorphism, thus \( U(4) \cap \text{Spin}(7) \) is a 15-dimensional connected Lie group with a simple Lie algebra. Observe that \( SU(4) \) is the commutator subgroup of \( U(4) \). Since its Lie algebra is simple, \( U(4) \cap \text{Spin}(7) \) is its own commutator subgroup, thus \( U(4) \cap \text{Spin}(7) \subset SU(4) \), and a dimension count tells us that these subgroups are equal. \( \square \)

We embed \( SU(4) \subset \text{SO}(8) \) via \( A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \). To embed \( SU(4) \) into \( \text{SO}(8) \) in this way, we want \( \mathbb{R}^4 \oplus \mathbb{R}^4 \cong \mathbb{C}^4 \) with \( (u, v) \mapsto u + iv \); this restricts our choice of ordered bases for \( \mathbb{O} \): we choose \( \{1, j, \varepsilon, j\varepsilon, i, k, i\varepsilon, -k\varepsilon\} \). The intersection of our two subgroups is \( G_2 \cap SU(4) = SU(3) \), and this time \( SU(3) \) in \( G_2 \) fixes \( i \) instead of \( k \).

We check that \( \text{Spin}(7) \) acts transitively on \( G_3^+ (\mathbb{R}^8) \): Let \( P_0 = \text{span}\{i, j, k\} \), an oriented three-plane through the origin. Let \( P \) be given by \( \text{span}\{v_1, v_2, v_3\} \), where \( v_1, v_2 \) and \( v_3 \) are
ordered orthonormal vectors. Without loss of generality we may assume $v_1, v_2 \in \text{Im}(\mathcal{O})$, since $P$ is a three-plane, so $\dim(P \cap \text{Im}(\mathcal{O})) \geq 3$. We know from the previous example that we can find an element $A \in G_2$ such that $A(i) = v_1, A(j) = v_2$. Let $x = A^{-1}(v_3)$. Observe that

\[ \langle x, i \rangle = \langle A^{-1}(v_3), i \rangle = \langle v_3, v_1 \rangle = 0 \]
\[ \langle x, j \rangle = \langle A^{-1}(v_3), j \rangle = \langle v_3, v_2 \rangle = 0. \]

We claim there exists $A' \in \text{Spin}(7)$ such that $A'(i) = i, A'(j) = j$, and $A'(k) = x$. This is because the subgroup of $\text{Spin}(7)$ fixing one unit octonian is conjugate to $G_2$, and the subgroup of $\text{Spin}(7)$ fixing two orthonormal octonians is conjugate to $\text{SU}(3)$, which acts transitively on $S^5(1) \subset \text{span}\{i, j\}^\perp$. The composition $A \circ A' \in \text{Spin}(7)$ is our map taking $i \mapsto v_1, j \mapsto v_2$, and $k \mapsto v_3$, so that $P_0$ goes to $P$, and this shows $\text{Spin}(7)$ acts transitively on $G_3^+(\mathbb{R}^8)$.

Next we must determine the isotropy subgroup $H$ of $P_0$. We claim that $H \subset G_2$. To see this, we note that if $A(P_0) = P_0$, then if $B$ is the element of $\text{SO}(7)$ in the definition of $\text{Spin}(7)$ corresponding to $A$, since $B(i)A(j) = A(k)$ we know that $B(i) \in \text{span}\{i, j, k\}$. Thus $A(1) = -B(i)A(i) \in \text{span}\{1, i, j, k\}$, and furthermore $A(1) \perp P_0$, so $A(1) = \pm 1$. Since $\text{Spin}(7)$ and $G_3^+(\mathbb{R}^8)$ are connected and simply connected, we know $H$ is connected, hence $A(1) = 1$. From the definition of $\text{Spin}(7)$ it follows that $A = B$ and this implies $H \subset G_2$. Furthermore, any element of $H$ takes 1 to itself and preserves the standard quaternionic subalgebra $\text{span}\{1, i, j, k\}$. Thus $H \subset \text{SO}(4) \subset G_2$, and by a dimension count $H = \text{SO}(4)$. 

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Now we are ready to find the isotropy representation. On the Lie algebra level we have

$$\text{spin}(7) = \text{span}\{E_{ij} + E_{4+i,4+j}, E_{i,4+j} + E_{j,4+i} \mid 1 \leq i < j \leq 4\}$$

$$\oplus \text{span}\{E_{i,4+i} - E_{48} \mid 1 \leq i \leq 3\}$$

$$\oplus \text{span}\{E_{27} - E_{45}, E_{23} + E_{58}, E_{24} - E_{57}, E_{28} + E_{35}, E_{56} - E_{78}, 2E_{25} - E_{38} + E_{47}\}.$$

The subalgebra corresponding to the isotropy subgroup is $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$:

$$\mathfrak{h} = \text{span}\{E_{37} - E_{48}, E_{34} + E_{78}, E_{38} + E_{47}\}$$

$$\oplus \text{span}\{2E_{56} + E_{34} - E_{78}, 2E_{26} - E_{48} - E_{37}, 2E_{25} - E_{38} + E_{47}\}.$$

Notice each copy of $\mathfrak{su}(2)$ is an ideal in $\mathfrak{h}$ and its basis vectors above are orthogonal with respect to the inner product on $\text{spin}(7)$ given by $Q(X, Y) = -\frac{1}{2}\text{tr}(XY)$. As usual we denote by $\mathfrak{p}$ the $Q$-orthogonal complement of $\mathfrak{h}$ in $\text{spin}(7)$.

There are two fibrations of our symmetric space $\text{Spin}(7)/\text{SO}(4)$. The first is

$$\text{G}_2/\text{SO}(4) \to \text{Spin}(7)/\text{SO}(4) \to \text{Spin}(7)/\text{G}_2 \cong S^7.$$ 

Let $\mathfrak{p}'$ be the subspace tangent to the fibre; let $\mathfrak{p}''$ be the subspace tangent to the base. Let $\theta_k$ denote the unique irreducible complex representation of $\mathfrak{su}(2)$ in $k$ dimensions; the first fibration tells us that $\mathfrak{p}' = [\theta_2 \hat{\otimes} \theta_4]_R$, since this is the representation of the symmetric space $\text{G}_2/\text{SO}(4)$ [W, p.287]. The isotropy representation of $\text{Spin}(7)/\text{G}_2$ is the seven-dimensional representation of $\text{G}_2 \subset \text{SO}(7)$. We restrict this representation to $\text{SO}(4)$, to see that $\mathfrak{p}'' = [\rho_3 \hat{\otimes} \text{Id}] \oplus [\theta_2 \hat{\otimes} \theta_2]_R$, where $\rho_3$ denotes the standard representation of $\mathfrak{so}(3)$ on $\mathbb{R}^3$. We let $\mathfrak{p}_1 = [\rho_3 \hat{\otimes} \text{Id}]$, and $\mathfrak{p}_2 = [\theta_2 \hat{\otimes} \theta_2]_R$, and $\mathfrak{p}_3 = [\theta_2 \hat{\otimes} \theta_4]_R$. We have $\text{dim}(\mathfrak{p}_1) = 3$, $\text{dim}(\mathfrak{p}_2) = 4$, and $\text{dim}(\mathfrak{p}_3) = 8$.

For the second fibration, we first need some explanation.
Lemma 1.5. The compact group \((\text{Spin}(4) \times \text{Spin}(3))/\{\pm(\text{Id}, \text{Id})\}\) satisfies
\[
\text{SO}(4) \subset (\text{Spin}(4) \times \text{Spin}(3))/\{\pm(\text{Id}, \text{Id})\} \subset \text{Spin}(7).
\]

Proof. Recall, for any \(n\), \(\text{Spin}(n)\) is the simply connected double cover of \(\text{SO}(n)\); let \(\pi : \text{Spin}(n) \to \text{SO}(n)\) denote the two-fold homomorphism. Because \(\pi_1(\text{SO}(k)) \to \pi_1(\text{SO}(n))\) is a surjection, \(\pi(\text{Spin}(k)) = \text{SO}(k)\) for all \(k \leq n\). Observe that \(\ker(\pi) = \{\pm(\text{Id}, \text{Id})\} = \text{Spin}(k) \cap \text{Spin}(n - k)\), thus it is \((\text{Spin}(k) \times \text{Spin}(n - k))/\{\pm(\text{Id}, \text{Id})\} \subset \text{Spin}(n)\). In \(\text{Spin}(7)\) we can consider the subgroup \((\text{Spin}(3) \times \text{Spin}(4))/\{\pm(\text{Id}, \text{Id})\}\). We know \(\text{Spin}(4) \cong \text{Spin}(3) \times \text{Spin}(3)\) (and \(\text{Spin}(3) \cong \text{SU}(2)\)). Thus \(\text{Spin}(7)\) has a subgroup \((\text{Spin}(3) \times \text{Spin}(3) \times \text{Spin}(3))/\{\pm(\text{Id}, \text{Id}, \text{Id})\}\). Our isotropy subgroup \(\text{SO}(4) \subset \text{Spin}(7)\) is exactly \((\Delta \text{Spin}(3) \times \text{Spin}(3))/\{\pm(\text{Id}, \text{Id})\}\), where \(\Delta \text{Spin}(3)\) is the diagonal subgroup of \(\text{Spin}(3) \times \text{Spin}(3)\) isomorphic to \(\text{Spin}(3)\). This can be seen via the restriction to \(\text{SO}(4)\) of the homomorphism from \(\text{Spin}(7)\) to \(\text{SO}(7)\) taking \(A\) to \(B\) (\(A\) and \(B\) in the definition of \(\text{Spin}(7)\)). \(\square\)

We obtain the following fibration:
\[
S^3 \cong \frac{\text{Spin}(4) \times \text{Spin}(3)}{\{\pm(\text{Id}, \text{Id})\}} / \frac{\text{Spin}(3) \times \Delta \text{Spin}(3)}{\{\pm(\text{Id}, \text{Id})\}} \to \frac{\text{Spin}(7)/\text{SO}(4)}{\text{Spin}(7)/\{\pm(\text{Id}, \text{Id})\}} \cong G^+_3(\mathbb{R}^8)
\]
\[
\cong G^+_3(\mathbb{R}^7).
\]

In the second fibration, it is \(p_1\) which is the subspace tangent to the fibre, while \(p_2 \oplus p_3\) is the subspace tangent to the base.

\[p_1 = \text{span}\{3E_{12} - E_{34} + E_{56} + E_{78}, 3E_{15} - E_{26} - E_{37} - E_{48}, 3E_{16} + E_{25} + E_{38} - E_{47}\}\]
\[p_2 = \text{span}\{3E_{13} + E_{24} + E_{57} - E_{68}, 3E_{14} - E_{23} + E_{58} + E_{67}, 3E_{17} - E_{28} + E_{35} + E_{46}, 3E_{18} + E_{27} - E_{36} + E_{45}\}\]
\[p_3 = \text{span}\{E_{23} + E_{67}, E_{24} + E_{68}, E_{27} + E_{36}, E_{28} + E_{46}, 2E_{57} - E_{24} + E_{68}, 2E_{58} + E_{23} - E_{67}, 2E_{35} + E_{28} - E_{46}, 2E_{45} - E_{27} + E_{36}\}.
\]
Since each of the $p_i$'s has a different dimension, they are inequivalent representations. This means any Spin(7)-invariant metric on $G^+_3(\mathbb{R}^8)$ is determined by an inner product on $\mathfrak{p}$ satisfying
\[
\langle \ , \rangle = x_1 Q|_{p_1} \perp x_2 Q|_{p_2} \perp x_3 Q|_{p_3}, \text{ for } x_1, x_2, x_3 > 0.
\]

From the fibrations we obtain the following Lie bracket relations:
\[
[p_1, p_1] \subset \mathfrak{h} \oplus p_1, \quad [p_1, p_2] \subset p_2 \oplus p_3,
\]
\[
[p_2, p_2] \subset \mathfrak{h} \oplus p_1, \quad [p_3, p_3] \subset \mathfrak{h}.
\]

Recall the scalar curvature formula from [W-Z2]:
\[
S = \frac{1}{2} \sum_i d_i b_i \frac{1}{x_i} - \frac{1}{4} \sum_{i,j,k} \left( \frac{k}{i,j} \right) \frac{x_k}{x_i x_j}.
\]

On Spin(7) we find that $b_i = 10$ for $i = 1, 2, 3$, and we know $d_1 = 3$, $d_2 = 4$, and $d_3 = 8$.

