THE GEOMETRY OF FILIFORM NILPOTENT LIE GROUPS

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Abstract. We study the geometry of a filiform nilpotent Lie group endowed with a left-invariant metric. We describe the connection and curvatures, and we investigate necessary and sufficient conditions for subgroups to be totally geodesic submanifolds. We also classify the one-parameter subgroups which are geodesics.

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1. Introduction

Nilpotent Lie groups endowed with left invariant metrics (*nilmanifolds*) arise naturally in many areas of mathematics, including algebra, dynamics and control theory, and in geometry they are studied in homogeneous geometry, spectral geometry, subriemannian geometry and harmonic analysis. There has been extensive study of the geometry of two-step nilmanifolds and their compact quotients. In his foundational work [Ebe94a, Ebe94b], Eberlein investigates nonsingular two-step nilmanifolds, describing curvatures, geodesics, totally geodesic submanifolds, and density of closed geodesics in compact quotients. By now the geometry of two-step nilmanifolds is well understood (see [Ebe04a, Ebe04b]), particularly groups with additional structure, such as those of Heisenberg type (see [BTV95]). The geometry of two-step nilmanifolds provides the setting for many of the examples of isospectral, non-isomorphic spaces (see [DGGW93, GG97]).

The geometry of higher-step nilpotent Lie groups is as yet unexplored. Gornet uses three-step nilmanifold geometry to construct examples with some prescribed spectral properties [Gor94, Gor96b, Gor96a]. Lauret analyzes preferred ("minimal") metrics on general nilmanifolds ([Lau01, Lau03, Lau05]). Soliton metrics on higher-step nilmanifolds have been studied in low dimensions ([Wil03]) and for several infinite families ([Lau02, Pay05]). It is time for a more thorough investigation of the geometric properties of generic higher-step nilmanifolds. This paper is an initial foray into the higher-step setting.

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We consider the class of nilpotent Lie groups called filiform. An n-dimensional nilpotent Lie group is *filiform* (threadlike) if the lower central series of its corresponding Lie algebra is as long as possible, having $n-2$ nontrivial subalgebras. That is, an n-dimensional filiform Lie algebra is $(n-1)$-step. A special family within the class of filiform nilpotent Lie groups is L_n. In L_n, one may choose a basis $\{X_i\}$ for the Lie algebra such that the nontrivial Lie bracket relations are given by $[X_1, X_i] = X_{i+1}$. Just as the groups of Heisenberg type can be viewed as model spaces for two-step nilgeometry, the filiform nilmanifolds L_n can be said to be model spaces for filiform nilgeometry. Notice L_3 is exactly the Heisenberg group. Thus, Heisenberg-type nilmanifolds are one way to generalize the Heisenberg group, maintaining the number of steps but enlarging the dimension of the top step, while the filiform group L_n is another way to generalize, where this time the number of steps increases as the dimension goes up.

The class of filiform nilpotent Lie groups has proved to be a rich source of examples, and especially counterexamples, in algebra. They play a key role in the study of characteristically nilpotent Lie algebras ([Kha02]). They show promise for playing an equally important role in geometry. For instance, the first example of a nilpotent Lie algebra admitting no affine structure is filiform [Ben92]. We believe that in geometry, filiform nilmanifolds will provide new examples and help shape our intuition about nilmanifold and solvmanifold geometry.

For a general higher-step nilmanifold, understanding the geometry is difficult; the connection and curvature are considerably more complicated than in the two-step setting. In contrast, for filiform Lie groups, the calculations are manageable, yet the class is still large. In fact, despite much research on their algebraic properties, filiform Lie algebras are not yet classified. Here we give a description of the geometry of filiform nilmanifolds. Our results give insight into similarities and differences between two-step and higher-step nilpotent geometry. This should allow others to further explore existence and nonexistence questions in geometry.

In Section 2 we define the classes $\mathcal{C}_n$ and $\mathcal{L}_n$ of filiform nilpotent Lie algebras with $\mathbb{N}$-gradings. The set $\mathcal{C}_n$ is relatively large: it includes several basic continuous families of examples ([Kha00]). Thanks to the grading, the computations that describe the geometry of the nilmanifolds in $\mathcal{C}_n$ and $\mathcal{L}_n$ are simplified.

In Sections 3 and 4, we compute the connection and curvatures for nilmanifolds arising from Lie algebras in the classes $\mathcal{C}_n$ and $\mathcal{L}_n$ respectively. To the best of our knowledge, this is the first detailed description of the basic geometry of nilmanifolds in the fundamental class $\mathcal{L}_n$. In Section 5, we classify geodesics that are one-parameter subgroups for elements of $\mathcal{C}_n$. We also find a restrictive characteristic of those higher-dimensional totally geodesic submanifolds that are subgroups. In Section 6, we completely classify totally geodesic subalgebras of filiform metric Lie algebras in the family $\mathcal{L}_n$, showing that the only such examples arise from flat abelian subgroups. We conclude in Section 7 with a comparison of filiform nilmanifold geometry: How do these "almost abelian" Lie groups compare geometrically to two-step nilpotent Lie groups? And to abelian (flat) space?

2. Preliminaries

2.1. **Geometry of Lie groups.** Let G be a simply connected Lie group endowed with a left-invariant metric g. We may identify the Riemannian manifold (G,g) with the metric Lie algebra $(\mathfrak{g},\langle\cdot,\cdot\rangle)$, where $\mathfrak{g}\cong T_eG$ is the Lie algebra of G, the tangent space at the identity. We let $\langle\cdot,\cdot\rangle$ denote the inner product on $\mathfrak{g}$ obtained by restricting our metric g to T_eG. Since the metric g is left-invariant, we identify the connection and curvature operators on (G,g) with the corresponding operators on the metric Lie algebra $(\mathfrak{g},\langle\cdot,\cdot\rangle)$.

Consider a Lie algebra $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ with an orthonormal basis $\{X_i\}_{i=1}^n$. Here we describe the Levi-Civita connection, sectional curvature K, and Ricci form ric (see [Ale75, Equations (2.1, 2.2, 2.3)]):

$$\nabla_X Y = \tfrac{1}{2}\left(\operatorname{ad}_X Y - \operatorname{ad}_X^* Y - \operatorname{ad}_Y^* X\right) \tag{2.1}$$

$$K(X\wedge Y) = ||\nabla_X Y||^2 - \langle \nabla_X X, \nabla_Y Y\rangle - \langle [Y,[Y,X]],X\rangle - ||[X,Y]||^2 \tag{2.2}$$

In the case that $\mathfrak{g}$ is nilpotent,

$$\operatorname{ric}(X,Y) = -\tfrac{1}{2}\textstyle\sum_i \langle [X,X_i],[Y,X_i]\rangle + \tfrac{1}{4}\sum_{i,j}\langle [X_i,X_j],X\rangle\langle [X_i,X_j],Y\rangle. \tag{2.3}$$

We break the connection into its skew-symmetric and symmetric parts (cf. [Bes87, 7.28]):

$$\nabla_X Y = \tfrac{1}{2}[X,Y] + U(X,Y) \tag{2.4}$$

where $U(X,Y)$ is the symmetric $(2,1)$ tensor

$$\langle U(X,Y),Z\rangle = \tfrac{1}{2}\langle [Z,X],Y\rangle + \tfrac{1}{2}\langle [Z,Y],X\rangle.$$

For $k=1,\ldots,n$, we define the symmetric $(2,0)$ tensor U_k to be the k^{th} component of U:

$$U_k(X,Y) = \langle U(X,Y),X_k\rangle.$$

Notice $\langle U(X,X),Z\rangle = \langle [Z,X],X\rangle$ and $U_k(X,X) = \langle [X_k,X],X\rangle$.

