

NEW HOMOGENEOUS EINSTEIN METRICS OF NEGATIVE RICCI CURVATURE

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ABSTRACT. We construct new homogeneous Einstein spaces with negative Ricci curvature in two ways: First, we give a method for classifying and constructing a class of rank one Einstein solvmanifolds whose derived algebras are two-step nilpotent. As an application, we describe an explicit continuous family of ten-dimensional Einstein manifolds with a two-dimensional parameter space, including a continuous subfamily of manifolds with negative sectional curvature. Secondly, we obtain new examples of non-symmetric Einstein solvmanifolds by modifying the algebraic structure of non-compact irreducible symmetric spaces of rank greater than one, preserving the (constant) Ricci curvature.

INTRODUCTION

The primary goal of this paper is the construction of new examples of homogeneous Einstein manifolds of negative Ricci curvature. The classical examples of Einstein metrics of negative Ricci curvature are the symmetric spaces of non-compact type. A number of other examples are known, e.g. [Al], [D], [GP-SV], [P-S], and [Wt2] (see [Hb2] for more extensive references).

A Riemannian solvmanifold is a solvable Lie group S together with a left-invariant Riemannian metric. All known examples of homogeneous Einstein manifolds of negative Ricci curvature are Riemannian solvmanifolds of Iwasawa type. See Section 1 for a precise definition; here we just note that a solvable Lie group of Iwasawa type is a semi-direct product of an abelian group A with a nilpotent normal subgroup N , the nilradical. We restrict our attention in this paper to solvmanifolds of Iwasawa type.

After providing the necessary background in Section 1, we consider in Section 2 the class of solvmanifolds for which N is two-step nilpotent, A is one-dimensional, and the action of A on N is of a particular type. The solvmanifolds in this class are sometimes referred to as Carnot solvmanifolds. This class of solvmanifolds includes (but is not limited to) the harmonic manifolds of negative Ricci curvature constructed by Damek and Ricci [DR]. P. Eberlein and J. Heber [EH2] gave a method for classifying all Einstein solvmanifolds of this type; see Theorem 2.5 below. For completeness, we include the proof of this result (which we observed independently). We then give an explicit classification in low dimensions with several new examples, including, in Section 3, an explicit continuous family of ten-dimensional Einstein manifolds with a two-dimensional parameter space. We prove that within the deformation there is a continuous family of non-isometric Einstein solvmanifolds with negative sectional curvature.

In Section 4, we obtain new examples of non-symmetric Einstein solvmanifolds by modifying the algebraic structure of the non-compact irreducible symmetric spaces of rank greater than one.

The study of homogeneous Einstein metrics is currently very active. While this work was in progress, we learned of recent work on Einstein solvmanifolds contained in the habilitation of Jens Heber [Hb2] and the thesis of Denise Hengesch [Hn]. Heber's extensive habilitation is a ground-breaking work, which allows Einstein solvmanifolds (which are non-compact) to be viewed as critical points of a certain functional and which delves deeply into questions of existence and uniqueness. Although our results in Section 4 were obtained independently of Heber's habilitation, they share a common theme with his work: Beginning with

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a given Einstein solvmanifold, we modify not the metric but rather the bracket structure on the associated Lie algebra in order to construct new examples of Einstein solvmanifolds. Our work in Sections 2 and 3 was motivated directly by Heber's habilitation. In his study of the moduli space of Einstein solvmanifolds, he shows that the moduli space of Einstein solvmanifolds near a given one may have very large dimension. In some cases one can compute the dimension explicitly; in other cases one gets lower bounds on the dimension. However, his computation of the dimension does not give explicit examples but rather guarantees that if there is one Einstein solvmanifold of a particular type, then (depending on the dimension and structure) one may be guaranteed a large moduli space. Our classification results and explicit examples in Sections 2 and 3 complement Heber's work. In particular, one can show using Heber's work together with Theorem 2.5 that the lowest dimension in which continuous families of Einstein Carnot solvmanifolds can exist is dimension 10; i.e., our examples in Section 3 have the lowest possible dimension.

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1. PRELIMINARIES

A solvable Lie group S together with a left-invariant Riemannian metric g is called a *Riemannian solvmanifold*. The left-invariant Riemannian metric g on S defines an inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra \mathfrak{s} of S , and conversely, any inner product on \mathfrak{s} gives rise to a unique left-invariant metric on S .

We will refer to a Lie algebra endowed with an inner product as a *metric Lie algebra*. Thus we have a one-to-one correspondence between simply-connected Riemannian solvmanifolds and solvable metric Lie algebras. We will say two metric Lie algebras $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$ and $(\mathfrak{s}', \langle \cdot, \cdot \rangle')$ are *isomorphic* if there exists a linear map $\tau : \mathfrak{s} \rightarrow \mathfrak{s}'$ which is both an inner product space isometry and a Lie algebra isomorphism.

Notation 1.1. Let (S, g) be a simply-connected Riemannian solvmanifold, and let \mathfrak{s} be the associated solvable metric Lie algebra. Write $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ with \mathfrak{n} the nilradical of \mathfrak{s} . The metric Lie algebra $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$ and the solvmanifold (S, g) are said to be of *Iwasawa type* if the following conditions hold:

- (i) \mathfrak{a} is abelian;
- (ii) All $\text{ad}(A)$, $A \in \mathfrak{a}$, are symmetric relative to the inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{s} and non-zero when $A \neq 0$;
- (iii) For some $A \in \mathfrak{a}$, the restriction of $\text{ad}(A)$ to \mathfrak{n} is positive-definite.

When (i) holds, the dimension of \mathfrak{a} is called the *algebraic rank* of \mathfrak{s} and of S .

The following condition for two Riemannian solvmanifolds of Iwasawa type to be isometric is a special case of Theorem 5.2 in [GW].

Proposition 1.2. *Two simply-connected solvmanifolds of Iwasawa type are isometric if and only if the associated metric Lie algebras are isomorphic.*

We next recall the formula for the Ricci tensor of the solvmanifold (S, g) , viewed as a bilinear form on \mathfrak{s} ; see Besse [Bes] for details.

Notation 1.3. Let $\{X_1, \dots, X_n\}$ be an orthonormal basis for \mathfrak{s} . Define $U : \mathfrak{s} \times \mathfrak{s} \rightarrow \mathfrak{s}$ by

$$\langle U(X, Y), Z \rangle = \frac{1}{2} \langle [Z, X], Y \rangle + \frac{1}{2} \langle [Z, Y], X \rangle$$

for $X, Y, Z \in \mathfrak{s}$. Set

$$H = \sum_i U(X_i, X_i).$$

Observe that H must lie in \mathfrak{a} and that

$$\langle H, X \rangle = \text{tr ad}(X)$$

for all $X \in \mathfrak{s}$. As an aside, we note that H is the mean curvature vector field for the embedding of the nilradical N in S . Thus, we will refer to H as the mean curvature vector field.

Let B denote the Killing form of \mathfrak{s} . For $X, Y \in \mathfrak{s}$, the Ricci curvature $\text{Ric}(X, Y)$ is given by

$$\text{Ric}(X, Y) = -\frac{1}{2} \sum_i \langle [X, X_i], [Y, X_i] \rangle - \frac{1}{2} B(X, Y) + \frac{1}{4} \sum_{i,j} \langle [X_i, X_j], X \rangle \langle [X_i, X_j], Y \rangle - \langle U(X, Y), H \rangle. \quad (1.1)$$

In particular,

$$\text{Ric}(X, X) = -\frac{1}{2} \text{tr ad}(X) \circ \text{ad}(X)^* - \frac{1}{2} B(X, X) + \frac{1}{4} \sum_{i,j} \langle [X_i, X_j], X \rangle^2 - \langle [H, X], X \rangle. \quad (1.2)$$

As pointed out in [Wt2] and easily seen from equation (1.1), if the manifold (S, g) is Einstein, then so is the solvmanifold associated with the subalgebra $\mathfrak{s}_0 = \mathbf{R}H + \mathfrak{n}$ of \mathfrak{s} with the induced inner product. Conversely, Heber [Hb2] showed how to construct Einstein solvmanifolds of higher algebraic rank from a given Einstein solvmanifold of Iwasawa type and of rank one. Thus in Section 2, we will focus on a class of solvmanifolds of algebraic rank one. In Section 4, it will be more convenient to work with higher rank solvmanifolds, however.