From the Lie bracket relations we know $(1_1, 2_2), (1_2, 3_3) \neq 0$; all other triples (except rearrangements) are zero. We find that $(1_1, 2_2) = 4$, and $(3_1, 3_2) = 8$.

We now have the scalar curvature function in $x_1, x_2, x_3$:
\[
S = \frac{25}{2x_1} + \frac{20}{x_2} + \frac{40}{x_3} - \frac{x_1}{x_2 x_3} - 4 \left( \frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} + \frac{x_3}{x_1 x_2} \right).
\]

We want the critical points of the scalar curvature function with the constraint equation of volume 1: $\tilde{S} = S - \lambda(x_1^3 x_2^4 x_3^8 - 1)$.

\[
\frac{\partial \tilde{S}}{\partial x_1} = -\frac{25}{2x_1^2} - \frac{1}{x_2^2} - \frac{4}{x_2 x_3} + \frac{4x_2}{x_1^2 x_3} + \frac{4x_3}{x_1^2 x_2} - 3\lambda x_1^2 x_2^4 x_3^8
\]
\[
\frac{\partial \tilde{S}}{\partial x_2} = -\frac{20}{x_2^2} + \frac{2x_1}{x_2^2} + \frac{4x_1}{x_2^2 x_3} - \frac{4}{x_1 x_2} + \frac{x_3}{x_1^2 x_2} - 4\lambda x_1^3 x_2^3 x_3^8
\]
\[
\frac{\partial \tilde{S}}{\partial x_3} = -\frac{40}{x_3^2} + \frac{4x_1}{x_2 x_3} + \frac{4x_2}{x_1 x_3^2} - \frac{4}{x_1 x_2} - 8\lambda x_1^3 x_2^4 x_3^7.
\]
A solution to \( \frac{\partial S}{\partial x_1} = \frac{\partial S}{\partial x_2} = \frac{\partial S}{\partial x_3} = 0 \) is equivalent to the simultaneous solution of the following two polynomials:

\[
10x_1x_2^2 - 10x_1x_2x_3 + x_1^2x_2 + x_1^2x_3 + 3x_2x_3^2 - 3x_2^3 = 0
\]

\[
-11x_1^2x_2 + 11x_2^2x_3 + 5x_2^3 - 25x_2^2x_3 + 30x_1x_2^2 - 2x_1x_3^2 = 0.
\]

We obtain three solutions, using Maple. The first is the symmetric solution: \( x_1 = \frac{3}{4}x_3, \ x_2 = \frac{1}{4}x_3 \). Let \( \tilde{p} \) denote the \( Q \)-orthogonal complement to \( \mathfrak{so}(3) \oplus \mathfrak{so}(5) \) in \( \mathfrak{so}(8) \); \( \tilde{p} = \text{span}\{E_{ij} \mid i = 2, 5, 6, \ j = 1, 3, 4, 7, 8\} \). (Of course we take \( \mathfrak{so}(3) \oplus \mathfrak{so}(5) \) corresponding to \( P_0 \).) We must project \( p_1, p_2, \) and \( p_3 \) to \( \tilde{p} \). In \( p_1 \), we take the basis element \( \frac{1}{2\sqrt{3}}(3E_{12} - E_{34} + E_{56} + E_{78}) \) of norm \( = \sqrt{x_1} \). It is projected to \( -\frac{\sqrt{3}}{2}E_{12} \) in \( \tilde{p} \), an element of norm \( = \frac{\sqrt{3}}{2} \). In \( p_2 \) we take the basis element \( \frac{1}{2\sqrt{3}}(3E_{13} + E_{24} + E_{57} - E_{68}) \) of norm \( = \sqrt{x_2} \), which projects to \( \frac{1}{2\sqrt{3}}(E_{24} + E_{57} - E_{68}) \) in \( \tilde{p} \), an element of norm \( = \frac{1}{2} \). Finally, the element \( \frac{1}{\sqrt{2}}(E_{23} + E_{67}) \) is already in \( \tilde{p} \), so norm \( = \sqrt{x_3} \) corresponds to norm \( = 1 \). Hence \( x_1 = \frac{3}{4}x_3, \ x_2 = \frac{1}{4}x_3 \) is indeed the symmetric metric.

The second and third solutions are \( x_2 = \eta x_3 \), and

\[
x_1 = \left( -\frac{629}{1980} + \frac{5689}{660}\eta - \frac{3799}{165}\eta^2 + \frac{13559}{495}\eta^3 - \frac{392}{33}\eta^4 \right)x_3,
\]

for \( \eta \) a positive root of the polynomial

\[
4704t^5 - 11788t^4 + 10400t^3 - 3315t^2 - 398t + 289.
\]

This polynomial has three real roots, of which two are positive and yield two positive values for \( x_1 \) and \( x_2 \) in terms of \( x_3 \). We give the approximate values, setting \( x_3 = 1 \):

\[
x_1 = -0.241854 \quad x_2 = -4.177304
\]

\[
x_1 = 0.425179 \quad x_2 = 0.902192
\]

\[
x_1 = 1.100300 \quad x_2 = 0.369813
\]
These are two new Einstein metrics on $G^+_3(\mathbb{R}^8)$.

Remark. None of these is a fibration metric, since the first fibration required $x_1 = x_2$, and the second required $x_2 = x_3$. Just as in the previous example in both fibrations the fibre and base space are isotropy irreducible, therefore Einstein. However, the isotropy representation of each base, when restricted to $SO(4)$, again decomposes into the sum of two irreducible subrepresentations, where the Casimir constant corresponding to $Q$ and to the restriction to $SO(4)$ of the isotropy representations for these homogeneous spaces differs. Again this implies the O’Neill tensor restricted to horizontal vectors is not a scalar multiple of $Q$. Using Besse’s Proposition 9.70 [Bes, p.253] we know our fibrations cannot give rise to Einstein metrics.

We verify that they are all distinct with the (scale-invariant) product: $(S)^{15}(V)^{12}$, where $S$ is the scalar curvature of the metric and $V$ is the volume of the metric. For the symmetric metric, $S = \frac{90}{x_3}$ and $V = \frac{3^4}{2\pi^4}x_3^{15}$, and so $(S)^{15}(V)^{12} \approx 1.84200 \times 10^{13}$. For the two new metrics, we find that $(S)^{15}(V)^{12} \approx 1.80936 \times 10^{13}$, and $(S)^{15}(V)^{12} \approx 1.61159 \times 10^{13}$, respectively. This shows that they are non-isometric.
In this chapter we consider some products of two irreducible symmetric spaces on which there is a simple compact Lie group acting transitively. The same classification by Oniščik of simple compact Lie algebras $\mathfrak{g}$ with Lie subalgebras $\mathfrak{g}', \mathfrak{g}''$ such that $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}''$ (from page 5) also gives a list of simple groups acting on products. Let $G$ be the simply connected compact Lie group corresponding to $\mathfrak{g}$ and let $G'$ and $G''$ be the Lie subgroups corresponding to $\mathfrak{g}'$ and $\mathfrak{g}''$, respectively. Then $G/(G' \cap G'') = G/G' \times G/G''$. Here are the products of compact irreducible symmetric spaces we obtain from Oniščik’s result:

\[
\begin{align*}
\text{SU}(2n)/\text{Sp}(n-1) &= S^{4n-1} \times \text{SU}(2n)/\text{Sp}(n) \\
\text{SU}(2n)/\text{Sp}(n-1)\text{U}(1) &= \mathbb{C}P^{2n-1} \times \text{SU}(2n)/\text{Sp}(n) \\
\text{SO}(2n+2)/\text{U}(n) &= S^{2n+1} \times \text{SO}(2n+2)/\text{U}(n+1) \\
\text{Spin}(8)/G_2 &= S^7 \times S^7 \\
\text{Spin}(7)/\text{SU}(3) &= S^7 \times S^6 \\
\text{Spin}(8)/U(3) &= S^7 \times G_2^+(\mathbb{R}^8) \\
\text{Spin}(8)/\text{SO}(4) &= S^7 \times G_3^+(\mathbb{R}^8) \\
\text{Spin}(7)/U(2) &= S^7 \times G_2^+(\mathbb{R}^7) .
\end{align*}
\]

In our analysis of most of these spaces, we found that the equations were too complicated to solve, either by hand or with the help of Maple. We summarize our results in the following table, denoting by $M = G/H$ the product of two irreducible symmetric spaces, and $\mathcal{M}_G = \{G$-homogeneous metrics $g$ on $M$ with $\sqcup \{\} = \infty\}$. 

31
<table>
<thead>
<tr>
<th>$M$</th>
<th>$G/H$</th>
<th>$\dim M_G$</th>
<th>no. Einstein</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^7 \times S^7$</td>
<td>$\text{Spin}(8)/G_2$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$S^7 \times S^6$</td>
<td>$\text{Spin}^+(7)/\text{SU}(3)$</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$S^7 \times G_2^+(\mathbb{R}^8)$</td>
<td>$\text{Spin}(8)/U(3)$</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

2.1. $S^7 \times S^7$

Just as $S^7$ is the unit sphere in $\mathbb{R}^8 \cong \mathbb{O}$, the Cayley numbers (or octonians), the product of two seven-spheres is a natural submanifold of $\mathbb{O} \times \mathbb{O}$. Since it is a product of symmetric spaces, $S^7 \times S^7$ is homogeneous; the simple Lie group $\text{Spin}(8)$ acts transitively on $S^7 \times S^7$ with isotropy subgroup $G_2$. One expects at least two distinct $\text{Spin}(8)$-invariant Einstein metrics: one the product metric, and the other induced by the Killing form [W-Z1, p. 575]. We find that it carries exactly these, and no others.