In the following lemma, we remind the reader of some basic properties of the tensors U and U_k for general metric Lie algebras (cf. [Bes87]).

Lemma 2.1. *Let $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ be a metric Lie algebra with orthonormal basis $\{X_i\}_{i=1}^n$.*

(i) *If X and Y are in the centralizer of X_k, $U_k(X,Y)=0$*
(ii) *For $\mathfrak{z}(\mathfrak{g})$ the center of $\mathfrak{g}$, $U|_{\mathfrak{z}(\mathfrak{g})}\equiv 0$.*

For any Riemannian manifold (M,g), a submanifold M' is said to be *totally geodesic* if for any vector fields X,Y in $\mathfrak{X}(M')$, $\nabla_X Y$ is in $\mathfrak{X}(M')$. Let $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ be a metric Lie algebra with its corresponding simply connected Lie group with left-invariant metric (G,g). We say a subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ that is closed under ∇ (i.e., for all X and Y in $\mathfrak{m}$, $\nabla_X Y$ is in $\mathfrak{m}$) is a *totally geodesic subalgebra* of $(\mathfrak{g},\langle\cdot,\cdot\rangle)$. The totally geodesic submanifolds which are closed subgroups M of G are in one-to-one correspondence with the subspaces $\mathfrak{m}$ of $\mathfrak{g}$ which are closed under ∇. If a subspace $\mathfrak{m}$ of $\mathfrak{g}$ is closed under ∇, then via the torsion formula for the connection, we know that $\mathfrak{m}$ is also closed under the Lie bracket; thus, $\mathfrak{m}$ is a subalgebra and $M' = \exp(\mathfrak{m})$ is a totally geodesic submanifold. Conversely, if a subgroup M is totally geodesic, then its Lie algebra $\mathfrak{m}$, viewed as a subalgebra of $\mathfrak{g}$, is closed under ∇.

In the following elementary lemma, we describe the totally geodesic subalgebra property in terms of the tensor U.

Lemma 2.2. *Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ be a metric Lie algebra and let $\mathfrak{m}$ be a subalgebra of $\mathfrak{g}$. Then the following are equivalent:*

(i) *$\mathfrak{m}$ is totally geodesic;*
(ii) *$U(X, Y)$ is in $\mathfrak{m}$ for all X and Y in $\mathfrak{m}$;*
(iii) *$U(X, X)$ is in $\mathfrak{m}$ for all X in $\mathfrak{m}$;*
(iv) *$\langle U(X, X), Z \rangle = \langle X, [X, Z] \rangle = 0$ for all X in $\mathfrak{m}$ and for all Z in $\mathfrak{m}^\perp$.*

Proof. Since $\mathfrak{m}$ is a subalgebra, Equation (2.4) implies that $\mathfrak{m}$ is closed under ∇ if and only if $U(X, Y)$ is in $\mathfrak{m}$ for all X and Y in $\mathfrak{m}$. Thus (i) and (ii) are equivalent. Clearly (ii) implies (iii). To see that (iii) implies (ii), we note that the vectors $U(X + Y, X + Y), U(X, X)$ and $U(Y, Y)$ are all in $\mathfrak{m}$ and U is symmetric, so $U(X, Y)$ is also in $\mathfrak{m}$. Finally we show the equivalence of (iii) and (iv). The vector $U(X, X)$ is in $\mathfrak{m}$ for all X in $\mathfrak{m}$ if and only if $\langle X, [X, Z] \rangle = 0$ for all Z in $\mathfrak{m}^\perp$, because $\langle U(X, X), Z \rangle = \langle X, [X, Z] \rangle$. □

Remark 2.3. If $\mathfrak{m}$ is one-dimensional, then $\mathfrak{m}$ is totally geodesic if and only if $U|_{\mathfrak{m}} \equiv 0$.

From Lemmas 2.1 and 2.2 we get the following nice result (obvious but worth mentioning).

Proposition 2.4. *Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ be a metric Lie algebra. If $\mathfrak{m}$ is an abelian subalgebra with $U|_{\mathfrak{m}} \equiv 0$, then $\mathfrak{m}$ is a totally geodesic subalgebra. Furthermore, $\mathfrak{m}$ is flat.*

We will see in Theorems 5.1 and 6.3 that these are the *only* totally geodesic subalgebras for filiform metric Lie algebras in the class $\mathcal{L}_n$.

2.2. **Filiform nilpotent Lie algebras.** For any Lie algebra $\mathfrak{g}$, the lower central series of $\mathfrak{g}$ is defined by $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{(j+1)} = [\mathfrak{g}, \mathfrak{g}^{(j)}]$. If $\mathfrak{g}^{(k)}$ is eventually trivial for some integer k, we say the Lie algebra $\mathfrak{g}$ is *nilpotent*. Let k be the smallest integer so that $\mathfrak{g}^{(k)}$ is trivial: then $\mathfrak{g}$ is said to be *k-step nilpotent*. A nilpotent Lie algebra which is k-step and dimension $k + 1$ is *filiform*.

Algebraists view filiform Lie algebras as almost abelian, having few nontrivial Lie brackets relative to dimension. Yet with their lower central series as long as possible, filiform Lie algebras are also the "least" nilpotent possible. The category of filiform Lie algebras is large: in dimension as low as seven, there are continuous families of nonisomorphic filiform Lie algebras. And though there has been a lot of research on their algebraic properties, filiform Lie algebras are not yet classified. Among filiform nilpotent Lie algebras, the algebra L_n (defined in Section 2) is the simplest. It has a codimension-one abelian ideal, thus it can be viewed as a high-step Lie algebra which is nearly abelian. Any filiform Lie algebra of dimension n can be viewed as a deformation of L_n via Lie algebra cohomology ([Ver70], see also [Mil04]).

We consider the family $\mathcal{C}_n$ of nilpotent metric Lie algebras $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ of dimension n ($n \geq 3$) whose nonvanishing Lie brackets are of the form

$$[X_i, X_j] = c_{i,j} X_{i+j} \tag{2.5}$$

where $\mathcal{B} = \{X_i\}_{i=1}^n$ is an orthonormal basis for $\mathfrak{n}$. A Lie algebra in $\mathcal{C}_n$ is filiform provided sufficiently many structure constants $c_{i,j}$ are nonzero. In what follows, we will assume the Lie algebra $\mathfrak{n}$ is nonabelian. Our focus is on those metric Lie algebras in $\mathcal{C}_n$ for which $c_{1,k} \neq 0$ for $1 < k < n-1$. We will denote this subfamily of filiform metric Lie algebras by $\mathcal{C}_n'$. The Jacobi identity for an element of $\mathcal{C}_n$ is equivalent to the following relation of the structure constants:

$$c_{i,j+k}c_{j,k} + c_{j,k+i}c_{k,i} + c_{k,i+j}c_{i,j} = 0 \qquad \text{for all } i, j, k. \tag{2.6}$$

For convenience, we define $c_{i,j}$ and X_i to be trivial if i or j fails to be in the set $\{1, 2, \ldots, n\}$.