Proposition 1.4. [Hb2] *Let (S, g) be an Einstein solvmanifold of Iwasawa type and let $H \in \mathfrak{s}$ be the mean curvature vector field defined in Notation 1.3. Then for some multiple λH of H , the eigenvalues of $\text{ad}(\lambda H)|_{\mathfrak{n}}$ are positive integers. Moreover, the distinct eigenvalues have no common divisors.*

If $\mu_1 < \mu_2 < \dots < \mu_m$ are the distinct eigenvalues of $\text{ad}(\lambda H)|_{\mathfrak{n}}$ with multiplicities d_1, \dots, d_m , respectively, Heber refers to the data $(\mu_1, \dots, \mu_m; d_1, \dots, d_m)$ as the *eigenvalue type* of the Einstein solvmanifold. In Section 2 we will consider Einstein solvmanifolds of Iwasawa type and of algebraic rank one with eigenvalue type $(1, 2; d_1, d_2)$ where d_1 and d_2 are arbitrary. This is a rich class of Riemannian solvmanifolds which contains the negatively curved rank one symmetric spaces and all the non-compact harmonic manifolds constructed by Damek and Ricci [DR]. In Section 4 we will consider Einstein solvmanifolds of the same eigenvalue types as the higher rank symmetric spaces.

2. TWO-STEP EXAMPLES

In the notation following Proposition 1.4, we now consider Einstein solvmanifolds of algebraic rank one and eigenvalue type $(1, 2; r, s)$ where the multiplicities r and s of the eigenvalues are arbitrary. All solvmanifolds considered here will be of Iwasawa type. We use the terminology of Notations 1.1 and 1.3. Letting \mathfrak{v} and \mathfrak{z} be the eigenspaces of $\text{ad}(\lambda H)|_{\mathfrak{n}}$ corresponding to the eigenvalues 1 and 2, respectively, the Jacobi identity implies that $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$ and that \mathfrak{z} is central in \mathfrak{n} . Thus \mathfrak{n} is either two-step nilpotent or abelian. (As we will see in Theorem 2.5, \mathfrak{n} must in fact be two-step nilpotent, not abelian, in order for the Einstein condition to hold.) Since $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$ is of Iwasawa type, $\text{ad}(H)$ is symmetric and thus \mathfrak{v} and \mathfrak{z} are orthogonal.

Notation 2.1. (i) Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be a two-step nilpotent metric Lie algebra, let \mathfrak{z} be a central subspace of \mathfrak{n} containing the derived algebra $[\mathfrak{n}, \mathfrak{n}]$, and let $\mathfrak{v} = \mathfrak{z}^\perp$ relative to $\langle \cdot, \cdot \rangle$. We can then define a linear map $j : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle)$ by

$$\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle \text{ for } X, Y \in \mathfrak{v}, Z \in \mathfrak{z}. \quad (2.1)$$

Equivalently,

$$j(Z)X = \text{ad}(X)^*(Z). \quad (2.2)$$

(ii) Conversely, given any two finite dimensional real inner product spaces \mathfrak{v} and \mathfrak{z} along with a linear map $j : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v})$, we can define a metric Lie algebra \mathfrak{n} as the inner product space direct sum of \mathfrak{v} and \mathfrak{z} with the alternating bilinear bracket map $[\cdot, \cdot] : \mathfrak{n} \times \mathfrak{n} \rightarrow \mathfrak{z}$ defined by declaring \mathfrak{z} to be central in \mathfrak{n} and using (2.1) to define $[X, Y]$ for $X, Y \in \mathfrak{v}$. Then \mathfrak{n} is two-step nilpotent if j is non-zero.

(iii) Given the data $(\mathfrak{v}, \mathfrak{z}, j)$ and thus a two-step nilpotent (or abelian if $j = 0$) metric Lie algebra $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ as in (ii), we can further define a Riemannian solvmanifold (S, g) of algebraic rank one and eigenvalue type $(1, 2; r, s)$ where $r = \dim(\mathfrak{v})$ and $s = \dim(\mathfrak{z})$ as follows: Let \mathfrak{a} be a one-dimensional inner product space and A a choice of unit vector in \mathfrak{a} . Define an inner product space $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$ by taking the orthogonal direct sum of \mathfrak{a} and \mathfrak{n} . Give \mathfrak{s} the unique Lie algebra structure for which \mathfrak{n} is an ideal (the nilradical) and such

that $\text{ad}(A)|_{\mathfrak{v}} = \frac{1}{2}\text{Id}$ and $\text{ad}(A)|_{\mathfrak{z}} = \text{Id}$. Then \mathfrak{s} is a metric solvable Lie algebra of rank one and eigenvalue type $(1, 2; r, s)$. We will refer to the associated simply-connected Riemannian solvmanifold (S, g) as the solvmanifold defined by the data triple $(\mathfrak{v}, \mathfrak{z}, j)$.

(iv) Data triples $(\mathfrak{v}, \mathfrak{z}, j)$ and $(\mathfrak{v}', \mathfrak{z}', j')$ will be said to be *equivalent* if there exist orthogonal transformations α of \mathfrak{v} and β of \mathfrak{z} such that

$$j'(\beta(Z)) = \alpha \circ j(Z) \circ \alpha^{-1}$$

for all $Z \in \mathfrak{z}$.

Remark 2.2. The choice of $\frac{1}{2}$ and 1 as the eigenvalues of A in Notation 2.1(iii), as opposed to, say, 1 and 2, is for convenience in the statement of Theorem 2.5 below. Up to scaling of the metric, all Riemannian solvmanifolds of algebraic rank one and eigenvalue type $(1, 2; r, s)$ arise as in Notation 2.1(iii).

Example 2.3. The classical $2n + 1$ -dimensional Heisenberg algebra is a two-step nilpotent Lie algebra with basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$ satisfying the bracket relations $[X_i, Y_i] = Z$ for $i = 1, \dots, n$ with all other brackets of basis elements being zero. The derived algebra is one-dimensional and is spanned by Z . If we choose the inner product for which the basis above is orthonormal, then $j(Z)X_i = Y_i$ and $j(Z)Y_i = -X_i$ for all i . The associated solvmanifold is the complex hyperbolic space of real dimension $2n + 2$.

A two-step nilpotent Lie algebra \mathfrak{n} , defined as in 2.1(iii) by data $(\mathfrak{v}, \mathfrak{z}, j)$, is said to be of *Heisenberg type* if $j(Z)^2 = -|Z|^2 \text{Id}$ for all $Z \in \mathfrak{z}$. The Heisenberg type algebras, first introduced by A. Kaplan [K], have been studied extensively. Damek and Ricci showed that the solvmanifolds associated as in 2.1(iii) to metric nilpotent Lie algebras of Heisenberg type are harmonic (and thus necessarily Einstein). We will refer to such solvmanifolds as Damek–Ricci manifolds. The fact that these manifolds are Einstein will also follow from Theorem 2.5 below.

Proposition 2.4. *The Riemannian solvmanifolds defined, as in Notation 2.1, by data triples $(\mathfrak{v}, \mathfrak{z}, j)$ and $(\mathfrak{v}', \mathfrak{z}', j')$ are isometric if and only if $(\mathfrak{v}, \mathfrak{z}, j)$ and $(\mathfrak{v}', \mathfrak{z}', j')$ are equivalent.*

Proof. Let \mathfrak{s} and \mathfrak{s}' be the metric solvable Lie algebras defined by these data triples and let \mathfrak{n} and \mathfrak{n}' be their nilradicals. By Theorem 1.2, we need only show that \mathfrak{s} and \mathfrak{s}' are isomorphic precisely when the data triples are equivalent. An elementary argument shows that if the data triples are equivalent, then the metric Lie algebras \mathfrak{n} and \mathfrak{n}' are isomorphic. Moreover, any isomorphism of the metric Lie algebras \mathfrak{n} and \mathfrak{n}' extends in an obvious way to an isomorphism of the metric Lie algebras \mathfrak{s} and \mathfrak{s}' . Conversely any isomorphism from \mathfrak{s} to \mathfrak{s}' must restrict to an isomorphism from \mathfrak{n} to \mathfrak{n}' which carries \mathfrak{v} to \mathfrak{v}' and \mathfrak{z} to \mathfrak{z}' . Such an isomorphism forces the data triples to be equivalent. \square

Theorem 2.5. [EH2] *A Riemannian solvmanifold defined by data $(\mathfrak{v}, \mathfrak{z}, j)$ as in Notation 2.1 is Einstein if and only if the following two conditions hold:*

- (i) *The map $j : \mathfrak{z} \rightarrow j(\mathfrak{z}) \subset \mathfrak{so}(\mathfrak{v})$ is a linear isometry relative to the Riemannian inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{z} and the inner product (\cdot, \cdot) on $\mathfrak{so}(\mathfrak{v})$ given by $(\alpha, \beta) = -\frac{1}{r} \text{tr}(\alpha\beta)$. (Note that $\mathfrak{so}(\mathfrak{v})$ is isomorphic to $\mathfrak{so}(r)$.)*
- (ii) *Letting $\{Z_1, \dots, Z_s\}$ be an orthonormal basis of \mathfrak{z} , then $\sum_{i=1}^s j(Z_i)^2$ is a scalar operator. (This condition is independent of the choice of orthonormal basis.)*

Remark 2.6. In the presence of condition (i), condition (ii) says that $\sum_{i=1}^s j(Z_i)^2 = -s \text{Id}$.