We begin by describing the homogeneous presentation of $S^7 \times S^7$, then we can determine $\mathcal{M}_{\text{Spin}(\mathbb{O})}$, the space of invariant metrics, and consider it for Einstein metrics. We have a natural matrix group representation for $\text{Spin}(8)$:

$$\text{Spin}(8) = \{(A, B, C) \in \text{SO}(8)^3 \mid A(x)B(y) = C(xy) \ \forall x, y \in \mathbb{O}\}.$$ 

The Triality Principle tells us that $\text{Spin}(8)$ is indeed a double cover of $\text{SO}(8)$, since a choice of $A \in \text{SO}(8)$ determines the corresponding $B$ and $C$, up to sign [M]. The Moufang identities give us three families of triples in $\text{Spin}(8)$: for each $z \in \text{Im}(\mathbb{O})$ with $\|z\| = 1$,

$$(R_z, L_zR_z, R_z), \ (L_zR_z, L_z, L_z), \ \text{and} \ (L_z, R_z, -L_zR_z) \in \text{Spin}(8).$$

We note some of the subgroups of $\text{Spin}(8)$:

$$\text{Spin}^+(7) = \{(A, B, C) \in \text{Spin}(8) \mid B = C\} \text{ generated by } \{(L_zR_z, L_z, L_z)\},$$

$$\text{Spin}^-(7) = \{(A, B, C) \in \text{Spin}(8) \mid A = C\} \text{ generated by } \{(R_z, L_zR_z, R_z)\},$$

and $G_2 = \{(A, B, C) \in \text{Spin}(8) \mid A = B = C\} = \text{Spin}^+(7) \cap \text{Spin}^-(7).$
For a triple \((A, B, C)\) in \(\text{Spin}^+(7) \subset \text{Spin}(8)\), notice that \(B = C\) implies \(A(1) = 1\), thus \(A \in \text{SO}(7)\). This shows that \(\text{Spin}^+(7)\) is indeed a double cover of \(\text{SO}(7)\). For a triple \((A, B, C)\) in \(\text{Spin}^-(7) \subset \text{Spin}(8)\), notice that \(A = C\) implies \(B(1) = 1\), so \(B \in \text{SO}(7)\), and \(\text{Spin}^-(7)\) double covers \(\text{SO}(7)\).

We define the action of \(\text{Spin}(8)\) on \(S^7 \times S^7\) via \((A, B, C) : (x, y) \mapsto (Ax, By)\). To show that this action is transitive we take any point \((x, y) \in S^7 \times S^7\) and construct a map from \((x, y)\) to \((1, 1)\). We can first find \((A, B, C) : (x, y) \mapsto (1, y')\) for some \(y'\), since \(A\) can be any element of \(\text{SO}(8)\). Next, since \(\text{Spin}^+(7)\) acts transitively on \(S^7\), there exists an element \((A', B', B')\) of \(\text{Spin}^+(7)\) mapping \((1, y') \mapsto (1, 1)\).

Next we determine the isotropy subgroup \(H \subset \text{Spin}(8)\) corresponding to the point \((1, 1)\). We know that \(\text{Spin}^+(7)\) fixes the first component of \((1, \ast)\), and \(\text{Spin}^-(7)\) fixes the second component of \((\ast, 1)\), hence the subgroup \(H\) fixing \((1, 1)\) satisfies \(H \subset \text{Spin}^+(7) \cap \text{Spin}^-(7) = G_2\). Every element of \(G_2\) takes \((1, 1)\) to itself, thus \(H = G_2\). This shows that \(\text{Spin}(8)/G_2 \cong S^7 \times S^7\).

Under the double covering homomorphism from \(\text{Spin}(8)\) to \(\text{SO}(8)\) \((A, B, C) \mapsto C\), the subgroups \(\text{Spin}^+(7)\) and \(\text{Spin}^-(7)\) are isomorphic to their images in \(\text{SO}(8)\). We use the homomorphism to identify the Lie algebras \(\text{spin}(8) \cong \text{so}(8)\). If we order our basis for the octonians in the following way: \(\{1, j, \varepsilon, j\varepsilon, i, k, i\varepsilon, -k\varepsilon\}\), then \(G_2\) is a subset of \(\text{SO}(7) \cong \begin{pmatrix} 1 & 0 \\ 0 & \text{SO}(7) \end{pmatrix} \subset \text{SO}(8)\). While \(g_2\) is invariant under the triality automorphism of \(\text{so}(8)\), the Lie subalgebras \(\text{so}(7)\), \(\text{spin}^+(7)\), and \(\text{spin}^-(7)\) (subalgebras of \(\text{so}(8)\)), which all contain \(g_2\), are interchanged.

We decompose \(\text{so}(8)\) into \(g_2 \oplus p\), where \(p\) is the orthogonal complement to \(g_2\) in \(\text{so}(8)\) with respect to the inner product \(Q(X, Y) = -\frac{1}{2} \text{tr}(XY)\). Using the following fibrations we
find that \( p \) is the sum of two equivalent copies of the standard orthogonal seven-dimensional representation of \( G_2 \), denoted by \( \varphi \).

\[
S^7 = \text{Spin}^\pm(7)/ G_2 \to \text{Spin}(8)/ G_2 \to \text{Spin}(8)/ \text{Spin}^\pm(7) = S^7.
\]

We could also consider the fibration

\[
\mathbb{RP}^7 = \text{SO}(7)/ G_2 \to \text{SO}(8)/ G_2 \to \text{SO}(8)/ \text{SO}(7) = S^7.
\]

These three subalgebras give three natural ways to decompose \( p \): We can choose \( p_1 \) so that \( g_2 \oplus p_1 \) is \( \text{spin}^+(7) \), \( \text{spin}^-(7) \), or \( \mathfrak{so}(7) \), then we set \( p_2 \) to be the \( Q \)-orthogonal complement to \( p_1 \) in \( p \). We choose to decompose \( p = p_1 \oplus p_2 \) so that \( \text{spin}^+(7) = g_2 \oplus p_1 \). On the Lie algebra level we have

\[
g_2 = \text{span}\{E_{37} - E_{48}, E_{34} + E_{78}, E_{38} + E_{47}, 2E_{56} + E_{34} - E_{78}, 2E_{26} - E_{48} - E_{37},
2E_{25} - E_{38} + E_{47}, E_{23} + E_{67}, E_{24} + E_{68}, E_{27} + E_{36}, E_{28} + E_{16},
2E_{57} - E_{24} + E_{68}, 2E_{58} + E_{23} - E_{67}, 2E_{35} + E_{28} - E_{46}, 2E_{45} - E_{27} + E_{36}\},
\]

\[
p_1 = \text{span}\{3E_{12} - E_{34} + E_{56} + E_{78}, 3E_{15} - E_{26} - E_{37} - E_{48}, 3E_{16} + E_{25} + E_{38} - E_{47},
3E_{13} + E_{24} + E_{57} - E_{68}, 3E_{14} - E_{23} + E_{58} + E_{67},
3E_{17} - E_{28} + E_{35} + E_{46}, 3E_{18} + E_{27} - E_{36} + E_{45}\},
\]

\[
p_2 = \text{span}\{E_{12} + E_{34} - E_{56} - E_{78}, E_{15} + E_{26} + E_{37} + E_{48}, E_{16} - E_{25} - E_{38} + E_{47},
E_{13} - E_{24} - E_{57} + E_{68}, E_{14} + E_{23} - E_{58} - E_{67},
E_{17} + E_{28} - E_{35} - E_{46}, E_{18} - E_{27} + E_{36} - E_{45}\}.
\]

We can now describe \( \mathcal{M}_{\text{Spin}(\gamma)} \) for \( S^7 \times S^7 \). Let \( x_1, x_2 > 0 \), let \( \lambda \) be any real number, and let \( J \) operate on \( p \) by interchanging \( p_1 \) and \( p_2 \): \( J = \begin{pmatrix} 0 & \text{Id}_7 \\ \text{Id}_7 & 0 \end{pmatrix} \) with respect to the ordered basis
for $p$ above. It is easy to see that $J$ is $\text{Ad}(G_2)$-equivariant and an isometry with respect to $Q$.

Since $p_1$ and $p_2$ are absolutely irreducible and equivalent, Schur’s lemma tells us that every $\text{Ad}(G_2)$-invariant inner product on $p$ satisfies $\langle \ , \ \rangle_{|p_i} = x_i Q_{|p_i}$, and $\langle p_1, p_2 \rangle = \lambda Q(J p_1, p_2)$. I.e., $\langle \ , \ \rangle = Q(g \ , \ )$, where $g = \left( \begin{array}{cc} x_1 \text{Id}_7 & \lambda \text{Id}_7 \\ \lambda \text{Id}_7 & x_2 \text{Id}_7 \end{array} \right)$. (We say $g_{ij}$ is the metric.) Thus $\mathcal{M}_{\text{Spin}(\mathbb{H})}$ is parametrized by $x_1, x_2 > 0$, $\lambda$.

We remark that while Schur’s lemma holds for complex representations, $\varphi$ is a real representation; we apply Schur’s lemma to its complexification, and since $\varphi$ is orthogonal, $\varphi \otimes \mathbb{C}$ is an irreducible complex representation of $G_2$. Thus the space of operators intertwining with $\varphi$ is one-dimensional: scalar multiples of $J$.

The decomposition of $p$ into $p_1 \oplus p_2$ is not unique, so in order to consider the space of all homogeneous metrics, we cannot restrict our search for Einstein metrics to diagonal ($\lambda = 0$) metrics. Thus we need a formula for the scalar curvature functional which does not assume we have an orthonormal basis to work with. We give such a formula for the general case of a non-diagonal metric on a homogeneous space $G/H$ with $G$ unimodular, $g = \mathfrak{h} \oplus p_1 \oplus \cdots \oplus p_r$.

We first rewrite the formula found in Besse [Bes, 7.39] in a form which shows plainly the result of a change of coordinates. The inner product is a biinvariant metric $Q$ and $\{X_i\}$ is a $Q$-orthonormal basis for $p$.

\[
S = -\frac{1}{4} \sum_{i,j} |[X_i, X_j]_p|^2 - \frac{1}{2} \sum_{i,j} B(X_i, X_j) \\
= -\frac{1}{4} \sum_{i,j} Q(C_i(X_j), pr_p \circ C_i(X_j)) - \frac{1}{2} \sum_{i,j} B(X_i, X_j) \\
= -\frac{1}{4} \sum_{i,j} Q(X_j, C_i \circ pr_p \circ C_i)(X_j)) - \frac{1}{2} \sum_{i,j} B(X_i, X_j), \text{ which can be written} \\
S = -\frac{1}{4} \sum_i \text{tr}(pr_p \circ C_i \circ pr_p \circ C_i) - \frac{1}{2} \sum_i \text{tr}(C_i \circ C_i),
\]
where \( pr_p \) is projection onto \( p \) and \( C_i = \text{ad} X_i \). In matrix form (completing our basis to one for \( g = h \oplus p \)), \( pr_p = \begin{pmatrix} 0 & 0 \\ 0 & \text{Id} \end{pmatrix} \) and \( C_i = \begin{pmatrix} 0 & a_i \\ -\alpha_i & \gamma_i \end{pmatrix} \), the matrix of structure constants. Therefore, with respect to an orthonormal basis, we have

\[
S = -\frac{1}{2} \sum_i \text{tr}(C_i \circ C_i) - \frac{1}{4} \sum_i \text{tr}(\gamma_i \circ \gamma_i).
\]