As a special case, in each dimension n, we have the filiform Lie algebra L_n is defined by the Lie brackets

$$[X_1, X_i] = X_{i+1}.$$

We give L_n the natural metric $\langle\cdot,\cdot\rangle$ such that $\{X_i\}$ is an orthonormal basis. Notice that L_3 is the Heisenberg algebra of dimension three. We let $\mathcal{L}_n$ denote the subfamily of $\mathcal{C}_n$ with $[X_1, X_i] = c_i X_{i+1}$ (where $c_i := c_{1,i}$) and no other nontrivial brackets. Let $\mathcal{L}_n'$ denote the subfamily of elements of $\mathcal{L}_n$ with $c_2, \ldots, c_{n-1}$ nonzero. Each $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ in $\mathcal{L}_n'$ is isomorphic to L_n as an algebra; a change in the structure constants is really a rescaling of lengths in each step. The subalgebra $\mathfrak{a} = \operatorname{span}\{X_2, \ldots, X_n\}$ is a codimension-one abelian ideal in $\mathfrak{n}$ orthogonal to X_1.

2.3. **Properties of graded nilpotent metric Lie algebras.** Let S be a subset of the real numbers. We say a Lie algebra $\mathfrak{g}$ is *S-graded* if there is a decomposition $\mathfrak{g} = \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$ where $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in S$. Each element of $\mathcal{C}_n$ is $\mathbb{N}$-graded, with one-dimensional subspaces $\mathfrak{g}_i$. Given a graded metric Lie algebra $\mathfrak{g}$, we say it has an *adapted basis* $\mathcal{B} = \{X_i\}$ with respect to the grading if each basis vector X_i lies in some $\mathfrak{g}_\alpha$. Let $\alpha(i)$ denote the index of the subspace containing X_i: $X_i \in \mathfrak{g}_{\alpha(i)}$. When $\mathfrak{g}$ is an element of $\mathcal{C}_n$, we have an orthonormal adapted basis $\mathcal{B}$, and $\alpha(i) = i$ for each i.

Lemma 2.5. *Suppose that $\mathfrak{n} = \bigoplus_\alpha \mathfrak{n}_\alpha$ is an n-dimensional $\mathbb{N}$-graded nilpotent metric Lie algebra. Let $\langle\cdot,\cdot\rangle$ be the inner product on $\mathfrak{n}$ and let $\mathcal{B} = \{X_i\}$ be an orthonormal adapted basis. Then for any $X_k \in \mathcal{B}$,*

(i) $U_k(X_k, X) = 0$ *for any* X, *and*
(ii) $U(X_k, X_k) = 0$.

Proof. We first prove (i): $U_k(X_k, X_i) = \langle U(X_k, X_i), X_k\rangle = \frac{1}{2}\langle [X_k, X_k], X_i\rangle + \frac{1}{2}\langle [X_k, X_i], X_k\rangle = \frac{1}{2}\langle [X_k, X_i], X_k\rangle$. While $[X_k, X_i]$ is in the $\alpha(i) + \alpha(k)$ eigenspace, X_k is in the $\alpha(k)$ eigenspace. Since $\alpha(i) > 0$, these spaces are distinct, hence orthogonal. Thus $U_k(X_k, X_i) = 0$ for all $i = 1, \ldots, n$.

To prove (ii), we note that for each i, $U_i(X_k, X_k) = \langle [X_i, X_k], X_k\rangle$ and $[X_i, X_k]$ is in the $\alpha(i) + \alpha(k)$ space, while X_k is in the $\alpha(k)$ space. □

3. Connection and Curvatures in the Family $\mathcal{C}_n$

We now examine the geometric properties of metric Lie algebras in the family $\mathcal{C}_n$. Consider an arbitrary element $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ of $\mathcal{C}_n$, a higher-step nilpotent metric Lie algebra, with an adapted orthonormal basis $\mathcal{B} = \{X_i\}$ for $\mathfrak{n}$, so that nontrivial brackets are of the form $[X_i, X_j] = c_{i,j} X_{i+j}$. We will often use that

$$\mathrm{ad}^*_{X_i} X_j = c_{i,j-i} X_{j-i} \quad \text{for all } 1 \le i, j \le n. \tag{3.1}$$

We see that the geometry is not too complicated.

Theorem 3.1. *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be a metric Lie algebra in the family $\mathcal{C}_n$. Let $\mathcal{B} = \{X_i\}$ be an adapted orthonormal basis. The connection is given by*

$$\nabla_{X_i} X_j = \tfrac{1}{2}(c_{i,j} X_{i+j} - c_{i,j-i} X_{j-i} - c_{j,i-j} X_{i-j})$$

and for $X = \sum_{i=1}^n x_i X_i$ and $Y = \sum_{i=1}^n y_i X_i$,

$$\nabla_X Y = \tfrac{1}{2} \sum_{k=1}^n \left(\sum_{i=1}^n \left(x_i y_{k-i} c_{i,k-i} - x_i y_{k+i} c_{i,k} - x_i y_{i-k} c_{i-k,k} \right) \right) X_k.$$

The connection has the following properties:

(i) *For any i and j, $\nabla_{X_i} X_j$ has at most two nonzero terms.*
(ii) *$\nabla_{X_i} X_i = 0$ for any i.*
(iii) *$\nabla_{X_n} X_{n-1}$ is a multiple of X_1, and $\nabla_{X_1} X_n$ is a multiple of X_{n-1}.*
(iv) *For $k > 1$ and X in $[\mathfrak{n}, \mathfrak{n}]$, $U_k(X_1, X) = \frac{1}{2} c_{k,1} \langle X_{k+1}, X\rangle$.*

Proof. The expression for $\nabla_{X_i} X_j$ comes from Equations (2.1), (2.5) and (3.1). The first three properties follow from the formula for $\nabla_{X_i} X_j$. For (iv), we note that $U_k(X_1, X) = \frac{1}{2}\langle [X_k, X_1], X\rangle + \frac{1}{2}\langle [X_k, X], X_1\rangle$, whereas X_1 is orthogonal to $[\mathfrak{n}, \mathfrak{n}]$. □

We use the curvature convention $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, so that $K(X \wedge Y) = \langle R(X,Y)Y, X\rangle$. As long as $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ is nonabelian, we see the scalar curvature is always negative, and there will necessarily be some positive sectional curvature.

Theorem 3.2. *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be a metric Lie algebra in the family $\mathcal{C}_n$. Let $\mathcal{B} = \{X_i\}$ be an adapted orthonormal basis. Then when $i < j$, the sectional curvature is given by*

$$K(X_i \wedge X_j) = \tfrac{1}{4}(c_{i,j-i}^2 - 3c_{i,j}^2). \tag{3.2}$$

Proof. This follows directly from Equation (2.2), where we substitute $\mathrm{ad}_{X_i} X_j$ from Equation (2.5) and $\nabla_{X_i} X_j$ from Theorem (3.1). □

Theorem 3.3. *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be a metric Lie algebra in the family $\mathcal{C}_n$. Let $\mathcal{B} = \{X_i\}$ be an adapted orthonormal basis. The Ricci form is given by*

$$\mathrm{ric}(X_i, X_i) = \tfrac{1}{4} \sum_{k=1}^n (c_{k,i-k}^2 - 2c_{i,k}^2)$$
$$\mathrm{ric}(X_i, X_j) = 0 \qquad \text{if } i \ne j.$$

The scalar curvature sc is

$$sc = -\tfrac{1}{4} \sum_{i=1}^n \sum_{j=1}^n c_{i,j}^2.$$

Proof. One obtains the Ricci curvature from Equation (2.3) and the scalar curvature from the Ricci curvature. □

For any $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$, a metric Lie algebra in the family $\mathcal{C}_n$, sectional curvatures will be both positive and negative. Observe that if we normalize $sc \equiv -1$, we see from Theorem 3.2 that there exists $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ with coordinate planes whose sectional curvature K is arbitrarily close to $-\frac{1}{2}$ and similarly, there exists $(\mathfrak{n}', \langle\cdot,\cdot\rangle')$ with coordinate planes whose sectional curvature K is arbitrarily close to $\frac{3}{2}$.