Proof. We have $\text{ad}(A)|_{\mathfrak{v}} = \frac{1}{2}\text{Id}$ and $\text{ad}(A)|_{\mathfrak{z}} = \text{Id}$. The mean curvature vector field H , defined in Notation 1.3, is given by

$$H = \frac{1}{2}(r + 2s)A.$$

Using formula (1.1) for the Ricci curvature, the fact that \mathfrak{a} , \mathfrak{v} and \mathfrak{z} are orthogonal with respect to the Riemannian inner product, and the fact that the Killing form B satisfies $B|_{\mathfrak{n} \times \mathfrak{s}} = 0$, one easily checks that the

subspaces \mathfrak{a} , \mathfrak{v} and \mathfrak{z} of \mathfrak{s} are mutually orthogonal with respect to the Ricci form (independently of whether the metric is Einstein). Thus we need only examine the restriction of the Ricci form to each of these three subspaces. In applying equation (1.2), it is convenient to choose an orthonormal basis $\{X_1, \dots, X_n\}$ of \mathfrak{s} so that each X_i lies in one of \mathfrak{a} , \mathfrak{v} or \mathfrak{z} .

from equation (1.2) and the facts that $\text{ad}(A)^* = \text{ad}(A)$ and that $A \perp [\mathfrak{s}, \mathfrak{s}]$, we see that

$$\text{Ric}(A, A) = -B(A, A) = -\frac{1}{4}(r + 4s). \quad (2.3)$$

Next let $X \in \mathfrak{v}$. We have

$$\text{ad}(X) \circ \text{ad}(X)^*(A) = 0 = \text{ad}(X) \circ \text{ad}(X)^*|_{\mathfrak{v} \cap X^\perp},$$

$$\text{ad}(X) \circ \text{ad}(X)^*(X) = \frac{1}{4}|X|^2 X,$$

and

$$\langle \text{ad}(X) \circ \text{ad}(X)^*(Z), Z \rangle = -\langle j(Z)^2 X, X \rangle$$

for $Z \in \mathfrak{z}$ (see equation (2.2)). Thus

$$-\frac{1}{2} \text{tr} \text{ad}(X) \circ \text{ad}(X)^* = -\frac{1}{8}|X|^2 + \frac{1}{2}S(X, X)$$

where

$$S(X, X) = \left\langle \sum_{i=1}^s j(Z_i)^2(X), X \right\rangle.$$

The third term in equation (1.2) equals $\frac{1}{8}|X|^2$. Thus equation (1.2) yields

$$\text{Ric}(X, X) = -\frac{1}{4}(r + 2s)|X|^2 + \frac{1}{2}S(X, X). \quad (2.4)$$

Finally, let $Z \in \mathfrak{z}$. The contribution to the third term in the equation (1.2) for $\text{Ric}(Z, Z)$ from those X_i 's forming an orthonormal basis of \mathfrak{v} is exactly $\frac{r}{4}|j(Z)|^2$ relative to the inner product (\cdot, \cdot) . The only other non-zero contribution to this term arises from the bracket of A with $Z/|Z|$. This latter contribution is exactly cancelled by the first term in equation (1.2). Indeed, since $\text{image}(\text{ad}(Z)) = \mathbf{R}Z$ and $\text{ad}(Z)^*(Z/|Z|) = -|Z|A$, we see that $\text{tr}(\text{ad}(Z) \circ \text{ad}(Z)^*) = |Z|^2$. Hence

$$\text{Ric}(Z, Z) = -\frac{1}{2}(r + 2s)|Z|^2 + \frac{r}{4}|j(Z)|^2. \quad (2.5)$$

Assume the metric is Einstein. Equation (2.4) then implies that $S = c_1 \langle \cdot, \cdot \rangle|_{\mathfrak{v} \times \mathfrak{v}}$ for some constant c_1 and thus condition (ii) holds. From equation (2.5), we see that there exists a constant c_2 such that $|j(Z)|^2 = c_2|Z|^2$. Comparing equations (2.3) and (2.5), we see that $c_2 = 1$. It then follows from the definition of S that $c_1 = -s$. Conversely, when conditions (i) and (ii) hold, we see from equations (2.3), (2.4), and (2.5) that the metric is Einstein. \square

Definition 2.7. We will say an s -dimensional subspace W of $\mathfrak{so}(r)$ is *uniform* if, relative to the inner product on $\mathfrak{so}(r)$ defined in Theorem 2.5, some (and hence every) orthonormal basis $\{\alpha_1, \dots, \alpha_s\}$ of W satisfies $\sum_{i=1}^s \alpha_i^2 = -c \text{Id}$ for some c (necessarily $c = s$). We will say two uniform subspaces of $\mathfrak{so}(r)$ are *equivalent* if they are conjugate by an orthogonal transformation of \mathbf{R}^r .

Corollary 2.8. *Up to scaling, isometry classes of Einstein solvmanifolds of algebraic rank one and eigenvalue type $(1, 2; r, s)$ are in one-to-one correspondence with equivalence classes of uniform s -dimensional subspaces of $\mathfrak{so}(r)$.*

Proof. If (S, g) is an Einstein solvmanifold with associated data triple $(\mathfrak{v}, \mathfrak{z}, j)$, then Theorem 2.5 implies that $W := j(\mathfrak{z})$ is a uniform subspace of $\mathfrak{so}(r)$, where $\mathfrak{so}(r)$ is identified with $\mathfrak{so}(\mathfrak{v})$. The equivalence class of W is independent of the choice of isomorphism between $\mathfrak{so}(r)$ and $\mathfrak{so}(\mathfrak{v})$. Moreover, any data triple equivalent to $(\mathfrak{v}, \mathfrak{z}, j)$ gives rise to a subspace of $\mathfrak{so}(r)$ equivalent to W . Conversely, given a uniform s -dimensional subspace of $\mathfrak{so}(r)$, define a data triple $(\mathfrak{v}, \mathfrak{z}, j)$ by taking \mathfrak{v} to be r -dimensional Euclidean space, \mathfrak{z} to be W equipped with the inner product (\cdot, \cdot) from $\mathfrak{so}(\mathfrak{z})$, and j to be the identity map. Then this data triple satisfies

the two conditions of Theorem 2.5 and thus the associated Riemannian solvmanifold is Einstein. Equivalent subspaces of $\mathfrak{so}(r)$ give rise to equivalent data triples. The corollary now follows from Theorem 2.5 and Proposition 2.4. \square

Proposition 2.9. *Let $r \in \mathbf{Z}^+$.*

- (i) *If W_1 and W_2 are orthogonal uniform subspaces of $\mathfrak{so}(r)$, then $W_1 + W_2$ is also uniform.*
- (ii) *If W_1 and W_2 are uniform subspaces of $\mathfrak{so}(r)$ with $W_1 \subset W_2$, then $W_2 \ominus W_1$ is uniform.*
- (iii) *If W is a subalgebra of $\mathfrak{so}(r)$ and if the restriction to W of the standard representation of $\mathfrak{so}(r)$ on \mathbf{R}^r has only one isotypic component (i.e., all the irreducible components are equivalent), then W is a uniform subspace of $\mathfrak{so}(r)$. In particular, $\mathfrak{so}(r)$ is uniform in itself.*
- (iv) *If W is uniform in $\mathfrak{so}(r)$, then so is the orthogonal complement of W in $\mathfrak{so}(r)$.*
- (v) *If r is odd, then $\mathfrak{so}(r)$ contains no uniform subspaces of dimension one or two or of codimension one or two.*

Proof. Statements (i) and (ii) are immediate from Definition 2.7 and (iv) follows from (ii) and (iii). To prove (iii), let W be a subalgebra of $\mathfrak{so}(r)$. The expression $\sum_{i=1}^s \alpha_i^2$ in Definition 2.7 is the Casimir operator for the representation of W on \mathbf{R}^r . The Casimir operator is constant on each isotypic component of the representation, so (iii) follows.