Now suppose we are given a new metric, \( \langle \cdot, \cdot \rangle = Q(g \cdot, \cdot) \). We change coordinates to obtain a corresponding orthonormal basis \( \{\tilde{X}_i\} \). After our change of coordinates \( \tilde{X}_i = A X_i \), we can express the new matrices of structure constants in terms of the change of basis matrix \( A \) and the original \( C_i \)'s, with \( \tilde{A} = \begin{pmatrix} \text{Id}_{\dim h} & 0 \\ 0 & A \end{pmatrix} \): \( \tilde{C}_k = \sum_i a_{ik} \tilde{A}^{-1} C_i \tilde{A} \) (cf. [Je2, p.1127]). Since \( AA^t = g^{-1} \), we can now express the scalar curvature as a function of \( g \) (where \( g^{ik} = (g^{-1})_{jk} \)). We reduce the first sum, the second sum reduces similarly.

\[
\tilde{C}_i \circ \tilde{C}_i = \sum_{j,k} a_{ji}(\tilde{A}^{-1} C_j \tilde{A})a_{ki}(\tilde{A}^{-1} C_k \tilde{A}) = \sum_{j,k} a_{ji} a_{ki} \tilde{A}^{-1} C_j C_k \tilde{A}
\]

\[
\sum_i \text{tr}(\tilde{C}_i \circ \tilde{C}_i) = \sum_{j,k} (AA^t)_{kl} \text{tr}(A^{-1} C_k C_l A) = \sum_{j,k} (AA^t)_{jk} \text{tr}(C_j C_k).
\]

Hence \( S(g) = -\frac{1}{2} \sum_i \text{tr}(\tilde{C}_i \circ \tilde{C}_i) - \frac{1}{4} \sum_i \text{tr}(\tilde{\gamma}_i \circ \tilde{\gamma}_i) \)

\[
= -\frac{1}{2} \sum_{j,k} (AA^t)_{jk} \text{tr}(C_j \circ C_k) - \frac{1}{4} \sum_{j,k} (AA^t)_{jk} \text{tr}(\gamma_j \circ (AA^t)^{-1} \circ \gamma_k (AA^t))
\]

\[
= -\frac{1}{2} \sum_{j,k} g^{jk} \text{tr}(C_j \circ C_k) - \frac{1}{4} \sum_{j,k} g^{jk} \text{tr}(\gamma_j \circ g \circ \gamma_k \circ g^{-1})
\]

\[
= -\frac{1}{2} \sum_{j,k} g^{jk} B(X_j, X_k) - \frac{1}{4} \sum_{j,k} g^{jk} \text{tr}(\gamma_j \circ g \circ \gamma_k \circ g^{-1}). \quad (*)
\]

Returning to our example of \( S^7 \times S^7 \), we use (*) and Maple to obtain

\[
S = -\frac{7(x_1^3 - 12x_1^2x_2 - 9x_1x_2^2 + 6x_1^2\lambda^2 + 18x_2\lambda^2)}{2(x_1x_2 - \lambda^2)^2}.
\]
We normalize to restrict to volume one, \( \tilde{S} = S - k(x_1 x_2 - \lambda^2)^7 \), where \( k \) is the Lagrange multiplier; then we can solve for the critical points, using Maple.

\[
\frac{\partial \tilde{S}}{\partial x_1} = \frac{-7(18x_1 x_2 \lambda^2 - 6\lambda^4 + 9x_1 x_2^3 - 27x_2^2 \lambda^2 + x_1^3 x_2 - 3x_1^2 \lambda^2)}{2(x_1 x_2 - \lambda^2)^3}, \\
\frac{\partial \tilde{S}}{\partial x_2} = \frac{7(-6x_1^3 x_2 + 9\lambda^4 + x_1^4)}{(x_1 x_2 - \lambda^2)^3}, \\
\frac{\partial \tilde{S}}{\partial x_2} = \frac{-14\lambda(-9x_1^2 x_2 + 3x_1 \lambda^2 + 9x_2 \lambda^2 + x_1^2)}{(x_1 x_2 - \lambda^2)^3}.
\]

We find these results:

\[
x_1 = x_2 \quad x_1 = 3x_2 \quad x_1 = \frac{3}{5} x_2 \\
\lambda = 0 \quad \lambda = 0 \quad \lambda = \pm \frac{1}{\sqrt{3}} x_1.
\]

The first solution is the metric induced by the Killing form (recall that \( Q \) is a multiple of the Killing form). We show that the third solution is a pair of metrics isometric to the product metric, in which the tangent spaces to \( S^7 \times \{1\} \) and \( \{1\} \times S^7 \) are orthogonal: with respect to \( Q \), the tangent spaces to the two spheres meet at angle \( \frac{2\pi}{3} \), so we expect that the product metric will be a non-diagonal solution. We can write \( \mathfrak{so}(8) = p_1 \oplus p_2^* \), where \( p_2^* \) denotes the \((Q\)-orthogonal\) complement to \( \mathfrak{g}_2 \) in \( \mathfrak{spin}^-(7) \). The product metric is

\[
\langle \langle p_1, p_1 \rangle \rangle = x_1^* Q(p_1, p_1) \\
\langle \langle p_2, p_2^* \rangle \rangle = x_2^* Q(p_2, p_2^*) \\
\langle \langle p_1, p_2^* \rangle \rangle = 0 \text{ and } x_1^* = x_2^*.
\]

When we project the representation space \( p_2^* \) to \( p_1 \oplus p_2 \), we see that under this projection the product metric corresponds to the non-diagonal metric. Here is a typical unit vector (with respect to \( Q \)) in \( p_2^* \), decomposed with respect to \( p_1 \oplus p_2 \):

\[
\frac{1}{2\sqrt{3}}(3E_{18} - E_{27} + E_{36} - E_{45}) = \frac{1}{4\sqrt{3}}(3E_{18} + E_{27} - E_{36} + E_{45}) + \frac{\sqrt{3}}{4}(E_{18} - E_{27} + E_{36} - E_{45}).
\]
This shows that while \( x_1^* = x_1 \) (since \( p_1 \) gets projected to itself), we have \( x_2^* = \frac{x_1^*}{4} + \frac{\sqrt{3} \lambda}{2} + \frac{3x_2}{4} \). The equality \( x_1^* = x_2^* \) induces the equality \( x_1 = \frac{x_1^*}{4} + \frac{\sqrt{3} \lambda}{2} + \frac{3x_2}{4} \), which simplifies to \( x_1 = x_2 + \frac{2 \lambda}{\sqrt{3}} \); this is our third metric exactly.

The second metric is not a new metric, but the product metric in an unexpected form; it is conjugated by \( R\left(\frac{\pi}{3}\right) \), rotation by \( \frac{\pi}{3} \).

\[
\begin{pmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\
\frac{\sqrt{3}}{2} & \frac{1}{2} & \lambda \\
3\lambda & 0 & \lambda
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\
0 & \lambda & \frac{5\lambda}{\sqrt{3}} \\
\lambda & -\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{pmatrix} =
\begin{pmatrix}
\lambda \sqrt{3} & \lambda \\
\lambda & \frac{5\lambda}{\sqrt{3}}
\end{pmatrix}
\]

The rotation by \( \frac{\pi}{3} \) is the action of the triality automorphism; it is an isometry of our space. (Notice that conjugating a second time by \( R\left(\frac{\pi}{3}\right) \) gives us \( \begin{pmatrix} \lambda \sqrt{3} & -\lambda \\ -\lambda & \frac{5\lambda}{\sqrt{3}} \end{pmatrix} \), which shows that the third solution is really only one metric, isometric to the product metric.) Thus there are exactly two Spin(8)-invariant Einstein metrics on \( S^7 \times S^7 \).

### 2.2. \( S^6 \times S^7 \)

Our next product of symmetric spaces is \( S^6 \times S^7 \), the unit spheres in \( \text{Im}(\mathbb{O}) \times \mathbb{O} \). The simple Lie group \( \text{Spin}^+(7) \) acts transitively with isotropy subgroup \( \text{SU}(3) \). The product metric is one invariant Einstein metric; we find there are exactly two others.

We know \( \text{Spin}^+(7) \) from the previous example, since \( \text{Spin}^+(7) \subset \text{Spin}(8) \), and we include a description of some useful subgroups of \( \text{Spin}^+(7) \):

\[
\text{Spin}^+(7) = \{(A, B, B) \in \text{SO}(8)^3 \mid A(x)B(y) = B(xy) \ \forall x, y \in \mathbb{O}\}
\]

\[
G_2 = \{(A, B, B) \in \text{Spin}^+(7) \mid A = B\}
\]

\[
\text{SU}(4) = \{(A, B, B) \in \text{Spin}^+(7) \mid A(i) = i\}.
\]

Recall that under the double covering homomorphism from \( \text{Spin}(8) \) to \( \text{SO}(8) \) sending \( (A, B, C) \mapsto C \), the Lie group \( \text{Spin}^+(7) \) is isomorphic to its image. Giving the Cayley numbers our usual ordering: \( \{1, j, e, je, i, k, ie, -ke\} \), we know that \( G_2 \) is a subset of
SO(7) ≃ \begin{pmatrix} 1 & 0 \\ 0 & \text{SO}(7) \end{pmatrix} \subset \text{SO}(8)$, and SU(4) is the standard embedding of $\text{SU}(4) \subset \text{SO}(8)$, with $X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$. (That is, SU(4) respects the complex structure induced from left multiplication by $i$.)

We begin by showing that $\text{Spin}^+(7)$ indeed acts transitively on $S^6 \times S^7$. Let $(x, y)$ be any point in $S^6 \times S^7$, we want to find an element of $\text{Spin}^+(7)$ mapping $(x, y)$ to $(i, 1)$. Since SU(4) acts transitively on $S^7$, we can find a map $(A, B, B) \in \text{SU}(4)$ such that $(A, B, B)(x, y) = (x', 1)$. Next we use that $G_2$ acts transitively on $S^6 \subset \text{Im}(\mathbb{O})$, leaving 1 fixed, to find a map $(A', A', A') \in G_2$ satisfying $(A', A', A')(x', 1) = (i, 1)$. Note that the definition of $\text{Spin}^+(7)$ implies the first component of $(1, *)$ is fixed, hence $x \in \text{Im}(\mathbb{O})$ implies $x' \in \text{Im}(\mathbb{O})$.

We next determine the isotropy subgroup $H$ corresponding to our point $(i, 1)$. For any element of $H$, we have $(A, B, B)(i, 1) = (i, 1)$, and $A(i) = i$ implies $(A, B, B) \in \text{SU}(4)$. Thus $H \subset \text{SU}(4)$. We also have $B(1) = 1$, so by definition $A(x)B(1) = B(x)$ for all $x$. Hence $A = B$, we have $(A, A, A) \in G_2$, and $H \subset G_2$. Finally, we recall that any subgroup of $G_2$ fixing an imaginary octonian is isomorphic to $\text{SU}(3)$, and in our case, $H = \text{SU}(3)$ is the subgroup of $\text{SU}(4)$ fixing the complex line spanned by $\{1, i\}$.