The full curvature tensor R is sparse: many of the quantities $\langle R(X_i, X_j)X_k, X_l\rangle$ vanish.

Theorem 3.4. *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be a metric Lie algebra in the family $\mathcal{C}_n$. Then the curvature tensor for $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ is given by*

$$R(X_i, X_j)X_k = \frac{1}{4}(AX_{i+j+k} + BX_{|j+k-i|} + CX_{|i+k-j|} + DX_{|i+j-k|}) \tag{3.3}$$

where

$$A = c_{i,j}c_{k,i+j}$$

$$B = \begin{cases} c_{i,j}^2 & \text{if } k = i \\ -c_{j,k}c_{i,j+k-i} - c_{k,i-k}c_{i-k,j+k-i} & \text{if } k < i < j+k \\ -c_{j,k}c_{i,j+k-i} + c_{i,k-i}c_{j,k-i} & \text{if } k > i \\ 0 & \text{if } k = i-j \\ -c_{j,i-j}c_{k,i-j-k} & \text{if } k < i-j \end{cases}$$

$$C = \begin{cases} -c_{i,j}^2 & \text{if } k = j \\ c_{i,k}c_{j,i-j+k} + c_{k,j-k}c_{j-k,i+k-j} & \text{if } k < j < i+k \\ c_{i,k}c_{j,i-j+k} - c_{j,k-j}c_{i,k-j} & \text{if } k > j \\ 0 & \text{if } k = j-i \\ c_{i,j-i}c_{k,j-i-k} & \text{if } k < j-i \end{cases}$$

$$D = \begin{cases} 2c_{i,j}^2 - c_{i,j-i}^2 & \text{if } k = i < j \\ 2c_{i,j}^2 - c_{j,i-j}^2 & \text{if } k = i > j \\ -2c_{i,j}^2 + c_{j,i-j}^2 & \text{if } k = j < i \\ -2c_{i,j}^2 + c_{i,j-i}^2 & \text{if } k = j > i \\ c_{i,j}c_{i+j,k-i-j} & \text{if } k > i+j \\ 0 & \text{if } k = i+j \\ 2c_{i,j}c_{k,i+j-k} - c_{i,j-k}c_{k,j-k} + c_{j,i-k}c_{k,i-k} & \text{if } k < i,j \\ 2c_{i,j}c_{k,i+j-k} - c_{i,j-k}c_{k,j-k} - c_{i,k-i}c_{k-i,i+j-k} & \text{if } i < k < j \\ 2c_{i,j}c_{k,i+j-k} + c_{j,i-k}c_{k,i-k} + c_{j,k-j}c_{k-j,i+j-k} & \text{if } j < k < i \\ 2c_{i,j}c_{k,i+j-k} - c_{i,k-i}c_{k-i,i+j-k} + c_{j,k-j}c_{k-j,i+j-k} & \text{if } i,j < k < i+j. \end{cases}$$

Proof. Using $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ and our expressions for the bracket and connection, and simplifying using the Jacobi identity as in Equation (2.6), one can derive

$$\begin{aligned} R(X_i,X_j)X_k = \tfrac{1}{4}\big[& (c_{i,j}c_{k,i+j}X_{i+j+k} \\ & + (-c_{j,k}c_{i,j+k-i} + c_{i,k-i}c_{j,k-i} - c_{k,i-k}c_{i-k,j-i+k})X_{-i+j+k} \\ & + c_{j,i-j}c_{i-k-j,k}X_{i-j-k} \\ & + (c_{j,k-j}c_{k-j,i+j-k} - c_{k,j-k}c_{i,j-k} - c_{i,k-i}c_{k-i,j-k+i} \\ & \qquad + c_{k,i-k}c_{j,i-k} + 2c_{i,j}c_{k,i+j-k})X_{i+j-k} \\ & + c_{i,j}c_{i+j,k-i-j}X_{-i-j+k} \\ & + (-c_{j,k-j}c_{i,k-j} + c_{k,j-k}c_{j-k,i-j+k} + c_{i,k}c_{j,i+k-j})X_{i-j+k} \\ & + c_{j-k-i,k}c_{j-i,i}X_{-i+j-k}\big]. \end{aligned}$$

The calculations are long; we will not reproduce them here. The reader may easily check that this expression has the correct symmetries for a curvature tensor, and it yields the same sectional curvatures as those given in Theorem 3.2. □

4. Connection and Curvatures for $\mathcal{L}_n$

In this section we move to the special case of a metric Lie algebra $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ in $\mathcal{L}_n$, the "model space" of filiform nilpotent Lie algebras. Recall that in this case, our nonzero structure constants $c_{i,j}$ must have either $i=1$ or $j=1$. In this section, we will write c_j for $c_{1,j}$ for all j. We note that $c_1 = c_n = 0$. We begin by specializing Theorem 3.1, describing the connection ∇ for an element of $\mathcal{L}_n$.

Theorem 4.1. *For any $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ a metric Lie algebra in $\mathcal{L}_n$ with adapted orthonormal basis $\mathcal{B} = \{X_i\}$, the connection is given by*

$$\begin{aligned} \nabla_{X_i}X_i &= 0 && \text{if } 1 \le i \le n \\ \nabla_{X_1}X_i &= -\tfrac{1}{2}(c_{i-1}X_{i-1} - c_iX_{i+1}) && \text{if } 2 \le i \le n \\ \nabla_{X_i}X_1 &= -\tfrac{1}{2}(c_{i-1}X_{i-1} + c_iX_{i+1}) && \text{if } 2 \le i \le n \\ \nabla_{X_i}X_j &= U_1(X_i,X_j)\,X_1 && \text{if } 2 \le i,j \le n. \end{aligned}$$

$$\nabla_X Y = \tfrac{1}{2}[X,Y] + \sum_{j=1}^{n-1} U_j(X,Y)X_j.$$

The tensors U and U_k have the following properties:

(i) *For any X, Y in $[\mathfrak{n},\mathfrak{n}]$ and $k>1$, $U(X,Y) = U_1(X,Y)X_1$ and $U_k(X,Y) = 0$.*
(ii) *For any X in $[\mathfrak{n},\mathfrak{n}]$ and $k>1$, $U_k(X,X_1) = \frac{1}{2}c_k\langle X_{k+1}, X\rangle$.*

(iii) *For any* $X = \sum_{i=1}^n x_i X_i$ *and* $Y = \sum_{i=1}^n y_i X_i$ *and* $k > 1$,

$$U_1(X,Y) = \tfrac{1}{2} \sum_{j=2}^{n-1} c_j(x_j y_{j+1} + y_j x_{j+1}),$$
$$U_k(X,Y) = -\tfrac{1}{2} \sum_{j=2}^{n-1} c_j(x_1 y_{j+1} + y_1 x_{j+1}).$$

The connection has the property that $\nabla_{X_1} X_2$ *is in* $\mathrm{span}\{X_3\}$, $\nabla_{X_1} X_n$ *is in* $\mathrm{span}\{X_{n-1}\}$, *and for* $k > 2$, $\nabla_{X_1} X_k$ *is in* $\mathrm{span}\{X_{k-1}, X_{k+1}\}$.