Finally we prove (v). Assume r is odd. It is clear that no one-dimensional subspace of $\mathfrak{so}(r)$ can be uniform in this case and thus, by (iv), no subspace of codimension one can be uniform. Next suppose W is a two-dimensional uniform subspace of $\mathfrak{so}(r)$. Let $\{a_1, a_2\}$ be an orthonormal basis of W . Since $a_1^2 + a_2^2$ is a scalar, each a_i^2 commutes with $a_1^2 + a_2^2$, and thus a_1^2 commutes with a_2^2 . Consequently, a_2^2 leaves invariant the 0-eigenspace of a_1^2 . Since this eigenspace is odd-dimensional and since the eigenspaces of a_2^2 for non-zero eigenvalues are even-dimensional, a_1^2 and a_2^2 must have a common 0-eigenvector, contradicting uniformity of W . Thus $\mathfrak{so}(r)$ contains no uniform subspaces of dimension two or codimension two. \square

Example 2.10. We classify the Einstein solvmanifolds of algebraic rank one and eigenvalue type $(1, 2; r, s)$ with $r = 2$ or $r = 3$. For $r = 2$, the algebra $\mathfrak{so}(2)$ is one-dimensional, so necessarily $s = 1$ and \mathfrak{n} is the three-dimensional Heisenberg algebra. As in Example 2.3, the associated Einstein manifold is symmetric. By Proposition 2.9, for $r = 3$, the only uniform subspace of $\mathfrak{so}(r)$ is $\mathfrak{so}(r)$ itself. The associated Einstein solvmanifold is not a symmetric or Damek–Ricci manifold in the terminology of Example 2.3.

We next classify the Einstein solvmanifolds of rank one and eigenvalue type $(1, 2; 4, s)$; equivalently, we classify the equivalence classes of uniform subspaces of arbitrary dimension in $\mathfrak{so}(4)$. The algebra $\mathfrak{so}(4)$ is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Viewing \mathbf{R}^4 as the space of quaternions \mathbf{H} , then $\mathfrak{so}(4)$ is identified with $P \oplus P$ where P is the space of purely imaginary quaternions. The element $(q, p) \in P \oplus P$ acts on \mathbf{H} as $L(q) + R(p)$ where $L(q)$ and $R(q)$ denote left and right multiplication by the quaternion q . The inner product on $\mathfrak{so}(4)$ agrees, up to scalar multiple, with the standard inner product on $P \oplus P$ given by $\langle (q, p), (q', p') \rangle = qq' + pp'$.

Lemma 2.11. *Suppose $\{\alpha_1, \dots, \alpha_s\}$ is a set of orthonormal vectors in $P \oplus P = \mathfrak{so}(4)$. Let B be the matrix whose i^{th} row is the vector α_i expressed in terms of the standard orthonormal basis of $P \oplus P$. Then the subspace W of $\mathfrak{so}(4)$ spanned by $\{\alpha_1, \dots, \alpha_s\}$ is uniform, if and only if the first three columns of B are orthogonal to the last three columns, where the columns are viewed as vectors in \mathbf{R}^s with its standard dot product.*

Proof. For $q, p \in P$, the linear transformation (q, p) of \mathbf{H} satisfies

$$(q, p)^2 = -(|q|^2 + |p|^2) \text{Id} + 2L(q)R(p), \quad (2.6)$$

where $|q|^2 = q\bar{q}$. Letting $\epsilon_1 = \mathbf{i}$, $\epsilon_2 = \mathbf{j}$, and $\epsilon_3 = \mathbf{k}$ be the standard basis vectors of P , write $\alpha_i = (q, 0) + (0, p) = \sum_{j=1}^3 [(a_{ij}\epsilon_j, 0) + (0, b_{ij}\epsilon_j)]$, so that the i^{th} row of B is given by $(a_{i1}, a_{i2}, a_{i3}, b_{i1}, b_{i2}, b_{i3})$.

Then from equation (2.6) we see that

$$\sum_{i=1}^s \alpha_i^2 = - \sum_{i=1}^s [|q_i|^2 + |p_i|^2] \text{Id} + \sum_{j,k=1}^s c_{jk} L(\epsilon_j) R(\epsilon_k)$$

where

$$c_{jk} = \sum_{i=1}^s a_{ij} b_{ik}.$$

Note that c_{jk} is the dot product of the j^{th} column of B with the $(3+k)^{\text{th}}$ column. The lemma follows from the fact that $\sum_{j,k=1}^s c_{jk} L(\epsilon_j) R(\epsilon_k)$ is a multiple of the identity operator only when all the c_{jk} are zero. \square

Recall that the action on $P \oplus P$ by conjugation of the special orthogonal group $\text{SO}(4)$ consists of pairs of rotations (R, S) sending (q, p) to $(R(q), S(p))$. Conjugation by an orthogonal matrix of determinant -1 is given by the composition of some such (R, S) with the map that interchanges p and q .

We will now classify the equivalence classes of uniform subspaces of $\mathfrak{so}(4)$ of dimension s for each $s = 1, \dots, 6$. By Proposition 2.9(iv), we need only classify the uniform subspaces of dimension less than or equal to 3. Except when indicated, the examples below are neither symmetric nor Damek-Ricci manifolds (see Example 2.3).

After conjugating W by an orthogonal matrix of determinant -1 if necessary, we may assume that the dimension of the subspace of \mathbf{R}^s spanned by the first three columns of B is greater than or equal to that of the subspace spanned by the last three columns.

s = 1 In this case we obtain only one solvmanifold, the 6-dimensional complex hyperbolic space. (Indeed, for any r , the only Einstein solvmanifolds of algebraic rank one and eigenvalue type $(1, 2; r, 1)$ are the complex hyperbolic spaces.)

s = 2 If columns 1–3 of B span a 2-dimensional space, then columns 4–6 are trivial. After applying an orthogonal transformation, we have $\alpha_1 = L(\epsilon_1)$ and $\alpha_2 = L(\epsilon_2)$. The resulting metric nilpotent Lie algebra \mathfrak{n} is of Heisenberg type, as defined in Example 2.3, and the associated Einstein solvmanifold is a Damek-Ricci manifold. The complementary four-dimensional uniform subspace is a subalgebra of $\mathfrak{so}(4)$ isomorphic to $\mathfrak{su}(2)$.

If columns 1–3 span a one-dimensional space, then so do columns 4–6. Using the fact that exactly two rows of B are independent, after applying an orthogonal transformation, we can arrange that $\alpha_1 = L(\epsilon_1)$ and $\alpha_2 = R(\epsilon_1)$.

Up to equivalence, these are the only two possibilities. Thus we obtain exactly two isometry classes of Einstein manifolds with $s = 2$ and exactly two with $s = 4$.

s = 3 A similar argument shows we get exactly two uniform subspaces up to equivalence: (i) $W = \{(q, 0) : q \in P\}$ (the associated solvmanifold is the quaternionic hyperbolic space); (ii) W has orthonormal basis $\alpha_1 = L(\epsilon_1)$, $\alpha_2 = L(\epsilon_2)$ and $\alpha_3 = R(\epsilon_1)$.

We have now seen that there are only isolated examples of Einstein manifolds of eigenvalue type $(\frac{1}{2}, 1; r, s)$ with $r \leq 4$. A case-by-case check shows that the same holds for $r = 5$, $s \leq 3$. In contrast, in the next section we will construct a continuous family of Einstein manifolds of type $(\frac{1}{2}, 1; 6, 3)$.

3. A TWO-PARAMETER FAMILY

We will now construct a family of uniform three-dimensional subspaces of $\mathfrak{so}(6)$ as follows: We first construct three mutually orthogonal three-dimensional uniform subspaces U , V , and W of $\mathfrak{so}(6)$ with orthonormal bases $\{A_1, A_2, A_3\}$ of U , $\{B_1, B_2, B_3\}$ of V and $\{C_1, C_2, C_3\}$ of W such that, for each j , A_j , B_j and C_j anti-commute. This condition implies that, for any $r, s, t \in \mathbf{R}$,

$$(rA_i + sB_i + tC_i)^2 = r^2 A_i^2 + s^2 B_i^2 + t^2 C_i^2.$$

Consequently, if $r^2 + s^2 + t^2 = 1$, then the set $\{rA_i + sB_i + tC_i : i = 1, 2, 3\}$ is orthonormal and spans a uniform subspace of $\mathfrak{so}(6)$. Of course, (r, s, t) and $(-r, -s, -t)$ determine the same uniform subspace. We thus obtain a family of uniform subspaces parameterized by $\mathbf{R}P^2 = S^2/\{\pm 1\}$. After explicitly constructing these uniform subspaces, we show that some are negatively curved. Then we test for equivalences among them. We can also replace some of the B_i and C_j by their negatives to get additional families of uniform subspaces.