Remark. We can also think of $S^6 \times S^7$ as the unit tangent bundle of $S^7$: at the point $1 \in \mathbb{O}$, the tangent space to the unit sphere is $\text{Im}(\mathbb{O})$, and $S^6 \in \text{Im}(\mathbb{O})$ is the space of all unit tangent vectors. Thus $(i, 1)$ can be thought of as the unit vector in the direction $i$ tangent to the unit sphere at 1.

We identify $\text{Spin}^+(7)$ with its image under the homomorphism from $\text{Spin}(8)$ to $\text{SO}(8)$. On the Lie algebra level, we can take $\mathfrak{p}$ to be the orthogonal complement to $\mathfrak{su}(3)$ in $\mathfrak{spin}^+(7)$ with respect to $Q(X, Y) = -\frac{1}{2}\text{tr}(XY)$, so that $\mathfrak{spin}^+(7) = \mathfrak{su}(3) \oplus \mathfrak{p}$. We have three
fibrations of our product space, which we use to decompose $p$:

$$S^6 = G_2 / SU(3) \to Spin^+(7) / SU(3) \to Spin^+(7) / G_2 = S^7$$

$$S^7 = SU(4) / SU(3) \to Spin^+(7) / SU(3) \to Spin^+(7) / SU(4) = S^6$$

$$S^1 = U(3) / SU(3) \to Spin^+(7) / SU(3) \to Spin^+(7) / U(3).$$

In the first fibration, the isotropy representation of the fibre is $[\mu_3]_{\mathbb{R}}$, where $\mu_k$ is the standard $k$ dimensional complex representation of $SU(k)$. The isotropy representation of the base is the orthogonal representation of $G_2$ in $SO(7)$, which when restricted to $SU(3)$ is $[\mu_3]_{\mathbb{R}} \oplus \text{Id}$. The second fibration shows that the isotropy representation of the fibre is $[\mu_3 \oplus \text{Id}]_{\mathbb{R}} = [\mu_3]_{\mathbb{R}} \oplus \text{Id}$, the sum of two irreducible subrepresentations. The isotropy representation of the base space is $[\mu_4]_{\mathbb{R}}$, and $\mu_4|_{SU(3)} = \mu_3$. In the third fibration, the fibre is one dimensional, with trivial isotropy representation. The base space is the symmetric space we discussed in chapter 1, section § with $n = 4$. The isotropy representation is $\mu_3 \oplus \wedge^2 \mu_3$. When we restrict from $U(3)$ to $SU(3)$, we find $\wedge^2 \mu_3 \cong \mu_3$. Thus we conclude $p = p_1 \oplus p_2 \oplus p_0$, with two equivalent representations $p_1 \simeq p_2 \simeq [\mu_3]_{\mathbb{R}}$ and a trivial, one-dimensional representation $p_0$. The decomposition of $p_1 \oplus p_2$ is not unique; we choose our decomposition so that $\mathfrak{su}(3) \oplus p_1 \oplus p_0 = \mathfrak{su}(4)$.

$$\mathfrak{su}(3) = \text{span}\{E_{37} - E_{48}, E_{34} + E_{78}, E_{38} + E_{47}, E_{23} + E_{67}, E_{24} + E_{68},$$

$$E_{27} + E_{36}, E_{28} + E_{46}, 2E_{26} - E_{48} - E_{37}\}$$

$$p_1 = \text{span}\{E_{12} + E_{56}, E_{13} + E_{57}, E_{14} + E_{58}, E_{16} + E_{25}, E_{17} + E_{35}, E_{18} + E_{45}\}$$

$$p_2 = \text{span}\{E_{56} - E_{12} + E_{34} - E_{78}, E_{57} - E_{13} - E_{24} + E_{68}, E_{58} - E_{14} + E_{23} - E_{67},$$

$$E_{25} - E_{16} - E_{38} + E_{47}, E_{35} - E_{17} + E_{28} - E_{46}, E_{45} - E_{18} - E_{27} + E_{36}\}$$

$$p_0 = \text{span}\{3E_{15} - E_{26} - E_{37} - E_{48}\}.$$
From the fibrations we know the following Lie bracket relations:

\[
[p_0, p_1] \subset p_1, \quad [p_0, p_2] \subset p_2, \quad [p_1, p_1] \subset \mathfrak{su}(3) \oplus p_0,
\]

\[
[p_1, p_2] \subset p_2, \quad [p_2, p_2] \subset \mathfrak{su}(3) \oplus p_0 \oplus p_1.
\]

We would like to consider all Spin$^\dagger$(7)-invariant metrics on $S^6 \times S^7$. We know any such metric satisfies $\langle\cdot,\cdot\rangle = Q(L,\cdot)$ for a symmetric linear operator $L$ from $\mathfrak{p}$ to $\mathfrak{p}$, commuting with $\text{Ad}(\text{SU}(3))$. We use Schur’s lemma to determine the number of parameters of $L$. As in the previous section, we must complexify the representation on $p_1$ and $p_2$ to find the dimension of the space of intertwining maps. Since $[\mu_3]_\mathbb{R}$ is a unitary representation, we find that when complexified it is no longer irreducible; it is $\mu_3 \oplus \mu_3^*$, the sum of two inequivalent irreducible representations. Thus the space of intertwining maps from $p_1$ to $p_2$ is two-dimensional. Any such metric can be written $\langle\cdot,\cdot\rangle = Q(g\cdot,\cdot)$, where we identify $g$ with the metric, and for any basis respecting the complex structure on $p_1 \oplus p_2$,

\[
g = \begin{pmatrix}
 x_1 \text{Id}_3 & 0 & \lambda_1 \text{Id}_3 & \lambda_2 \text{Id}_3 & 0 \\
 0 & x_1 \text{Id}_3 & -\lambda_2 \text{Id}_3 & \lambda_1 \text{Id}_3 & 0 \\
 \lambda_1 \text{Id}_3 & \lambda_2 \text{Id}_3 & x_2 \text{Id}_3 & 0 & 0 \\
 \lambda_2 \text{Id}_3 & -\lambda_1 \text{Id}_3 & 0 & x_2 \text{Id}_3 & 0 \\
 0 & 0 & 0 & 0 & x_3
\end{pmatrix}.
\]

The space of Spin$^\dagger$(7)-invariant metrics on $S^6 \times S^7$ is parametrized by $x_1, x_2, x_3, \lambda_1,$ and $\lambda_2$. Before searching for Einstein metrics, we can simplify. The normalizer of SU(3) in Spin$^\dagger$(7) is U(3), with Lie algebra $\mathfrak{su}(3) \oplus p_0$ (since $\mathfrak{su}(3)$ and $p_0$ commute). Thus conjugation by any element of $N(\text{SU}(3))/\text{SU}(3) = U(1)$ is a diffeomorphism preserving $\mathfrak{p}$. This gives us a one-parameter family of isometries of $S^6 \times S^7$. We may use these isometries to reduce the number of parameters of $g$. If $X_{13}$ is the given $Q$-unit vector spanning $p_0$ and \{\(X_i\)\}_{i=1}^6 is a
basis for \( p_1 \), \( \{ X_j \}_{j=7}^{12} \) is a compatible basis for \( p_2 \) (e.g., the given basis), we have

\[
[X_{13}, X_i] = \frac{2}{\sqrt{3}} X_{i+3} \quad \text{for } 1 \leq i \leq 3, \quad [X_{13}, X_i] = -\frac{2}{\sqrt{3}} X_{i-3} \quad \text{for } 4 \leq i \leq 6, \\
[X_{13}, X_j] = -\frac{1}{\sqrt{3}} X_{j+3} \quad \text{for } 7 \leq j \leq 9, \quad [X_{13}, X_j] = \frac{1}{\sqrt{3}} X_{j-3} \quad \text{for } 10 \leq j \leq 12.
\]

Set \( g'(t) = \text{Ad}(\exp tX_{13}) \cdot g \), then \( g \) and \( g'(t) \) are isometric, and we can find a \( t \) such that \( \lambda_2' = 0 \).

From (*), we obtain the scalar curvature equation for \( S^7 \times S^6 \):

\[
S = \frac{-6 x_1^2 x_3 + x_1^2 (60 x_2 x_3 - x_2^2) + 6 x_1 (8 x_2^2 x_3 - 3 x_2 + 2 x_3) \lambda_1^2}{2 (x_1 x_2 - \lambda_1^2)^2 x_3} + \frac{18 (\lambda_1^2 - 4 x_2 x_3) \lambda_1^2 + 4 (\lambda_1^2 - x_2^2) x_3 + \lambda_2^2 (2 x_2^2 + 9 (\lambda_2^2 + 2 \lambda_1^2 - x_1 x_2 - 2 x_1 x_3 - 4 x_2 x_3))}{2 (x_1 x_2 - \lambda_1^2)^2 x_3}.
\]

We solve for the critical points of the normalized scalar curvature, so that we restrict to those metrics of volume one: \( \tilde{S} = S - k(x_1 x_2 - \lambda_1^2 - \lambda_2^2)^6 x_3 \), where \( k \) is the Lagrange multiplier. Notice that \( \tilde{S} \) is a function of \( \lambda_2^2 \), hence not only is every homogeneous metric isometric to one with \( \lambda_2 = 0 \), but setting \( \lambda_2 = 0 \) we do not miss any critical points. Using Maple, we obtained the following results:

\[
\begin{align*}
x_2 &= \frac{13}{6} x_1 & x_2 &= \frac{1}{2} x_1 & x_2 &= \zeta x_1 \\
x_3 &= \frac{3}{2} x_1 & x_3 &= \frac{9}{7} x_1 & x_3 &= \left(\frac{-72 \zeta^2 + 120 \zeta - 33}{7}\right) x_1 \\
\lambda_1 &= \frac{1}{\sqrt{2}} x_1 & \lambda_1 &= 0 & \lambda_1 &= 0,
\end{align*}
\]

where \( \zeta \) is a real, positive solution to \( 24 \zeta^3 - 28 \zeta^2 + 5 \zeta - 5 = 0 \). (This gives \( x_2 \sim 1.144 x_1 \) and \( x_3 \sim 1.437 x_1 \).)

We find that the first metric is the product metric: we show that it is the only metric of the three with the symmetric metric on \( S^7 \). Since we have \( S^7 = \text{SU}(4)/\text{SU}(3) \), which is not a symmetric pair, we must project. Recall we chose \( p_1 \) so that \( \mathfrak{su}(4) = \mathfrak{su}(3) \oplus p_1 \oplus p_0 \); we will
project a typical element in \( p_1 \) and a typical element in \( p_0 \) to \( \tilde{p} \), the \( Q \)-orthogonal complement to \( \mathfrak{so}(7) \) in \( \mathfrak{so}(8) \). In \( p_1 \), \( \frac{1}{\sqrt{2}}(E_{12} + E_{56}) \mapsto \frac{1}{\sqrt{2}}E_{12} \), while in \( p_0 \), \( \frac{1}{2\sqrt{3}}(E_{15} - E_{26} - E_{37} - E_{48}) \mapsto \frac{1}{2\sqrt{3}}E_{15} \). Thus \( x_1 \mapsto \frac{1}{2} \) and \( x_3 \mapsto \frac{3}{4} \), so the symmetric metric satisfies \( \frac{3}{2}x_1 = x_3 \); this is the first metric above, exactly.