Proof. The equations for the quantities $\nabla_{X_i} X_j$ come from Theorem 3.1, letting $c_{1,i} = c_i$ and $c_{i,1} = -c_i$ for $2 \leq i \leq n-1$ and letting all other $c_{i,j}$ be zero. In the expression for $\nabla_X Y$, the sum goes from 1 to $n-1$ since $U_n \equiv 0$ by Lemma 2.1. The first property of U is a special case of property (ii) in Theorem 2.1, and the second is a special case of the third property in Theorem 3.1. The expression for U_1 is found using the definition of U_k. The last assertion is clear from the formulas for $\nabla_{X_i} X_j$. □

As a special case of Theorem 3.4, we get the curvature tensor for the metric Lie algebras in the family $\mathcal{L}_n$.

Theorem 4.2. *For any* $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ *a metric Lie algebra in* $\mathcal{L}_n$ *with adapted orthonormal basis* $\mathcal{B} = \{X_i\}$, *the curvature tensor is given by*

$$R(X_1, X_j)X_1 = \tfrac{1}{4}\left[c_{j-2}c_{j-1}X_{j-2} + (3c_j^2 - c_{j-1}^2)X_j + c_j c_{j+1} X_{j+2}\right]$$

$$R(X_1, X_j)X_k = \begin{cases} -\frac{1}{4}(c_{j-2}c_{j-1})X_1 & \text{if } k = j-2 > 1 \\ \frac{1}{4}(c_{j-1}^2 - 3c_j^2)X_1 & \text{if } k = j > 1 \\ -\frac{1}{4}(c_j c_{j+1})X_1 & \text{if } k = j+2 > 1 \\ 0 & \text{otherwise.} \end{cases}$$

When $i, j, k \neq 1$,

$$\begin{aligned} R(X_i, X_j)X_k &= \tfrac{1}{4}[(-\delta_{k,j+1}c_j c_{i-1} - \delta_{k,j-1}c_{i-1}c_{j-1})\,X_{i-1} + (-\delta_{k,j-1}c_{j-1}c_i - \delta_{k,j+1}c_j c_i)\,X_{i+1} \\ &\quad + (\delta_{k,i-1}c_{i-1}c_{j-1} + \delta_{k,i+1}c_i c_{j-1})\,X_{j-1} + (\delta_{k,i-1}c_{i-1}c_j + \delta_{k,i+1}c_i c_j)\,X_{j+1}] \\ &= \begin{cases} -\frac{1}{4}(c_{i-1}c_j X_{i-1} + c_i c_j X_{i+1}) & \text{if } k = j+1 \text{ and } j \neq i-2 \\ \frac{1}{4}(c_{i-1}c_{i-3}X_{i-3} - c_i c_{i-2} X_{i+1}) & \text{if } k = j+1 \text{ and } j = i-2 \\ \frac{1}{4}(-c_{i-1}c_{i+1}X_{i-1} + c_i c_{i+2} X_{i+3}) & \text{if } k = j-1 \text{ and } j = i+2 \\ -\frac{1}{4}(c_{i-1}c_{j-1}X_{i-1} + c_i c_{j-1} X_{i+1}) & \text{if } k = j-1 \text{ and } j \neq i+2 \\ \frac{1}{4}(c_i c_{j-1} X_{j-1} + c_i c_j X_{j+1}) & \text{if } k = i+1 \text{ and } j \neq k \pm 1 \\ \frac{1}{4}(c_{i-1}c_{j-1}X_{j-1} + c_{i-1}c_j X_{j+1}) & \text{if } k = i-1 \text{ and } j \neq k \pm 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We next find the sectional curvatures for elements of $\mathcal{L}_n$.

Theorem 4.3. *Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be a metric Lie algebra in $\mathcal{L}_n$ with adapted orthonormal basis $\mathcal{B} = \{X_i\}$. The sectional curvature of the plane spanned by X_i and X_j (where $i < j$) is*

$$K(X_i \wedge X_j) = \langle R(X_i, X_j)X_j, X_i \rangle = \begin{cases} \frac{1}{4}(c_{j-1}^2 - 3c_j^2) & \text{if } i = 1 \\ \frac{1}{4}c_i^2 & \text{if } i \neq 1, j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

For any orthonormal X and Y each orthogonal to X_1,

$$K(X \wedge Y) = U_1(X, Y)^2 - U_1(X, X)U_1(Y, Y).$$

Proof. To prove the first part, we may assume $i < j$. By Theorem 3.2, the sectional curvature is given by $K(X_i \wedge X_j) = \frac{1}{4}(c_{i,j-i}^2 - 3c_{i,j}^2)$, where $c_{l,m} = 0$ except if $l = 1$ or $m = 1$. To find the sectional curvature for an arbitrary X and Y in $X_1^\perp$, we use Equation (2.2), relating sectional curvature to U and the Lie bracket. Since $X_1^\perp$ is abelian, the bracket terms in Equation (2.2) vanish and U reduces to U_1. □

As a corollary, we find that a subspace is flat ($K \equiv 0$) exactly when the tensor U_1 vanishes.

Corollary 4.4. *Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be a metric Lie algebra in $\mathcal{L}_n$. For any subspace $\mathfrak{m}$ of $\mathfrak{n}$, $K|_\mathfrak{m} \equiv 0$ if and only if $U_1|_\mathfrak{m} \equiv 0$.*

Using Theorem 4.3 we find the Ricci curvature for elements of $\mathcal{L}_n$. These appeared first in [Lau02].

Theorem 4.5 ([Lau02])**.** *Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be a metric nilpotent Lie algebra in $\mathcal{L}_n$. Then the Ricci curvature and scalar curvature are given by*

$$\mathrm{ric}(X_i, X_j) = \begin{cases} -\frac{1}{2}\sum_{k=2}^{n-1} c_k^2 & \text{if} \quad i = j = 1 \\ \frac{1}{2}(c_{i-1}^2 - c_i^2) & \text{if} \quad i = j > 1 \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

$$sc = -\frac{1}{2}\sum_{k=2}^{n-1} c_k^2.$$

In $\mathcal{L}_n$ a key to understanding sectional curvature and geodesics is in analyzing the restriction of U_1 to the orthogonal complement of X_1. When this tensor is represented by a matrix relative to the adapted basis, the matrix is a symmetric tridiagonal matrix with zeroes on the diagonal. The next lemma describes properties of such a matrix.

Lemma 4.6. *Let $A = (a_{ij})$ be an $m \times m$ symmetric matrix so that $a_{ij} = a_{ji}$ is nonzero if and only if $|i - j| = 1$. Let $p(x)$ denote the characteristic polynomial of A. Then A has the following properties:*

(i) *When m is even, there exists a polynomial q such that $p(x) = q(x^2)$ and A has rank m. When m is odd, there exists a polynomial q such that $p(x) = xq(x^2)$ and A has rank $m - 1$.*

(ii) *The eigenvalues of the matrix A are distinct, and nonzero eigenvalues come in real pairs of the form $\pm a$.*
(iii) *If V is a subspace of $\mathbb{R}^m$ of dimension k such that $v^t A w = 0$ for all v and w in V, then $2k \leq m$ if m is even and $2k \leq m+1$ if m is odd.*

Proof. Let A be an $m \times m$ matrix satisfying the hypotheses of the lemma. For simplicity, write a_i for $a_{i,i+1} = a_{i+1,i}$ for $i = 1$ to $m-1$. Let $p_k(x)$ denote the determinant of the $k \times k$ minor A_k in the upper left corner of A.