To construct U , V , and W , let $\tau : \mathfrak{u}(3) \rightarrow \mathfrak{so}(6)$ be the inclusion given as follows: The algebra $\mathfrak{u}(3)$ consists of all matrices of the form $X + \sqrt{-1}Y$ where X is a skew-symmetric real 3×3 matrix and Y is a symmetric real 3×3 matrix. Define

$$\tau(X + \sqrt{-1}Y) = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}.$$

Define the subspaces U_0 , V_0 and W_0 of $\mathfrak{u}(3)$ to be, respectively, the subspace of real skew-symmetric matrices, the subspace of purely imaginary symmetric matrices with zero diagonal, and the subspace of purely imaginary diagonal matrices. Let U , V , and W be the images of U_0 , V_0 and W_0 , respectively, under τ . Set

$$X_1 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad X_2 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let $A_i = \tau(X_i)$. Then $\{A_1, A_2, A_3\}$ is an orthonormal basis of U . Define Y_1, Y_2 and Y_3 analogously to X_1, X_2 and X_3 but with all -1 's replaced by 1 's, and let $B_i = \tau(\sqrt{-1}Y_i)$. Then $\{B_1, B_2, B_3\}$ is an orthonormal basis of V . For an orthonormal basis of W , let C_i be the image under τ of the diagonal matrix whose i^{th} diagonal entry is $\sqrt{-3}$ and whose other diagonal entries are zero. With these bases in hand, one easily checks that U , V and W are uniform subspaces of $\mathfrak{so}(6)$ and that A_i, B_i and C_i anti-commute.

For given (r, s, t) with $r^2 + s^2 + t^2 = 1$, let $W(r, s, t) = \text{span}\{D_i = rA_i + sB_i + tC_i \mid i = 1, 2, 3\}$. The D_i 's give an orthonormal basis for $W(r, s, t)$. Since $W(-r, -s, -t) = W(r, s, t)$, we have a two-dimensional moduli space of uniform subspaces of $\mathfrak{so}(6)$, parametrized by $\mathbf{R}P^2$. Each uniform subspace gives rise to an Einstein solvmanifold as in Corollary 2.8.

Proposition 3.1. *The deformation includes a continuous family of Einstein solvmanifolds with negative sectional curvature.*

Proof. We show that the Einstein solvmanifold corresponding to the uniform subspace $W(1, 0, 0)$ has negative sectional curvature. By continuity, $W(r, s, t)$ must also have negative curvature for all (r, s, t) sufficiently close to $(1, 0, 0)$.

Recall that the sectional curvature of the two-plane in \mathfrak{s} spanned by orthonormal vectors X and Y is given by

$$K(X, Y) = -\frac{3}{4}|[X, Y]|^2 - \frac{1}{2}\langle [X, [X, Y]], Y \rangle - \frac{1}{2}\langle [Y, [Y, X]], X \rangle + |U(X, Y)|^2 - \langle U(X, X), U(Y, Y) \rangle$$

where U is defined as in Notation 1.3. (See Besse [Bes], equation (7.30).)

By Proposition 2.3.1 of Leukert [L], in order to show that S has negative sectional curvature, it suffices to show that $K(P, Q) < -\frac{3}{4}|[P, Q]|^2$, whenever P and Q are orthonormal vectors in the nilradical \mathfrak{n} . By rotating the basis $\{P, Q\}$ within the plane spanned by these vectors, we obtain a new orthonormal basis $\{X + Z, Y + W\}$, where X and Y are orthogonal vectors in $\mathfrak{v} = \mathbf{R}^6$ and Z and W are orthogonal vectors in \mathfrak{z} . We can then rewrite the sectional curvature as

$$K(X + Z, Y + W) = -\frac{3}{4}|[X, Y]|^2 - \left(\frac{1}{2}|X|^2 + |Z|^2\right)\left(\frac{1}{2}|Y|^2 + |W|^2\right) - \langle j(Z)X, j(W)Y \rangle + \frac{1}{4}|j(Z)Y + j(W)X|^2.$$

To see that $K(X + Z, Y + W) < -\frac{3}{4}|[X, Y]|^2$, we need only show that

$$\left(\frac{1}{2}|X|^2 + |Z|^2\right)\left(\frac{1}{2}|Y|^2 + |W|^2\right) > -\langle j(Z)X, j(W)Y \rangle + \frac{1}{4}|j(Z)Y + j(W)X|^2. \quad (3.1)$$

The uniform subspace $W(1, 0, 0)$ of $\mathfrak{so}(6)$ was obtained by diagonally embedding $\mathfrak{so}(3)$ into $\mathfrak{so}(6)$. In [L], Leukert proved that the seven-dimensional Riemannian solvmanifold constructed from $\mathfrak{so}(3)$ (as a uniform subspace of itself) is negatively curved by verifying inequality (3.1) in this case. (The metric on his solvmanifold was rescaled but this does not affect the computations.) It is straightforward to verify that inequality (3.1) in our case follows from the case proven by Leukert. \square

We next verify that this deformation of Einstein manifolds is non-trivial:

Proposition 3.2. *The set of isometry classes of the Einstein manifolds $W(r, s, t)$ defined above is parameterized by the quotient of \mathbf{RP}^2 by a finite equivalence relation.*

Lemma 3.3. *Let $W = W(r, s, t)$ be a uniform subspace of $\mathfrak{so}(6)$ as above. The centralizer of W is one-dimensional, spanned by $C_1 + C_2 + C_3$.*

Proof. The eigenvalues of D_i^2 are $-t^2$ and $-(r^2 + s^2)/2$. First assume $t^2 \neq (r^2 + s^2)/2$. The eigenspace of D_i^2 corresponding to $-t^2$ is the non-zero eigenspace of C_i . For $i = 1$, this is $\{\mathbf{e}_1, \mathbf{e}_4\}$; for $i = 2$, it is $\{\mathbf{e}_2, \mathbf{e}_5\}$; for $i = 3$, it is $\{\mathbf{e}_3, \mathbf{e}_6\}$, where $\{\mathbf{e}_1, \dots, \mathbf{e}_6\}$ is the standard basis of \mathbf{R}^6 . Any element X in $\mathfrak{so}(6)$ which commutes with all of W must leave each of three subspaces invariant. It then follows that X is a linear combination of C_1, C_2, C_3 . It is easy to check that the only combinations which commute with all of W are those for which all coefficients are equal.

When $t^2 = (r^2 + s^2)/2$, the centralizer of W is still spanned by $C_1 + C_2 + C_3$. This is easily checked (e.g. using Maple). \square

Lemma 3.4. *Let $W(r, s, t)$ be a uniform subspace of $\mathfrak{so}(6)$ as above. The angle θ between $W(r, s, t)$ and its centralizer satisfies $\cos \theta = |t|$.*

Proof. By Lemma 3.3, θ is the angle between W and $C_1 + C_2 + C_3$. Any unit vector in $W(r, s, t)$ is of the form $xD_1 + yD_2 + zD_3$, for $x, y, z \in \mathbf{R}$ satisfying $x^2 + y^2 + z^2 = 1$. Since $|C_1 + C_2 + C_3| = \sqrt{3}$, we have

$$\cos \theta = \max \frac{\langle C_1 + C_2 + C_3, xD_1 + yD_2 + zD_3 \rangle}{|C_1 + C_2 + C_3| \cdot |xD_1 + yD_2 + zD_3|} = \max \frac{|t(x + y + z)|}{\sqrt{3}} = |t|.$$

\square

Lemma 3.5. *Let $W(r, s, t)$ be a uniform subspace of $\mathfrak{so}(6)$ as above. Then the minimal angle θ between $W(r, s, t)$ and $[W(r, s, t), W(r, s, t)]$ satisfies*

$$\cos \theta = \frac{|r|\sqrt{1-t^2}}{\sqrt{1+3t^2-2|t^2+\sqrt{2}st|}}.$$

Proof. We compute the generating elements of $[W(r, s, t), W(r, s, t)]$:

$$[D_1, D_2] = \frac{r^2 + s^2}{\sqrt{2}} A_3 + t(r(B_2 - B_1) + s(A_1 - A_2)) \quad (3.2)$$

$$[D_3, D_1] = \frac{r^2 - s^2}{\sqrt{2}} A_2 + \sqrt{2}rsB_2 + t(-r(B_1 + B_3) + s(A_1 + A_3)) \quad (3.3)$$

$$[D_2, D_3] = \frac{r^2 + s^2}{\sqrt{2}} A_1 + t(r(B_2 - B_3) + s(A_3 - A_2)) \quad (3.4)$$

We find the non-zero inner products of these basis elements of $[W(r, s, t), W(r, s, t)]$ with basis elements of $W(r, s, t)$:

$$\langle [D_i, D_j], D_k \rangle = \frac{r(r^2 + s^2)}{\sqrt{2}} \quad \text{for } (i, j, k) \text{ cyclic permutations of } (1, 2, 3).$$

Let $x, y, z, u, v, w \in \mathbf{R}$ satisfy $x^2 + y^2 + z^2 = 1$ and $u^2 + v^2 + w^2 = 1$. Then

$$\langle x[D_1, D_2] + y[D_3, D_1] + z[D_2, D_3], uD_1 + vD_2 + wD_3 \rangle = (xw + yv + zu) \frac{r(r^2 + s^2)}{\sqrt{2}}. \quad (3.5)$$