The second metric is a fibration metric, coming from the fibration

\[
S^1 = U(3)/SU(3) \to Spin^+(7)/SU(3) \to Spin^+(7)/U(3).
\]

Recall from our earlier discussion \( B = Spin^+(7)/U(3) \cong SO(8)/U(4) \). We found two \( Spin^+(7) \)-invariant Einstein metrics on the base space; we will show that the symmetric metric induces an Einstein metric on \( S^6 \times S^7 \) and the other does not.

With the symmetric metric, \( x_2 = \frac{1}{2}x_1 \), we see that for \( X \) a unit vector in \( p_1 \), and \( Y \) a unit vector in \( p_2 \), the O’Neill tensor (of our Riemannian submersion) satisfies

\[
|A_X|^2 = \frac{1}{3x_1^2} \quad \text{and} \quad |A_Y|^2 = \frac{1}{12x_2^2} = \frac{4}{12x_1^2} = \frac{1}{3x_1^2}.
\]

Thus the O’Neill tensor is a constant multiple of \( Q \). Consider the 1-parameter family of metrics \( g_t = g_B + tg_F \) \((t > 0)\) obtained by scaling in the direction of the fibre and keeping the metric fixed in the directions parallel to the base. A constant O’Neill tensor implies there is an Einstein metric in the 1-parameter family. (Since the fibre is flat, we expect only one.) Proposition 9.70 in [Bes, p.253] implies \( t = \frac{9}{7}x_1 \), which is exactly our second metric.

Why doesn’t the other \( Spin^+(7) \)-invariant Einstein metric on \( Spin^+(7)/U(3) \) induce an Einstein metric on \( S^6 \times S^7 ? \) We must again consider the O’Neill tensor. This time we substitute \( x_2 = \frac{3}{4}x_1 \):

\[
|A_X|^2 = \frac{1}{3x_1^2}, \quad \text{and} \quad |A_Y|^2 = \frac{1}{12x_2^2} = \frac{4}{27x_1^2}.
\]
We do not get a constant multiple of $Q$, thus by [Bes, Prop.9.70] there can be no Einstein metrics arising from this fibration.

Our three metrics cannot be isometric. We compare a scale-invariant constant and show that the constants are distinct, thus the metrics are distinct. Our constant is $(S)^{\frac{12}{7}}(V)^{\frac{1}{2}}$, where $S$ is the scalar curvature, and $V$ is the volume, for each metric. The first metric yields $(S)^{\frac{12}{7}}(V)^{\frac{1}{2}} \cong 1.245982 \times 10^{11}$. The second metric yields $(S)^{\frac{12}{7}}(V)^{\frac{1}{2}} \cong 1.0350064 \times 10^{11}$; the third metric yields $(S)^{\frac{12}{7}}(V)^{\frac{1}{2}} \cong 9.308607 \times 10^{10}$.

2.3. $S^7 \times G_2^+(\mathbb{R}^8)$

The next product of symmetric spaces we consider is $S^7 \times G_2^+(\mathbb{R}^8)$, the product of the seven-sphere with the Grassmannian of oriented two-planes in $\mathbb{R}^8$. We can write this space homogeneously as $\text{Spin}(8)/\text{U}(3)$. We know there is at least one homogeneous Einstein metric: the product metric. We will show that there are exactly two distinct $\text{Spin}(8)$-invariant Einstein metrics on $S^7 \times G_2^+(\mathbb{R}^8)$.

We again identify $\mathbb{R}^8$ with the Cayley numbers, $\mathbb{O}$, to describe $\text{Spin}(8)$ as a matrix group acting on $\mathbb{O} \times \mathbb{O}$. Recall

$$\text{Spin}(8) = \{(A, B, C) \in \text{SO}(8)^3 \mid A(x)B(y) = C(xy) \forall x, y \in \mathbb{O}\}.$$

For $(x, P) \in S^7(1) \times G_2^+(\mathbb{R}^8)$ we have the action $(A, B, C) : (x, P) \mapsto (A(x), B(P))$. We first show that this action is transitive and find the isotropy subgroup, then we describe the space of homogeneous metrics on $S^7 \times G_2^+(\mathbb{R}^8)$. To see that the action is transitive, we take any element $(x, P)$ in $S^7(1) \times G_2^+(\mathbb{R}^8)$, and we construct a map in $\text{Spin}(8)$ taking $(x, P)$ to $(1, P_0)$, where $P_0$ is the oriented two-plane span$\{1, i\}$. Recall, we have the following
subgroups of Spin(8):

$$\text{Spin}^+(7) = \{(A, B, C) \in \text{Spin}(8) \mid B = C\}$$

$$\text{Spin}^-(7) = \{(A, B, C) \in \text{Spin}(8) \mid A = C\}.$$

Since \(\text{Spin}^-(7)\) acts transitively on \(S^7 \times \{1\}\), there is an element \((A, B, A)\) in \(\text{Spin}^-(7)\) taking \((x, P) \mapsto (1, P')\). Similarly, we know from a previous example that \(\text{Spin}^+(7)\) acts transitively on \(\{1\} \times G_3^+(\mathbb{R}^8)\), thus \(\text{Spin}^+(7)\) acts transitively on \(\{1\} \times G_2^+(\mathbb{R}^8)\). Hence there is a map in \(\text{Spin}^+(7)\) taking \((1, P') \mapsto (1, P_0)\). Their composition sends \((x, P) \mapsto (1, P') \mapsto (1, P_0)\).

Next we show that the isotropy subgroup \(H\) of \(\text{Spin}(8)\) corresponding to \((1, P_0)\) is \(U(3) \cong S(U(1) U(3)) \subset SU(4) \subset \text{Spin}^+(7)\). By noting that for \((A, B, C) \in \text{Spin}(8)\), \(A(1) = 1\) implies that \(B = C\), we see that \(H \subset \text{Spin}^+(7)\). Then \(B(P_0) = P_0\) means for some angle \(\theta\), \(B(1) = e^{i\theta}\) and \(B(i) = ie^{i\theta}\), hence \(A(i) = i\). Since \(A(i) = i, H \subset SU(4)\). In fact, we have shown \(H \subset S(U(1) U(3))\). By a dimension count, we conclude \(H = S(U(1) U(3))\).

As in the previous examples we identify \(\text{spin}(8)\) with \(\mathfrak{so}(8)\) via the differential of the map taking \((A, B, C) \mapsto C\). On the Lie algebra level we have \(\mathfrak{so}(8) = \mathfrak{u}(3) \oplus \mathfrak{p}\), where \(\mathfrak{p}\) is the orthogonal complement to \(\mathfrak{u}(3)\) with respect to our usual comparison metric \(Q(X, Y) = -\frac{1}{2} \text{tr}(XY)\). We have three fibrations of our space; using them we decompose \(\mathfrak{p}\) into a sum of irreducible representations of \(\mathfrak{u}(3)\):

\[
\mathbb{C}P^3 \cong SU(4)/U(3) \to \text{Spin}(8)/U(3) \to \text{Spin}(8)/SU(4) \cong V_2(\mathbb{R}^8),
\]

\[
G_2^+(\mathbb{R}^8) \cong \text{Spin}^+(7)/U(3) \to \text{Spin}(8)/U(3) \to \text{Spin}(8)/\text{Spin}^+(7) \cong S^7,
\]

\[
S^7 \cong U(4)/U(3) \to \text{Spin}(8)/U(3) \to \text{Spin}(8)/U(4) \cong G_2^+(\mathbb{R}^8).
\]

Let \(\rho_k\) denote the standard representation of \(\text{SO}(k)\) and let \(\mu_k\) denote the standard representation of \(U(k)\). In the first fibration, the fibre is an irreducible symmetric space, with
isotropy representation $\mathfrak{p}_1 = [\mu_1 \otimes \mu_3]_\mathbb{R}$. The base space is isomorphic to $\text{SO}(8)/\text{SO}(6)$; we know that the isotropy representation of the base is $\rho_6 \oplus \rho_6 \oplus \text{Id}$. When we restrict this to $\text{U}(3)$, we obtain the decomposition $\mathfrak{p}_2 \oplus \mathfrak{p}_3 \oplus \mathfrak{p}_0 = [\mu_3]_\mathbb{R} \oplus [\mu_3]_\mathbb{R} \oplus \text{Id}.

In the second fibration, although the fibre is an irreducible symmetric space, $\text{Spin}^+(7)$ is not the full isometry group, so the isotropy representation is reducible: considering $\text{U}(3) \subset \text{SU}(4) \subset \text{Spin}^+(7)$ we see that the tangent space to the fibre is $\mathfrak{p}_1 \oplus \mathfrak{p}_2 = [\mu_1 \otimes \mu_3]_\mathbb{R} \oplus [\mu_3]_\mathbb{R}$. The base space, $\text{SO}(8)/\text{SO}(7)$, is also a symmetric space; its isotropy representation is $\rho_7$. When we restrict $\rho_7$ to $\text{U}(3) \subset \text{SO}(6) \subset \text{SO}(7)$, we get $\mathfrak{p}_3 \oplus \mathfrak{p}_0 = [\mu_3]_\mathbb{R} \oplus \text{Id}.

In the third fibration, the fibre is a symmetric space, but not a symmetric pair; the isotropy representation is $\mathfrak{p}_1 \oplus \mathfrak{p}_0 = [\mu_1 \otimes \mu_3]_\mathbb{R} \oplus \text{Id}$. The base space is isomorphic to $\text{SO}(8)/\text{SO}(2) \text{SO}(6)$, an irreducible symmetric space with isotropy representation $\rho_2 \otimes \rho_6$, which when restricted to $\text{U}(3)$ gives $\mathfrak{p}_3 \oplus \mathfrak{p}_0 = [\mu_3]_\mathbb{R} \oplus [\mu_3]_\mathbb{R}$. We conclude that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \oplus \mathfrak{p}_0$.