We will show by induction that for $k \geq 1$,

$$p_k(x) = xp_{k-1}(x) - a_{k-1}^2 p_{k-2}(x),$$

(where we let $a_0 = 0$, $p_{-1}(x) = 0$ and $p_0(x) = 1$), and that when k is even $p_k(x)$ is an even polynomial with k nonzero roots, while when k is odd, $p_k(x)$ is x times an even polynomial with $k-1$ nonzero roots.

First, we consider the case that $k = 1$. Then $A_1 = \begin{pmatrix} 0 \end{pmatrix}$, so $p_1(x) = x = xp_0(x) - a_0^2 p_{-1}(x)$. When $k = 2$, the matrix $A_2 = \begin{pmatrix} 0 & a_1 \\ a_1 & 0 \end{pmatrix}$. We see that $p_2(x) = x^2 - a_1^2$, an even polynomial, and $p_2(x) = xp_1(x) - a_1^2 p_0(x)$. We also see that $p_2(x)$ has two nonzero real roots, $\pm a_1$.

Now assume that the statement holds for $i = 1, 2, \ldots, k$, and consider the matrix

$$xI - A_{k+1} = \begin{pmatrix} x & -a_1 & 0 & 0 & \cdots & 0 & 0 \\ -a_1 & x & -a_2 & 0 & \cdots & 0 & 0 \\ 0 & -a_2 & x & -a_3 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & -a_{k-2} & x & -a_{k-1} & 0 \\ 0 & 0 & \cdots & 0 & -a_{k-1} & x & -a_k \\ 0 & 0 & \cdots & 0 & 0 & -a_k & x \end{pmatrix}.$$

To find the characteristic polynomial, we expand the determinant up the rightmost column:

$$p_{k+1}(x) = xp_k(x) - (-a_k)(-a_k p_{k-1}(x)) = xp_k(x) - a_k^2 p_{k-1}(x),$$

as desired. If $k+1$ is even, then k is odd and $k-1$ is even, and from the inductive hypothesis, each of $xp_k(x)$ and $p_{k-1}(x)$ is an even polynomial. Thus $p_{k+1}(x)$ is even also. We see that x^2 divides $xp_k(x)$, but $p_{k-1}(x)$ has a nontrivial constant term because zero is not a root of $p_{k-1}(x)$. Therefore zero is not a root of $p_{k+1}(x)$. If on the other hand $k+1$ is odd, k is even and $k-1$ is odd; hence $xp_k(x)$ and $p_{k-1}(x)$ are each a product of x and an even polynomial, which means $p_{k+1}(x)$ is of this form also. Thus zero is a root of $p_{k+1}(x)$. But zero is not a root of $p_k(x) = \det A_k$, so zero is a root of multiplicity at most one for $p_{k+1}(x) = \det A_{k+1}$. This completes the induction.

Since A is symmetric it has real roots. We know that if a is a root of an even polynomial then so is $-a$, hence the form of the characteristic polynomial ensures that the nonzero eigenvalues of A are in pairs. When m is even, zero is not a root of $p_m(x)$, so A is nonsingular. When m is odd, zero is a root of $p_m(x)$ with multiplicity one, so A is has corank one.

Now we show that there are no repeated roots. It is not hard to show using induction and the summation formula for matrix multiplication that for $k \geq 1$, the matrix A^k has a nonzero entry a_{ij} with $|i-j| = k$ and all entries a_{ij} with $|i-j| > k$ are zero. Therefore, the minimal polynomial for A must be of degree m. Hence the minimal and characteristic polynomials are equal. As A is diagonalizable, there are no repeated roots.

Finally, we prove the third property. Let V be a subspace of $\mathbb{R}^m$ of dimension k such that $v^tAw = 0$ for all v and w in V. Suppose that w and v are vectors so that $w = Av$ and both w and v are in V. Then $w^tw = (Av)^tw = v^tAw = 0$. Thus V and AV intersect only at 0. When m is even and A is nonsingular, this implies that $2k \leq m$. When m is odd and A is corank one, it is possible that V contains a zero eigenvector, and then $2k \leq m+1$. □

5. Geodesics and totally geodesic submanifolds of $\mathcal{C}'_n$

In this section, we return to the larger class, $\mathcal{C}'_n$, defined in Section 2. Here we use the information in Section 3 to determine which one-parameter subgroups are geodesics. First we classify those geodesics which are orbits of one-parameter subgroups.

Theorem 5.1. *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be in $\mathcal{C}'_n$ and let Y be a nontrivial vector in $\mathfrak{n}$. Then* $\operatorname{span}\{Y\}$ *is a totally geodesic subalgebra if and only if*

(i) *Y is a multiple of X_1, or*
(ii) *$\langle X_1, Y\rangle = 0$ and $U(Y,Y) = 0$, or*
(iii) *$Y = a_1X_1 + a_2X_2$ where $a_1, a_2 \neq 0$.*

Proof. There are three possibilities: Y is a multiple of X_1, Y is orthogonal to X_1, or Y has components in both the X_1 and $X_1^\perp$ directions. By Lemma 2.2, $\operatorname{span}\{Y\}$ is a totally geodesic subalgebra if and only if $U(Y,Y) = 0$. We see that $U(X_1,X_1) = 0$, so for the first case, any multiple of X_1 generates a geodesic subalgebra. In the case that Y is orthogonal to X_1, we simply need $U(Y,Y) = 0$.

Consider $Y = a_1X_1 + X$, where $a_1 \neq 0$ and $X = \sum_{i=2}^n a_iX_i \neq 0$ (the third case). Since X_1 is orthogonal to $[\mathfrak{n},\mathfrak{n}]$, $U(a_1X_1 + X, a_1X_1 + X) = 0$ if and only if for all k,

$$\begin{aligned} 0 &= \langle \operatorname{ad}^*_{a_1X_1+X}(a_1X_1+X), X_k\rangle \\ &= \langle a_1X_1 + X, \operatorname{ad}_{a_1X_1+X} X_k\rangle \\ &= \langle X, [a_1X_1 + X, X_k]\rangle \\ &= \langle \sum_{i=2}^n a_iX_i, [\sum_{i=1}^n a_iX_i, X_k]\rangle. \end{aligned}$$

Assume that $U(Y,Y) = 0$. We will use the previous condition to inductively show that $a_{n-i} = 0$ for all $i = 0,\ldots,n-3$. We first confirm the inductive hypothesis in the base case when $i = 0$. Let $k = n-1$ in the above equation:

$$0 = \langle \sum_{i=2}^n a_iX_i, [\sum_{i=1}^n a_iX_i, X_{n-1}]\rangle = \langle \sum_{i=2}^n a_iX_i, a_1c_{1,n-1}X_n\rangle = a_1a_nc_{1,n-1}.$$

Since $a_1 \neq 0$ and $c_{1,n-1} \neq 0$, the coefficient a_n is forced to be zero.