Since $|uD_1 + vD_2 + wD_3| = 1$ and

$$|x[D_1, D_2] + y[D_3, D_1] + z[D_2, D_3]| = \sqrt{(r^2 + s^2) \left(\frac{r^2 + s^2 + 4t^2}{2} + 2(xz + xy + zy)(t^2 + \sqrt{2}st) \right)},$$

we have

$$\cos \theta = \max \frac{|(xw + yv + zu)r\sqrt{r^2 + s^2}|}{\sqrt{(r^2 + s^2 + 4t^2) + 4(xz + xy + yz)(t^2 + \sqrt{2}st)}}. \quad (3.6)$$

The maximum is attained when $(w, v, u) = (x, y, z)$ and $xz + xy + yz = \pm 1/2$ with the sign chosen so that $(xz + xy + yz)(t^2 + \sqrt{2}st) \leq 0$. This gives

$$\cos \theta = |r| \sqrt{\frac{r^2 + s^2}{r^2 + s^2 + 4t^2 - 2|t^2 + \sqrt{2}st|}}. \quad (3.7)$$

Replacing $r^2 + s^2$ by $1 - t^2$, we obtain the expression in the lemma. \square

We can now prove Proposition 3.2. Suppose $W(r', s', t')$ is equivalent to $W(r, s, t)$. From Lemma 3.3, we know that $W(r, s, t)$ and $W(r', s', t')$ share the same centralizer. Since the angle between the uniform space and its centralizer is preserved under conjugation by elements of $O(6)$, it follows from Lemma 3.4 that $|t| = |t'|$ and thus also $r^2 + s^2 = (r')^2 + (s')^2$. Since the angle in Lemma 3.5 is also preserved under conjugation, it follows that r' and s' are determined up to finitely many possibilities.

We remark that the finite equivalence relation in Proposition 3.2 is non-trivial. For example the element of $O(6)$ with the 3×3 identity matrix in the upper right and lower left 3×3 blocks and zeros elsewhere conjugates $W(r, s, t)$ to $W(r, -s, -t)$.

4. EXAMPLES: IRREDUCIBLE SYMMETRIC SPACES OF TYPE IV

We now obtain new examples of Einstein manifolds by modifying the Lie algebra structure of the solvable Lie algebras associated with the non-compact symmetric spaces of rank greater than one. We first recall the structure of the isometry groups of the symmetric spaces of non-compact type. See Helgason [H] for more details.

Notation 4.1. The identity component of the full isometry group of an irreducible symmetric space M of non-compact type is a simple Lie group G_0 , and the isotropy subgroup K at a point of M is a maximal compact subgroup of G_0 . The Lie algebra \mathfrak{g}_0 of G_0 admits an Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ where \mathfrak{k} is the Lie algebra of K , \mathfrak{a} is abelian, \mathfrak{n} is nilpotent and $\mathfrak{s} := \mathfrak{a} + \mathfrak{n}$ is solvable with nilradical \mathfrak{n} . The Lie subgroup S of G_0 with Lie algebra \mathfrak{s} acts simply transitively on M , and M may be identified with S , endowed with a left-invariant metric.

The elements of $\text{ad}_{\mathfrak{s}}(\mathfrak{a})$ are symmetric and \mathfrak{n} decomposes into root spaces \mathfrak{n}_{α} , $\alpha \in \Sigma$, where Σ , the set of positive roots of \mathfrak{a} in \mathfrak{g}_0 , is a subset of the dual space \mathfrak{a}^* of \mathfrak{a} . Relative to the inner product on \mathfrak{s} defined by the symmetric space metric on S (identified with M), the root spaces are mutually orthogonal and there exists an orthonormal basis of the subspace \mathfrak{n} of \mathfrak{s} such that:

- Each basis vector lies in some \mathfrak{n}_{β} .
- The bracket of any two basis vectors is a scalar multiple of another basis vector.
- If X, Y , and U are vectors in the basis with $Y \neq U$, then $[X, Y] \perp [X, U]$.

Notation 4.2. Let \mathfrak{g} be the complexification of \mathfrak{g}_0 and let $\mathfrak{g}^{\mathbf{R}}$ be \mathfrak{g} viewed as a real Lie algebra. The dimension of $\mathfrak{g}^{\mathbf{R}}$ is twice that of \mathfrak{g} . View \mathfrak{g}_0 , and thus also \mathfrak{s} , as subalgebras of $\mathfrak{g}^{\mathbf{R}}$. Choose a basis \mathcal{B} for \mathfrak{n} satisfying the three conditions above. Consider a new subspace of $\mathfrak{g}^{\mathbf{R}}$ with basis \mathcal{B}' , where \mathcal{B}' is obtained from \mathcal{B} by replacing some of the vectors $X \in \mathcal{B}$ by $\sqrt{-1}X$. If the new subspace \mathfrak{n}' is closed under the bracket operation in \mathfrak{g} , then \mathfrak{n}' will be a nilpotent algebra. The brackets of the basis vectors in \mathfrak{n}' just differ by a sign from the corresponding brackets in \mathfrak{n} . Endow \mathfrak{n}' with the inner product for which \mathcal{B}' is orthonormal. The subalgebra \mathfrak{a} of $\mathfrak{g}^{\mathbf{R}}$ normalizes \mathfrak{n}' ; indeed, \mathfrak{n}' decomposes into root spaces \mathfrak{n}'_{α} of \mathfrak{a} , where α varies over Σ . Letting $\mathfrak{s}' = \mathfrak{a} + \mathfrak{n}'$, then \mathfrak{s}' is a solvable Lie algebra with nilradical \mathfrak{n}' . We will say \mathfrak{s} and \mathfrak{s}' are *associated* subalgebras of $\mathfrak{g}^{\mathbf{R}}$. Give \mathfrak{s}' the inner product for which $\mathfrak{s}' = \mathfrak{a} + \mathfrak{n}'$ is the orthogonal direct sum of the inner product spaces \mathfrak{a} and \mathfrak{n}' . This inner product on \mathfrak{s}' defines a left-invariant Riemannian metric on the associated simply connected Lie group S' . The Riemannian manifold S' will be referred to as a *Riemannian solvmanifold associated with the symmetric space M* .

Theorem 4.3. *If S' is a Riemannian solvmanifold associated with a symmetric space M as in Notation 4.2, then S' is Einstein with the same Einstein constant as M .*

Proof. We write Ric for the Ricci tensor on the symmetric solvmanifold S , identified with M , and Ric' for the Ricci tensor of S' . View Ric and Ric' as quadratic forms on \mathfrak{s} and \mathfrak{s}' , respectively. We denote by τ the linear isomorphism from \mathfrak{s} to \mathfrak{s}' which restricts to the identity on \mathfrak{a} and sends the basis \mathcal{B} of \mathfrak{n} to the basis \mathcal{B}' of \mathfrak{n}' in the obvious way. Using the expression for the Ricci curvature given in equation (1.1), we will show that $\text{Ric}(X, Y) = \text{Ric}'(\tau(X), \tau(Y))$ for all $X, Y \in \mathfrak{s}$.

Let H be the mean curvature vector of \mathfrak{s} as in Notation 1.3. From the fact that $[A, \tau(X)] = \tau([A, X])$ for all $A \in \mathfrak{a}$ and $X \in \mathfrak{s}$, we see that $\tau(H) = H$ is also the mean curvature vector of \mathfrak{s}' and that $\langle U(X, Y), H \rangle = \langle U(\tau(X), \tau(Y)), H \rangle$ for all $X, Y \in \mathfrak{s}$, where U is defined in Notation 1.3. Also τ is an isometry between the Killing forms B of \mathfrak{s} and B' of \mathfrak{s}' since the two Killing forms agree on \mathfrak{a} and are zero on the nilradicals \mathfrak{n} and \mathfrak{n}' , respectively. It follows that $\text{Ric}(X, Y) = \text{Ric}'(\tau(X), \tau(Y))$ whenever $X \in \mathfrak{a}$ and $Y \in \mathfrak{s}$. Moreover, from the properties of \mathcal{B} indicated by bullets, we see that $\text{Ric}(X, Y) = \text{Ric}'(\tau(X), \tau(Y))$ when $X, Y \in \mathcal{B}$. \square

Before applying this theorem, we discuss conditions under which two Riemannian solvmanifolds associated to a symmetric space are isometric.