This decomposition is not unique, since $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are equivalent representations of $\mathfrak{u}(3)$. We choose the decomposition to be orthogonal with respect to $Q$ and so that $\mathfrak{su}(4) = \mathfrak{u}(3) \oplus \mathfrak{p}_1$, $\mathfrak{u}(4) = \mathfrak{u}(3) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_0$, and $\mathfrak{spin}^+(7) = \mathfrak{u}(3) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

$$\mathfrak{u}(3) = \text{span}\{E_{37} - E_{48}, E_{34} + E_{78}, E_{38} + E_{47}, E_{23} + E_{67}, E_{24} + E_{68}, E_{27} + E_{36}, E_{28} + E_{46}, 2E_{26} - E_{48} - E_{37} - E_{26} - E_{37} - E_{48}\},$$

$$\mathfrak{p}_1 = \text{span}\{E_{16} + E_{25}, E_{17} + E_{35}, E_{18} + E_{45}, E_{12} + E_{56}, E_{13} + E_{57}, E_{14} + E_{58}\},$$

$$\mathfrak{p}_2 = \text{span}\{E_{16} - E_{25} + E_{38} - E_{47}, E_{17} - E_{35} + E_{46} - E_{28}, E_{18} - E_{45} + E_{27} - E_{36}, E_{12} - E_{56} - E_{34} + E_{78}, E_{13} - E_{57} + E_{24} - E_{68}, E_{14} - E_{58} - E_{23} + E_{67}\},$$

$$\mathfrak{p}_3 = \text{span}\{E_{12} - E_{56} + E_{34} - E_{78}, E_{13} - E_{57} - E_{24} + E_{68}, E_{14} - E_{58} + E_{23} - E_{67}\}.$$
\[ E_{16} - E_{25} - E_{38} + E_{47}, E_{17} - E_{35} - E_{46} + E_{28}, E_{18} - E_{45} - E_{27} + E_{36}, \]

\[ p_0 = \text{span}\{E_{15} + E_{26} + E_{37} + E_{48}\}. \]

Any invariant metric \( \langle \cdot, \cdot \rangle \) must satisfy \( \langle \cdot, \cdot \rangle = Q(g \cdot, \cdot) \), where we identify \( g \) with the metric, and \( g \) is a positive definite symmetric linear operator which commutes with the adjoint action of \( U(3) \). As in the previous example, we need to use Schur’s lemma to parametrize the space of maps between the equivalent representations \( p_2 \) and \( p_3 \). We have \( p_2 \cong p_3 \cong [\mu_3]_\mathbb{R} \), which is unitary. When we complexify \( [\mu_3]_\mathbb{R} \), we get the sum \( \mu_3 \oplus \mu_3^* \). Thus we have two dimensions of intertwining maps: the generators are the map interchanging \( p_2 \) and \( p_3 \), and a complex structure on each, with respect to the ordered basis above. That is,

\[
g = \begin{pmatrix}
  x_1 \text{Id}_3 & 0 & 0 & 0 & 0 & 0 \\
  0 & x_2 \text{Id}_3 & 0 & \lambda_1 \text{Id}_3 & \lambda_2 \text{Id}_3 & 0 \\
  0 & 0 & x_2 \text{Id}_3 & -\lambda_2 \text{Id}_3 & \lambda_1 \text{Id}_3 & 0 \\
  0 & \lambda_1 \text{Id}_3 & -\lambda_2 \text{Id}_3 & x_3 \text{Id}_3 & 0 & 0 \\
  0 & \lambda_2 \text{Id}_3 & \lambda_1 \text{Id}_3 & 0 & x_3 \text{Id}_3 & 0 \\
  0 & 0 & 0 & 0 & 0 & x_4
\end{pmatrix}.
\]

The scaling factors \( x_1, x_2, x_3, x_4, \lambda_1 \), and \( \lambda_2 \) parametrize the space of homogeneous metrics on \( \text{Spin}(8)/U(3) \). We have a six-parameter family of \( \text{Spin}(8) \)-invariant metrics, but without loss of generality, we can simplify in the following way: We see that the normalizer of \( U(3) \) in \( \text{Spin}(8) \) is \( U(1) \cdot U(3) \subset U(4) \); its corresponding Lie algebra is \( u(3) \oplus p_0 \) (since \( u(3) \) and \( p_0 \) commute). As in the previous example, conjugation by any element of \( U(1) = N(U(3))/U(3) \) is a diffeomorphism fixing \( p \), which gives us a one-parameter group of isometries of homogeneous metrics. With the given basis for \( p_2 \) and \( p_3 \), \( \{X_i\}_{i=7}^{12} \) and \( \{X_i\}_{i=13}^{18} \), these Lie bracket relations hold: (\( X_{19} \) is the given \( Q \)-unit vector whose span is \( p_0 \).)

\[
[X_{19}, X_i] = X_{i+6} \text{ for } X_i \in p_2 \text{ and } [X_{19}, X_j] = -X_{j-6} \text{ for } X_j \in p_3.
\]
Hence via \( \text{Ad}(\exp tX_{19}) \cdot g \), any homogeneous metric is isometric to one with \( \lambda_1 = 0 \). We get the following scalar curvature from (+).

\[
S = 3 \left( \frac{x_1 x_4 (2 \lambda_2^2 - (x_2^2 + x_3^2)) + 8 x_4 (2 \lambda_2^2 - x_2 x_3) (\lambda_2^2 - x_2 x_3) + 4 (x_1 + 2 x_4) \lambda_1^4}{x_1 x_4 (x_2 x_3 \lambda_1^2 - \lambda_2^2)^2} 
\right.
\]

\[
+ \frac{x_1 (12 x_4 (x_2 + x_3) (\lambda_2^2 - x_2 x_3) + \lambda_2^2 (x_4^2 - (x_2 - x_3)^2) + x_2 x_3 (x_4^2 + (x_2 - x_3)^2))}{x_1 x_4 (x_2 x_3 - \lambda_1^2 - \lambda_2^2)^2}
\]

\[
\left. + \frac{(x_1 (x_2^2 + x_3^2 + 3 (x_2 x_3 + 4 x_2 x_4 + 4 x_3 x_4)) - 2 x_4 (x_1^2 + 8 x_2 x_3 + 12 \lambda_2^2)) \lambda_1^2}{x_1 x_4 (x_2 x_3 - \lambda_1^2 - \lambda_2^2)^2} \right).
\]

We normalize for volume one, \( \tilde{S} = S - k(x_1^6(x_2 x_3 - \lambda_1^2 - \lambda_2^2)^6 x_4) \). Observe that \( \tilde{S} \) is in fact a function of \( \lambda_1^2 \). Hence without loss of generality we look for the critical points of the scalar curvature for \( \lambda_1 = 0 \), using Maple. We find that there are two real solutions with every \( x_i > 0 \).

\[
x_1 = \frac{2 x_3}{3} \quad x_1 = \left( \frac{91 \zeta^4 + 54 \zeta^2 - 1}{8} \right) x_3
\]

\[
x_2 = \frac{x_3}{3} \quad x_2 = x_3
\]

\[
x_4 = \frac{2 x_3}{3} \quad x_4 = \left( \frac{91 \zeta^4 - 2 \zeta^2 + 7}{7} \right) x_3
\]

\[
\lambda_2 = \frac{x_3}{3} \quad \lambda_2 = \zeta \lambda_3,
\]

where \( \zeta \) is a real solution to \( 91 \zeta^6 + 131 \zeta^4 + 41 \zeta^2 - 7 = 0 \). (Approximately, \( x_1 \simeq 0.853 x_3 \), \( x_4 \simeq 1.154 x_3 \), and \( \lambda_2 \simeq 0.347 x_3 \).)

We know that these are not isometric: the first metric yields a scale-invariant constant \( (S)_{\mathcal{F}} (V)^{\frac{1}{2}} = 4.3401636 \times 10^{18} \) while the second metric yields \( (S)_{\mathcal{F}} (V)^{\frac{1}{2}} = 2.8743704 \times 10^{18} \).

We remark that the first metric is the product metric: there the Grassmannian carries the symmetric metric. Since in our fibration, we do not have the symmetric pair, we will project \( p_1 \oplus p_2 \), the tangent space to \( G_2^+ (\mathbb{R}^8) \) with the given presentation, onto \( \tilde{p} \), the tangent space to \( G_2^+ (\mathbb{R}^8) \) with the symmetric pair presentation. I.e., \( \tilde{p} \) is the \( Q \)-orthogonal complement to \( \mathfrak{s} \mathfrak{o}(2) \oplus \mathfrak{s} \mathfrak{o}(6) \) in \( \mathfrak{s} \mathfrak{o}(8) \). We see that \( p_1 \subset \tilde{p} \), so that a unit element in \( p_1 \)
projects to a unit element in $\tilde{p}$. On the other hand, we write a unit element of $p_2$, such as
\[ \frac{1}{2}(E_{16} - E_{25} + E_{38} - E_{47}), \]
to respect the decomposition $\mathfrak{so}(8) = \tilde{p} \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(6)$. We get
\[ \frac{1}{2}(E_{16} - E_{25}) + \frac{1}{2}(E_{38} - E_{47}), \]
and so a unit element in $p_2$ projects to an element of length $\frac{1}{2}$. Hence the symmetric metric is the metric in which $2x_2 = x_1$, this is the first metric. Thus there are exactly two Spin(8)-invariant Einstein metrics on $S^7(1) \times G_2^+(\mathbb{R}^8)$. 
Chapter 3: $V_2(\mathbb{R}^{n+1})$

The Stiefel manifold $V_2(\mathbb{R}^{n+1})$ of two-flags in Euclidean $n + 1$-space can be written homogeneously as $V_2(\mathbb{R}^{n+1}) = \text{SO}(n+1)/\text{SO}(n-1)$. Although it is not a symmetric space, $V_2(\mathbb{R}^{n+1})$ inherits an $\text{SO}(n+1)$-invariant Einstein metric from the Grassmannian $G_2^+(\mathbb{R}^{n+1})$ via the following fibration:

$$
S^1 \longrightarrow V_2(\mathbb{R}^{n+1}) \\
\downarrow \\
G_2^+(\mathbb{R}^{n+1}).
$$

I.e., there is a metric on $V_2(\mathbb{R}^{n+1})$ so that the fibration map is a Riemannian submersion with totally geodesic fibres. In the one-parameter family of metrics $g(t) = g_B + tg_F$ ($t > 0$) on $V_2(\mathbb{R}^{n+1})$ obtained by scaling the metric in the direction of the fibre $F = S^1$ and holding it fixed in the direction of the base $B = G_2^+(\mathbb{R}^{n+1})$, there is one Einstein metric [Bes, §9.77].

We show that, up to scaling, this is the only $\text{SO}(n+1)$-invariant Einstein metric $V_2(\mathbb{R}^{n+1})$ carries.

To see that $\text{SO}(n+1)$ acts transitively on $V_2(\mathbb{R}^{n+1})$, let us identify $V_2(\mathbb{R}^{n+1})$ with $T_1S^n$, the unit tangent bundle of the sphere. A two-flag is a choice of a line and a two-plane containing that line, $\mathcal{F} : \text{span}\{\perp\} \subset \text{span}\{\perp, \parallel\}$. We may assume that $v$ and $w$ are orthonormal, and thus $v \in S^n$ and $w \in S^{n-1} \subset T_vS^n$. To see that $\text{SO}(n+1)$ acts transitively, we will send $\mathcal{F}_t : \text{span}\{\perp\} \subset \text{span}\{\perp, t\perp\}$ (in the standard basis) to $\mathcal{F}$. We use a matrix with $v$ as the first column vector and $w$ as the second column vector, then fill in the rest of the columns to complete $v$ and $w$ to an orthonormal basis for $\mathbb{R}^{n+1}$ with the same orientation as the standard basis. The embedding $\text{SO}(n-1) \cong \begin{pmatrix} \text{Id}_2 & 0 \\ 0 & \text{SO}(n-1) \end{pmatrix} \subset \text{SO}(n+1)$ corresponds to the isotropy subgroup fixing the flag $\mathcal{F}_t$. 