Now assume that $a_{n-i} = 0$ for $i = 0, \ldots, r$, where $r < n-3$. Then $X = \sum_{i=2}^{n-r-1} a_i X_i$. Note that $n > n-r-2 > 1$, so $c_{1,n-r-2} \neq 0$. We set $k = n-r-2$ in the equation above:

$$\begin{aligned} 0 &= \langle \sum_{i=2}^{n-r-1} a_i X_i, [\sum_{i=1}^{n-r-1} a_i X_i, X_{n-r-2}]\rangle \\ &= \langle \sum_{i=2}^{n-r-1} a_i X_i, \sum_{i=1}^{n-r-1} a_i c_{i,n-r-2} X_{n-r+i-2}\rangle \\ &= \langle a_{n-r-1} X_{n-r-1}, a_1 c_{1,n-r-2} X_{n-r-1}\rangle \\ &= a_1 a_{n-r-1} c_{1,n-r-2}. \end{aligned}$$

Since $a_1 \neq 0$ and $c_{1,n-r-2} \neq 0$, the coefficient $a_{n-r-1} = 0$. Thus the inductive hypothesis holds for $i = r+1$. This proves that if $U(Y,Y) = 0$ then $Y = a_1 X_1 + a_2 X_2$.

Conversely, when $Y = a_1 X_1 + a_2 X_2$, we find $\langle Y, [Y, X_k]\rangle = \langle a_2 X_2, [a_1 X_1 + a_2 X_2, X_k]\rangle = 0$ for each k, thus $U(Y,Y) = 0$. □

Example 5.2. Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be in $\mathcal{C}_n'$. For any $i = 1, \ldots, n$, the one-dimensional subalgebra $\operatorname{span}\{X_i\}$ is a totally geodesic subalgebra of $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$, since $U(X_i, X_i) = 0$.

Next we describe a property of totally geodesic subalgebras of dimension two or more for elements of $\mathcal{C}_n'$.

Theorem 5.3. *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be in $\mathcal{C}_n'$, and let $\mathfrak{m}$ be a nontrivial totally geodesic subalgebra of $\mathfrak{n}$ of dimension two or more. Then $\langle X_1, \mathfrak{m}\rangle = 0$ and $U_1(X,Y) = 0$ for all X and Y in $\mathfrak{m}$.*

Proof. Let $\mathfrak{m}$ be a totally geodesic subalgebra of $\mathfrak{n}$ with dimension at least two ($\mathfrak{m} \neq \mathfrak{n}$). First we will show that $\mathfrak{m}$ is orthogonal to X_1. Suppose not: $\langle X_1, \mathfrak{m}\rangle \neq 0$. For some element X of $\mathfrak{m}$, $\langle X, X_1\rangle = 1$. Since $\dim \mathfrak{m} > 1$, we can find another $Y \in \mathfrak{m}$ independent of X. Define k to be the minimum so that $\langle X_k, \mathfrak{m} \cap X_1^\perp\rangle \neq 0$. Note that $k > 1$. Choose Y an element of $\mathfrak{m} \cap X_1^\perp$ achieving this minimum value of k, so $\langle Y, X_k\rangle \neq 0$. Each of the vectors $\operatorname{ad}_X Y, \operatorname{ad}_X^2 Y, \ldots, \operatorname{ad}_X^{n-k} Y$ is in $\mathfrak{m}$. For each $i = 1, \ldots, n-k$, the least j for which the quantity $\langle \operatorname{ad}_X^i Y, X_j\rangle$ is nonzero is when $j = k+i$. Therefore, for each $i = 1, \ldots, n-k$, $\operatorname{ad}_X^i Y$ is nontrivial, and the set $\{\operatorname{ad}_X^i Y\}_{i=0}^{n-k}$ is linearly independent. The set $\{\operatorname{ad}_X^i Y\}_{i=0}^{n-k}$ is contained in the span of the set $\{X_k, X_{k+1}, \ldots, X_n\}$, and has cardinality $n-k+1$. Hence $\mathfrak{m} = \langle X\rangle \oplus \langle X_k, X_{k+1}, \ldots, X_n\rangle$.

Suppose $k = 2$, then the dimension of $\mathfrak{m}$ is $\dim(\mathfrak{m}) = n$ and so $\mathfrak{m} = \mathfrak{n}$, a contradiction. Next, suppose $2 < k < n$, then X_{n-1} and X_n are in $\mathfrak{m}$, and because $\mathfrak{m}$ is totally geodesic, $\nabla_{X_n} X_{n-1} = \frac{1}{2} c_{1,n-1} X_1$ is also in $\mathfrak{m}$. Once we know X_1 is in $\mathfrak{m}$, then $\nabla_{X_1} X_k = -\frac{1}{2} c_{1,k-1} X_{k-1} + \frac{1}{2} c_{1,k} X_{k+1}$ is also in $\mathfrak{m}$. This contradicts the minimality of k. Finally, suppose $2 < k = n$, so that $\mathfrak{m} = \langle X\rangle \oplus \langle X_n\rangle$. Then

$$\nabla_X X_n = \nabla_{X_1} X_n + \nabla_{X - X_1} X_n = -\tfrac{1}{2} c_{1,n-1} X_{n-1} + U(X - X_1, X_n)$$

is in $\mathfrak{m}$. This vector is nonzero, having a nontrivial component in the X_{n-1} direction, as

$$U_{n-1}(X - X_1, X_n) = \tfrac{1}{2}\langle [X_{n-1}, X - X_1], X_n\rangle + \tfrac{1}{2}\langle [X_{n-1}, X_n], X - X_1\rangle = 0.$$

Since $\nabla_X X_n$ is orthogonal to X_n, it must be a multiple of X. Let $a = \langle X, X_{n-1}\rangle$. We use the formula for the connection in Theorem 3.1 to find that $\langle \nabla_X X_n, X_1\rangle = \frac{1}{2} a c_{1,n-1}$, and we showed above that $\langle \nabla_X X_n, X_{n-1}\rangle = -\frac{1}{2} c_{1,n-1}$. In X, the ratio of the coefficients of X_{n-1} and X_1 is a, while in $\nabla_X X_n$ it is $-1/a$. In order for these to hold simultaneously, we need $a^2 = -1$, impossible. Thus if $\mathfrak{m}$ is a nontrivial totally geodesic subalgebra of $\mathfrak{n}$ of dimension greater than one, $\langle X_1, \mathfrak{m}\rangle = 0$.

To conclude the proof we observe that for a totally geodesic $\mathfrak{m}$ orthogonal to X_1, we have $U_1(X, Y) = 0$ for all X and Y in $\mathfrak{m}$, by Lemma 2.2. □

Example 5.4. Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be in $\mathcal{C}'_n$. The subalgebra $\mathfrak{m}_2 = \operatorname{span}\{X_i \mid i \text{ is even}\}$ is totally geodesic. More generally, for any integer k, the subalgebra $\mathfrak{m}_k$ spanned by basis vectors with subscripts that are multiples of k is also a totally geodesic subalgebra. This follows directly from the the connection form in Theorem 3.1. If $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ is in $\mathcal{L}_n$, then the subalgebras $\mathfrak{m}_k$ of $\mathfrak{n}$ $(k > 1)$ are flat, because $\nabla_X Y = 0$ for all X and Y in $\mathfrak{m}_k$. If $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ is in $\mathcal{C}_n$, then the subalgebras $\mathfrak{m}_k$ of $\mathfrak{n}$ $(k > 1)$ are flat if and only if they are abelian.

We note that $\mathfrak{m}_k$ is isometrically isomorphic to a metric Lie algebra in the family $\mathcal{L}'_d$, where $d = \dim(\mathfrak{m}_k)$.

6. Geodesics and totally geodesic submanifolds of $\mathcal{L}_n$

In Theorem 5.1, we characterized the one-dimensional totally geodesic subalgebras of elements of $\mathcal{C}'_n$. We now use the expression for U_1 in Theorem 4.1 to specialize to $\mathcal{L}'_n$ as follows:

Corollary 6.1. *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be in $\mathcal{L}'_n$. Then span$\{Y\}$ is totally geodesic if and only if $U_1(Y, Y) = 0$.*

Example 6.2. In the metric Lie algebra $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ in $\mathcal{L}_4$ with $c_2 = c_3 = 1$, the span of $X_2 - X_4$ is totally geodesic by Theorem 5.1.