Proposition 4.4. *In the notation of Notation 4.1 and 4.2, suppose that S' and S'' are two Riemannian solvmanifolds associated with M . Then S' is isometric to S'' if and only if there exists an automorphism $\phi : \Sigma \rightarrow \Sigma$ of the root system Σ (i.e., a Weyl group element) and an isomorphism τ between the metric Lie algebras \mathfrak{n}' and \mathfrak{n}'' such that $\tau(\mathfrak{n}'_{\alpha}) = \mathfrak{n}''_{\phi(\alpha)}$ for all $\alpha \in \Sigma$.*

Proof. By Proposition 1.2, S' and S'' are isometric if and only if there exists a metric Lie algebra isomorphism $\tau : \mathfrak{s}' \rightarrow \mathfrak{s}''$. Such an isomorphism τ must carry \mathfrak{n}' to \mathfrak{n}'' and thus \mathfrak{a} to \mathfrak{a} . The restriction of τ to \mathfrak{a} must then induce an automorphism of Σ and the restriction of τ to \mathfrak{n}' must satisfy the last statement of the proposition. \square

Example 4.5. Suppose \mathfrak{s}' and \mathfrak{s}'' are two solvable subalgebras of $\mathfrak{g}^{\mathbf{R}}$ associated with \mathfrak{s} . If for each $\alpha \in \Sigma$, we have either $\mathfrak{n}''_{\alpha} = \mathfrak{n}'_{\alpha}$ or $\mathfrak{n}''_{\alpha} = \sqrt{-1}\mathfrak{n}'_{\alpha}$, then the associated Riemannian solvmanifolds are isometric. To see this, choose a base $\{\alpha_1, \dots, \alpha_k\}$ for the root system Σ . Write \mathfrak{n}_j for \mathfrak{n}_{α_j} . Every root $\alpha \in \Sigma$ can be written uniquely as a linear combination of $\{\alpha_1, \dots, \alpha_k\}$ with non-negative integer coefficients. Since $[\mathfrak{n}_{\alpha}, \mathfrak{n}_{\beta}] = \mathfrak{n}_{\alpha+\beta}$ whenever $\alpha, \beta, \alpha + \beta \in \Sigma$, the \mathfrak{n}_j generate \mathfrak{n} as a Lie algebra. A similar statement holds in the algebras \mathfrak{n}' and \mathfrak{n}'' .

Let

$$\mathcal{A} = \{j \in \{1, \dots, k\} : \mathfrak{n}''_j = \sqrt{-1}\mathfrak{n}'_j\}.$$

For $\alpha \in \Sigma$, expressed as a linear combination $\alpha = \sum_{i=1}^k b_i \alpha_i$, define the *restricted height* $rh(\alpha)$ by

$$rh(\alpha) = \sum_{j \in \mathcal{A}} b_j.$$

The fact that \mathfrak{n}' and \mathfrak{n}'' are both closed under the bracket operation in $\mathfrak{g}^{\mathbf{R}}$ then forces \mathfrak{n}''_{α} to satisfy

$$\mathfrak{n}''_{\alpha} = \sqrt{-1}^{rh(\alpha)} \mathfrak{n}'_{\alpha}.$$

The linear map from \mathfrak{n}' to \mathfrak{n}'' whose restriction to each \mathfrak{n}'_{α} is given by multiplication by $\sqrt{-1}^{rh(\alpha)}$ is then a Lie algebra isomorphism and inner product space isometry. This map extends to an isomorphism from \mathfrak{s}' to \mathfrak{s}'' by acting as the identity on \mathfrak{a} . By Proposition 1.2, we conclude that S'' is isometric to S' . In the notation of Proposition 4.4, ϕ is the identity.

Remark 4.6. Example 4.5 shows that given any subalgebra \mathfrak{s}' of $\mathfrak{g}^{\mathbf{R}}$ associated with \mathfrak{s} and any subset \mathcal{A} of $\{1, \dots, k\}$, where k is the rank of \mathfrak{s} , then we can define another subalgebra \mathfrak{s}'' of $\mathfrak{g}^{\mathbf{R}}$ associated to \mathfrak{s} by setting $\mathfrak{n}''_{\alpha} = \sqrt{-1}^{rh(\alpha)} \mathfrak{n}'_{\alpha}$ in the notation of the example. The associated Riemannian solvmanifolds S' and S'' are then isometric. This construction will be useful in uniqueness arguments below.

We now apply Theorem 4.3. The irreducible symmetric spaces of negative Ricci curvature may be divided into two types by considering the structure of the complex Lie algebra \mathfrak{g} . We have either:

- (i) \mathfrak{g} is a complex simple Lie algebra, or
- (ii) \mathfrak{g} is isomorphic to the direct sum of two copies of a simple complex Lie algebra.

Helgason [H] calls these two types of symmetric spaces ‘‘Types III and IV’’, respectively, with types I and II referring to symmetric spaces of positive Ricci curvature. The book [H] is an excellent source for details on the facts about the root systems of semisimple Lie algebras used below.

We first consider case (ii); i.e., we assume that \mathfrak{g} consists of two copies of a complex simple Lie algebra \mathfrak{f} . In this case, \mathfrak{g}_0 is isomorphic to $\mathfrak{f}^{\mathbf{R}}$. Let J denote the complex structure on \mathfrak{g}_0 arising from \mathfrak{f} . When a subspace V of \mathfrak{g}_0 is invariant under J , we will denote by $V_{\mathbf{C}}$ the complex subspace of \mathfrak{f} defined by V . In the notation above, the subspace \mathfrak{a} of \mathfrak{g}_0 is invariant under J , and $\mathfrak{a}_{\mathbf{C}}$ is a Cartan subalgebra of \mathfrak{f} . Moreover, each root $\alpha \in \Sigma$ commutes with J and thus defines a complex linear functional, again denoted α , on $\mathfrak{a}_{\mathbf{C}}$. With this identification and with a suitable choice of ordering on the set of roots of $\mathfrak{a}_{\mathbf{C}}$ on \mathfrak{f} , the set Σ forms the system of positive roots of $\mathfrak{a}_{\mathbf{C}}$ in \mathfrak{f} , and for each α , we have $\mathfrak{f}_{\alpha} = (\mathfrak{n}_{\alpha})_{\mathbf{C}}$. In particular, since the root spaces \mathfrak{f}_{α} have complex dimension one, the root spaces \mathfrak{n}_{α} have real dimension two. The almost complex structure J on \mathfrak{s} is skew-symmetric relative to the inner product defined by the symmetric space metric on S .

One may choose a unit vector X_{α} in each \mathfrak{n}_{α} , $\alpha \in \Sigma$, so that

$$[X_{\alpha}, X_{\beta}] = N_{\alpha, \beta} X_{\alpha + \beta}$$

for some non-zero constants $N_{\alpha, \beta}$ whenever $\alpha, \beta \in \Sigma$, with the convention that $X_{\alpha + \beta} = 0$ when $\alpha + \beta$ is not a root. The collection of vectors

$$\mathcal{B}_0 = \{X_{\alpha}, JX_{\alpha} : \alpha \in \Sigma\}$$

forms an orthonormal basis for \mathfrak{n} . The brackets satisfy

$$[X_{\alpha}, X_{\beta}] = N_{\alpha, \beta} X_{\alpha + \beta} = -[JX_{\alpha}, JX_{\beta}] \quad \text{and} \quad [X_{\alpha}, JX_{\beta}] = N_{\alpha, \beta} JX_{\alpha + \beta} = [JX_{\alpha}, X_{\beta}]. \quad (4.1)$$

Theorem 4.7. *Let M be a symmetric space of Type IV; i.e., assume its isometry algebra satisfies condition (ii). If \mathfrak{g} has rank greater than one, then, up to isometry, there exists a unique non-symmetric Einstein solvmanifold S' associated with M . Its Lie algebra is spanned by \mathfrak{a} together with $\{X_{\alpha}, JX_{\alpha} : \alpha \in \Sigma\}$. If \mathfrak{g} has rank one, there are no non-symmetric Einstein manifolds associated with M .*

Proof. If \mathfrak{g} has rank one, then \mathfrak{n} is abelian. Any solvable algebra associated with \mathfrak{s}' will also have abelian nilradical, and it follows from Proposition 4.4 that the associated solvmanifold will be isometric to the symmetric space.