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On the Lie algebra level, recall $\mathfrak{so}(k)$ is the space of $k \times k$ skew-symmetric matrices, and we have \[
\begin{pmatrix}
0 & 0 \\
0 & \mathfrak{so}(k-1)
\end{pmatrix} \subset \mathfrak{so}(n+1). 
\] Using the inner product $Q(X,Y) = -\frac{1}{2} \text{tr}(XY)$, we choose an orthogonal complement $\mathfrak{p}$ to $\mathfrak{so}(n-1)$ in $\mathfrak{so}(n+1)$. Any $\text{SO}(n+1)$-invariant metric on $V_2(\mathbb{R}^{n+1})$ is equivalent to an $\text{Ad} \mathfrak{so}(n+1)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$. Decomposing $\mathfrak{p}$ into irreducible subrepresentations of $\text{SO}(n-1)$, we find $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$:

\[
\begin{align*}
\mathfrak{p}_0 &= \text{span}\{E_{12}\}, \\
\mathfrak{p}_1 &= \text{span}\{e_i = E_{1,2+i} \mid 1 \leq i \leq n-1\}, \\
\mathfrak{p}_2 &= \text{span}\{f_i = E_{2,2+i} \mid 1 \leq i \leq n-1\}.
\end{align*}
\]

The decomposition is not unique, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are equivalent representations of $\text{SO}(n-1)$; we have chosen them so that $\mathfrak{so}(n) = \mathfrak{so}(n-1) \oplus \mathfrak{p}_2$. We have the following Lie bracket relations:

\[
\begin{align*}
[E_{12}, e_i] &= -f_i, & [E_{12}, f_i] &= e_i, \\
[e_i, f_j] &= -\delta_{ij} E_{12}, & [e_i, e_j] &= [f_i, f_j] &= -E_{1+2,j+2} \in \mathfrak{so}(n-1).
\end{align*}
\]

Let $\sigma : \mathfrak{p}_1 \to \mathfrak{p}_2$ be the linear map such that $e_i \mapsto f_i$ for all $i$. Then $\sigma$ is $\text{Ad} \mathfrak{so}(n-1)$-equivariant, and an isometry with respect to $Q$. Since $\mathfrak{p}_1 \simeq \mathfrak{p}_2$ is a totally irreducible representation of $\text{SO}(n-1)$, every intertwining map between $\mathfrak{p}_1$ and $\mathfrak{p}_2$ is a scalar multiple of $\sigma$, by Schur’s Lemma. This implies that for any $\text{Ad} \mathfrak{so}(n-1)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$, we have $\langle e_i, f_j \rangle = \lambda \delta_{ij}$ for some constant $\lambda \in \mathbb{R}$. Thus the metric $\langle \cdot, \cdot \rangle$, in terms of this basis for $\mathfrak{p}$, has the form

\[
(g_{ij}) = \begin{pmatrix} x_0 & 0 & 0 \\
0 & x_1 \text{Id}_{n-1} & \lambda \text{Id}_{n-1} \\
0 & \lambda \text{Id}_{n-1} & x_2 \text{Id}_{n-1} \end{pmatrix}.
\]
The action of $SO(n-1)$ on $\mathfrak{p}_0$ is trivial; this corresponds to a copy of $SO(2)$ in $SO(n+1)$ commuting with $SO(n-1)$. If $N(SO(n-1))$ is the normalizer of $SO(n-1)$ in $SO(n+1)$, we have $SO(2) = N(SO(n-1))/SO(n-1)$, with tangent algebra $\mathfrak{p}_0$. Using this $SO(2)$, we can find a $\theta$ such that $e'_i = \cos \theta e_i - \sin \theta f_i$ and $f'_i = \sin \theta e_i + \cos \theta f_i$, and $\langle e'_i, f'_i \rangle = 0$. Hence (by a change of basis if necessary), we can assume our metric is diagonal. The new Lie bracket relations are identical to the old ones, and this gives an orthogonal basis for $\mathfrak{p}$ with respect to both $Q$ and $\langle , \rangle$.

We check that the Ricci tensor is diagonal with respect to our orthonormal basis for $\langle , \rangle$. Then we consider the scalar curvature equation, from [W-Z2, (1.3)].

$$S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \left( \begin{array}{c} k \\ i_j \end{array} \right) \frac{x_k}{x_i x_j}.$$

We know from the Lie bracket relations that $\left( \begin{array}{c} 1 \\ 23 \end{array} \right) = 1$, all other triples (except rearrangements) are zero. Furthermore $d_1 = d_2 = n - 1$, $d_0 = 1$, and $b_i = 2(n - 1)$ for $i = 0, 1, 2$. Thus we have

$$S = (n - 1) \left( \frac{n - 1}{x_1} + \frac{n - 1}{x_2} + \frac{1}{x_0} \right) - \frac{n - 1}{2} \left( \frac{x_1}{x_2 x_0} + \frac{x_2}{x_1 x_0} + \frac{x_0}{x_1 x_2} \right).$$

We normalize for volume 1, $\tilde{S} = S - \lambda (x_1^{n-1} x_2^{n-1} x_0 - 1)$, where $\lambda$ is the Lagrange multiplier. We find

$$\frac{\partial \tilde{S}}{\partial x_1} = (n - 1) \left( - \frac{n - 1}{x_1^2} + \frac{1}{2} \left( \frac{-1}{x_2 x_0} + \frac{x_2}{x_1 x_0} + \frac{x_0}{x_1 x_2} \right) - \lambda x_1^{n-2} x_2^{n-1} x_0 \right)$$

$$\frac{\partial \tilde{S}}{\partial x_2} = (n - 1) \left( - \frac{n - 1}{x_2^2} + \frac{1}{2} \left( \frac{-1}{x_1 x_0} + \frac{x_1}{x_2 x_0} + \frac{x_0}{x_1 x_2} \right) - \lambda x_1^{n-1} x_2^{n-2} x_0 \right)$$

$$\frac{\partial \tilde{S}}{\partial x_0} = (n - 1) \left( - \frac{1}{x_0^2} + \frac{1}{2} \left( \frac{-1}{x_1 x_2} + \frac{x_1}{x_2 x_0} + \frac{x_2}{x_1 x_0} \right) \right) - \lambda x_1^{n-1} x_2^{n-1}. $$
Solving for critical points we obtain the following equations:

\[ x_0^2 - x_1^2 + x_2^2 - 2(n - 1)x_2x_0 = (n - 1)((x_1 - x_2)^2 - x_0) \]

\[ (n - 1)(x_1 - x_2)x_0 + x_2^2 - x_1^2 = 0. \]

We conclude that if \( x_1 = x_2 \) then \( x_0 = 2\left(\frac{n-1}{n}\right)x_1 \). If \( x_1 \neq x_2 \), there are no solutions. Thus we have exactly one \( \text{SO}(n + 1) \)-homogeneous Einstein metric on \( V_2(\mathbb{R}^{n+1}) \), up to scaling.
Oniščik in fact lists more triples of Lie algebras \((\mathfrak{g}, \mathfrak{g}', \mathfrak{g}'')\), but the extra triples can be obtained by combining the information given on page 5. For example, in addition to \(\text{SO}(2n)/\text{SO}(2n-1) = \text{SU}(n)/\text{SU}(n-1)\) and \(\text{SO}(4n)/\text{SO}(4n-1) = \text{Sp}(n)/\text{Sp}(n-1)\), he lists \(\text{SU}(2n)/\text{SU}(2n-1) = \text{Sp}(n)/\text{Sp}(n-1)\), which follows from the inclusions \(\text{Sp}(n) \subset \text{SU}(2n) \subset \text{SO}(4n)\). Many of the triples on his list come from the subgroups of \(\text{SO}(8)\): In addition to \(\text{SO}(7) \subset \text{SO}(8)\), we have

\[
\text{Sp}(2) \subset \text{Sp}(2) \text{U}(1) \subset \text{Sp}(2) \text{Sp}(1) \subset \text{SO}(8),
\]

\[
\text{U}(2) \subset \text{SU}(3) \subset \text{SU}(4) \subset \text{U}(4) \subset \text{SO}(8),
\]

and \(\text{SO}(4) \subset \text{G}_2 \subset \text{Spin}(7) \subset \text{SO}(8)\).

There are two copies of \(\text{Spin}(7)\) in \(\text{SO}(8)\) and an outer automorphism of \(\text{SO}(8)\) of order three, called the triality automorphism, which interchanges the \(\text{Spin}(7)\)'s. On the Lie algebra level, it interchanges the \(\mathfrak{spin}(7)\)'s and the standard embedding of \(\mathfrak{so}(7)\), yielding equalities like the following:

\[
\text{SO}(8)/\text{Spin}(7) = \text{SO}(6)/\text{SU}(3) \quad \text{(with double cover } \text{SO}(8)/\text{SO}(7) = \text{SU}(4)/\text{SU}(3))
\]

\[
\text{SO}(8)/\text{Spin}(7) = \text{SO}(5)/\text{SU}(2) \quad \text{(with double cover } \text{SO}(8)/\text{SO}(7) = \text{Sp}(2)/\text{Sp}(1)).
\]

We include some intersections of subgroups of \(\text{SO}(8)\) and related equalities:

\[
\text{G}_2 = \text{SO}(7) \cap \text{Spin}(7) \quad \Rightarrow \text{SO}(8)/\text{SO}(7) = \text{Spin}(7)/\text{G}_2;
\]

\[
\text{Sp}(1) \text{Sp}(1) = \text{SO}(7) \cap \text{Sp}(2) \text{Sp}(1) \quad \Rightarrow \text{SO}(8)/\text{SO}(7) = \text{Sp}(2) \text{Sp}(1)/\text{Sp}(1) \text{Sp}(1);
\]

\[
\text{SU}(3) = \text{SO}(7) \cap \text{SU}(4) \quad \Rightarrow \text{SO}(8)/\text{SO}(7) = \text{SU}(4)/\text{SU}(3).
\]

Here are the non-symmetric homogeneous spaces on Oniščik’s list [O1]:

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\[ \frac{\text{SO}(7)}{\text{SO}(5)} = \frac{G_2}{\text{SU}(2)} = V_2(\mathbb{R}^7) \]
\[ \frac{\text{SO}(8)}{\text{SO}(6)} = \frac{\text{Spin}(7)}{\text{SU}(3)} = V_2(\mathbb{R}^8) \]
\[ \frac{\text{SO}(8)}{\text{SO}(5)} = \frac{\text{Spin}(7)}{\text{SU}(2)} = V_3(\mathbb{R}^8) \]
\[ \frac{\text{SO}(8)}{\text{SO}(2) \text{SO}(5)} = \frac{\text{Spin}(7)}{\text{SO}(2) \text{SU}(2)} \]
\[ \frac{\text{SO}(16)}{\text{Spin}(9)} = \frac{\text{SO}(15)}{\text{Spin}(7)} \]
\[ \frac{\text{SO}(2n)}{\text{SU}(n)} = \frac{\text{SO}(2n - 1)}{\text{SU}(n - 1)} \]
\[ \frac{\text{SO}(4n)}{\text{Sp}(n)} = \frac{\text{SO}(4n - 1)}{\text{Sp}(n - 1)} \]
\[ \frac{\text{SO}(4n)}{\text{Sp}(n) \text{U}(1)} = \frac{\text{SO}(4n - 1)}{\text{Sp}(n - 1) \text{U}(1)} \]
\[ \frac{\text{SO}(4n)}{\text{Sp}(n) \text{Sp}(1)} = \frac{\text{SO}(4n - 1)}{\text{Sp}(n - 1) \text{Sp}(1)}. \]
References