The next theorem shows that for elements of $\mathcal{L}'_n$, the only totally geodesic subalgebras of dimension two or more are the types we are guaranteed to find by Proposition 2.4.

Theorem 6.3. *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be in $\mathcal{L}'_n$, and let $\mathfrak{m}$ be a nontrivial subspace of $\mathfrak{n}$ of dimension two or more. Then $\mathfrak{m}$ is a totally geodesic subalgebra of $\mathfrak{n}$ if and only if $\mathfrak{m}$ is in the maximal abelian subalgebra of $\mathfrak{n}$ and $U(X, Y) = 0$ for all X and Y in $\mathfrak{m}$. Furthermore, any such nontrivial $\mathfrak{m}$ is flat.*

Proof. Let $\mathfrak{m}$ be a subspace of dimension two or more. First suppose that $\mathfrak{m}$ is orthogonal to X_1 and $U_1(X, Y) = 0$ for all X and Y in $\mathfrak{m}$. Then $\mathfrak{m}$ is a trivially a subalgebra because it is abelian, and $\mathfrak{m}$ is totally geodesic since $U(X, Y) = U_1(X, Y) = 0$ for all X and Y in $\mathfrak{m}$. By Proposition 2.4, $\mathfrak{m}$ is flat.

The converse follows from Theorem 5.3. □

Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be in $\mathcal{L}'_n$. In the next theorem we analyze which subspaces are totally geodesic subalgebras of dimension two or more, as described in Theorem 6.3. Let A denote the matrix representing the restriction of U_1 to $X_1^\perp$. Then A satisfies the hypotheses of Lemma 4.6, so we can let $X'_2, \ldots, X'_n$ denote eigenvectors for A with distinct real eigenvalues $\lambda_2, \ldots, \lambda_n$. Let

$$\begin{aligned} C_A &= \Big\{ \sum_{i=2}^n x_i X'_i \mid \sum_{i=2}^n \lambda_i x_i^2 = 0 \Big\} \\ &= \Big\{ \sum_{i=2}^n x_i X'_i \mid (\lambda_2, \ldots, \lambda_n) \perp (x_1^2, \ldots, x_n^2) = 0 \Big\}. \end{aligned}$$

This cone is a codimension-one subset of $X_1^\perp$.

Theorem 6.4. *Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be in $\mathcal{L}'_n$. Suppose $\mathfrak{m}$ is a totally geodesic subspace contained in $X_1^\perp$. Then $\mathfrak{m}$ is a subspace of C_A. The maximal dimension of $\mathfrak{m}$ is $\lfloor \frac{n}{2} \rfloor$, and this maximal dimension is always achieved. Conversely, any subspace of C_A is a totally geodesic subalgebra.*

Proof. Let X be an arbitrary element of $X_1^\perp$. Then $U_1(X, X) = 0$ if and only if $0 = X^T A X$. For an arbitrary X, $\sum_{i=2}^n x_i X'_i$, we get

$$0 = X^T A X = X^T A \sum_{i=2}^n x_i X'_i = X^T \sum_{i=2}^n x_i \lambda_i X'_i = \sum_{i=2}^n \lambda_i x_i^2$$

Thus, totally geodesic subalgebras correspond to subspaces of the cone.

We show that the maximum dimension is achieved. If $n = 2k+1$, then $\dim X_1^\perp = 2k$ and if $n = 2k+2$, $\dim X_1^\perp = 2k+1$. Let $\pm\mu_1, \ldots, \pm\mu_k$ be the distinct nonzero eigenvalues of A as guaranteed by Lemma 4.6, with eigenvectors U_i and V_i for μ_i and $-\mu_i$ respectively. Let Z denote the zero eigenvector in the case that n is even and $n-1$ is odd.

If $n = 2k+1$, the set spanned by $\{U_i + V_i\}_{i=1}^k$ is a k-dimensional totally geodesic subalgebra. When $n = 2k+2$, the set spanned by $\{Z\} \cup \{U_i + V_i\}_{i=1}^k$ is of dimension $k+1$ and totally geodesic. In both cases, this is because $\sum_{i=2}^n \lambda_i x_i^2 = \sum_{i=1}^k (\mu_i - \mu_i) = 0$. □

Example 6.5. Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be the metric Lie algebra in $\mathcal{L}'_5$ with $c_2 = c_3 = c_4 = 2$. The restriction of U_1 to $X_1^\perp$ is represented by the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with respect to the basis $\{X_i\}_{i=2}^5$. The characteristic polynomial of the matrix is $p(x) = x^4 - 3x^2 + 1 = (x^2 + x - 1)(x^2 - x - 1)$, and its eigenvalues are $\pm\tau$ and $\pm\tau^{-1}$, where

$\tau = \frac{1+\sqrt{5}}{2}$. The vectors

$$V_\tau = \begin{bmatrix} 1 \\ \tau \\ \tau \\ 1 \end{bmatrix}, V_{-\tau} = \begin{bmatrix} 1 \\ -\tau \\ \tau \\ -1 \end{bmatrix}, V_{\tau^{-1}} = \begin{bmatrix} 1 \\ \tau^{-1} \\ -\tau^{-1} \\ -1 \end{bmatrix}, \text{and } V_{-\tau^{-1}} = \begin{bmatrix} 1 \\ -\tau^{-1} \\ -\tau^{-1} \\ 1 \end{bmatrix}$$

have eigenvalues $\tau, -\tau, \tau^{-1}$ and $-\tau^{-1}$ respectively. The subspace $\mathfrak{m}$ spanned by $X = V_\tau + V_{-\tau}$ and $Y = V_{\tau^{-1}} + V_{-\tau^{-1}}$ is a totally geodesic subalgebra of $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$. This is because an arbitrary element of $\mathfrak{m}$ is of the form

$$aX + bY = aV_\tau + aV_{-\tau} + bV_{\tau^{-1}} + bV_{-\tau^{-1}}.$$

Thus, as required in Theorem 6.4, $(a^2, a^2, b^2, b^2) \cdot (\tau, -\tau, \tau^{-1}, -\tau^{-1}) = 0$.

7. Conclusion

We conclude with some general observations comparing filiform nilpotent geometry to two-step nilmanifold geometry. Filiform geometry is the opposite of two-step geometry in several ways. Using the formula for volume growth $\sum_{i=1}^d i\,\mathrm{rank}(\mathfrak{n}_i/\mathfrak{n}_{i+1})$ given in [Bas72], we see that volume growth in a filiform nilmanifold of dimension n is polynomial of degree $1 + \frac{n(n-1)}{2}$. This is the largest possible degree polynomial volume growth for n-dimensional nilmanifolds. By contrast, volume growth in two-step nilmanifolds is smaller, of degree $n + \dim[\mathfrak{n},\mathfrak{n}] < 2n$. Secondly, for filiform nilmanifolds in $\mathcal{L}_n$, all totally geodesic subalgebras are abelian and flat, which is not necessary in two-step nilgeometry. Furthermore, totally geodesic subalgebras are abundant in filiform nilmanifolds, whereas in the two-step case they may even fail to exist. Thirdly, filiform nilmanifolds have small isometry groups: in fact, there are only finitely many nontranslational isometries for a filiform Lie algebra of dimension four or more ([Gor02]).

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