Now assume that \mathfrak{g} has rank greater than one. The fact that \mathfrak{s}' , as defined in the theorem, is a solvable subalgebra of \mathfrak{g} associated with \mathfrak{s} is immediate from the bracket relations (4.1). Thus Theorem 4.3 guarantees that S' is Einstein. The fact that S' is not isometric to S can be seen either from Proposition 1.2 or by noting

that S' has some positive sectional curvature. To see the latter fact, recall that the sectional curvature of the two plane in \mathfrak{s}' spanned by a pair of orthonormal vectors X and Y is given by

$$K(X, Y) = -\frac{3}{4}|[X, Y]|^2 - \frac{1}{2}\langle [X, [X, Y]], Y \rangle - \frac{1}{2}\langle [Y, [Y, X]], X \rangle + |U(X, Y)|^2 - \langle U(X, X), U(Y, Y) \rangle \quad (4.2)$$

where U is defined as in Notation 1.3. (See Besse [Bes], equation 7.30.) Since \mathfrak{n} is not abelian, we can choose two basic roots α and β in Σ such that $\alpha + \beta$ is a root. Let $X = X_\alpha + JX_\alpha$ and $Y = X_\beta - JX_\beta$. By equation (4.1), $[X, Y] = 0$. Moreover, $U(X, Y) = 0$ since α and β are basic roots. Thus the only non-zero term in the right-hand-side of equation (4.2) is the last term, and we need to show that

$$\langle U(X, X), U(Y, Y) \rangle < 0. \quad (4.3)$$

Equation (4.3) follows from the structure theory of semisimple Lie algebras of non-compact type. Given a root vector $\delta \in \Sigma$, let $H_\delta \in \mathfrak{a}$ be the unique vector satisfying $\delta(Z) = \langle Z, H_\delta \rangle$ for all $Z \in \mathfrak{a}$. Then $\langle H_\delta, H_\epsilon \rangle \leq 0$ with strict inequality holding when $\delta + \epsilon \in \Sigma$. (See [H].) In our situation, we have $U(X, X) = |X|^2 H_\alpha$ since $[Z, X] = \alpha(Z)X$ for all $Z \in \mathfrak{a}$, and similarly $U(Y, Y) = |Y|^2 H_\beta$. Thus equation (4.3) follows and we conclude that $K(X, Y) > 0$.

We next turn to the uniqueness statement.

The construction in Notation 4.2 of solvmanifolds associated with M depends a priori on a choice of orthonormal basis \mathcal{B} of \mathfrak{n} satisfying the conditions in Notation 4.1. We first show under the hypotheses of Theorem 4.7 that all solvmanifolds associated as in Notation 4.2 with M are isometric to ones constructed using the particular basis \mathcal{B}_0 given above. Let \mathcal{C} be another orthonormal basis of \mathfrak{n} satisfying the conditions in Notation 4.1. \mathcal{C} contains an orthonormal basis of each root space \mathfrak{n}_α . This basis may be expressed in the form $\{U_\alpha, JU_\alpha\}$ for some vectors $U_\alpha \in \mathfrak{n}_\alpha$. Within the complex algebra \mathfrak{g} , we can write $U_\alpha = e^{i\theta_\alpha} X_\alpha$ for some $\theta_\alpha \in [0, 2\pi)$. We then have

$$[U_\alpha, U_\beta] = e^{i(\theta_\alpha + \theta_\beta - \theta_{\alpha+\beta})} N_{\alpha, \beta} U_{\alpha+\beta}.$$

The constants of structure of \mathfrak{n} are real, so $e^{i(\theta_\alpha + \theta_\beta - \theta_{\alpha+\beta})} = \pm 1$. Choose a base $\{\alpha_1, \dots, \alpha_k\}$ for the root system Σ . Each root α can be written uniquely in the form $\sum_{1 \leq j \leq k} b_j \alpha_j$. Define a linear isomorphism τ from \mathfrak{n} to itself by sending X_α to $\epsilon_\alpha U_\alpha$ and JX_α to $\epsilon_\alpha JU_\alpha$ where $\epsilon_\alpha = \pm 1$ is given by $\epsilon_\alpha = e^{i((\sum_{1 \leq j \leq k} b_j \theta_{\alpha_j}) - \theta_\alpha)}$. Then τ is easily seen to be a metric Lie algebra isomorphism. In particular, the basis $\mathcal{C}' = \{\epsilon_\alpha U_\alpha, \epsilon_\alpha JU_\alpha : \alpha \in \Sigma\}$ satisfies exactly the same bracket relations as \mathcal{B}_0 and thus any solvable metric Lie algebra \mathfrak{s}' associated with \mathfrak{s} which is constructed using \mathcal{C}' is isomorphic to one constructed using \mathcal{B} and conversely. Finally, since \mathcal{C}' can be obtained from \mathcal{C} simply by multiplying some elements of \mathcal{C} by -1 , the metric Lie algebras associated to \mathfrak{s} which can be constructed using \mathcal{C} are precisely the ones that can be constructed using \mathcal{C}' . Consequently, up to isometry, our construction of Riemannian solvmanifolds associated to M is independent of the choice of basis satisfying the conditions of Notation 4.1, and we can restrict attention to the basis \mathcal{B}_0 .

Thus suppose \mathfrak{s}'' is a solvable subalgebra of $\mathfrak{g}^{\mathbf{R}}$ constructed as in Notation 4.2 using the basis \mathcal{B}_0 of \mathfrak{n} . We need to show that the associated Riemannian solvmanifold is isometric either to the symmetric space M or to the solvmanifold S' of Theorem 4.7. Choose a base $\{\alpha_1, \dots, \alpha_k\}$ for the root system Σ and write \mathfrak{n}_j for \mathfrak{n}_{α_j} . Let

$$\mathcal{A} = \{j \in \{1, \dots, k\} : \mathfrak{n}_j'' \neq \mathfrak{n}_j\}.$$

In view of Remark 4.6, for $j \in \mathcal{A}$ we may assume that \mathfrak{n}_j'' is obtained from \mathfrak{n}_j by multiplying one of X_j or JX_j , but not both, by $\sqrt{-1}$, where X_j denotes X_{α_j} . Moreover, by further multiplying the eigenspace by $\sqrt{-1}$ if necessary and again referring to Remark 4.6, we may assume that $X_j'' = X_j$ for all j , that $JX_j'' = X_j$ when $j \in \mathcal{A}$, and that $JX_j'' = JX_j$ otherwise. But if $j \in \mathcal{A}$, $i \notin \mathcal{A}$, and $\alpha_j + \alpha_i \in \Sigma$, then from the bracket relations (4.1), we see that it is impossible to define $\mathfrak{n}_{\alpha_j + \alpha_i}''$ in such a way that \mathfrak{n}'' is closed under the bracket operation. We conclude that Σ decomposes into the direct sum of two root systems, one with base $\{\alpha_j : j \in \mathcal{A}\}$ and the other with base $\{\alpha_j : j \notin \mathcal{A}\}$. This contradicts the irreducibility of the symmetric

space M , unless \mathcal{A} is either empty or all of $\{1, \dots, k\}$. These two cases give, respectively, the symmetric space or the manifold S' defined in Theorem 4.7. \square

We next consider the case (i); i.e., we assume that \mathfrak{g} is simple. (See Notation 4.2.) Letting \mathfrak{m}_0 be the centralizer of \mathfrak{a} in \mathfrak{k} , then the complexification \mathfrak{h} of $\mathfrak{a} + \mathfrak{m}_0$ is a Cartan subalgebra of \mathfrak{g} . The Lie algebra \mathfrak{g} decomposes into root spaces

$$\mathfrak{g} = \Sigma_{\beta \in \Delta} \mathfrak{g}_\beta$$

where Δ is a subset of \mathfrak{h}^* . For $\beta \in \Delta$, let $\bar{\beta}$ denote the restriction of β to \mathfrak{a} . Then

$$\mathfrak{n} = \mathfrak{g}_0 \cap \Sigma_{\bar{\beta} \in \Sigma} \mathfrak{g}_\beta.$$

Each \mathfrak{g}_β is a one-dimensional complex subspace of \mathfrak{g} . However, the \mathfrak{n}_α may have dimension greater than one, since there may be several roots β such that $\bar{\beta} = \alpha$. The elements $\text{ad}(X)$, $X \in \mathfrak{m}_0$ normalize each root space \mathfrak{n}_α . In case \mathfrak{m}_0 is trivial, i.e., \mathfrak{h} is the complexification of \mathfrak{a} , then \mathfrak{g}_0 is said to be a *normal real form* of \mathfrak{g} . In this case, all the root spaces \mathfrak{n}_α are one-dimensional. If \mathfrak{m}_0 is non-trivial, then some of the root spaces will have higher dimension.

Proposition 4.8. *If each \mathfrak{n}_α is one-dimensional, i.e., if \mathfrak{g}_0 is a normal real form of \mathfrak{g} , then any Einstein manifold associated to M is isometric to M .*

The proposition is an immediate consequence of Example 4.5.

We now show by specific construction that in most cases of classical irreducible symmetric spaces M for which \mathfrak{g}_0 is not a normal real form of \mathfrak{g} , there exist non-symmetric Einstein manifolds associated with M .

Recall that the classical symmetric spaces are given as follows:

Symmetric Space $M = G/H$	Rank(M)	Rank(G)
$\text{SO}(p, q)/\text{SO}(p)\text{SO}(q)$	$\min(p, q)$	$\lfloor \frac{p+q}{2} \rfloor$
$\text{SU}(p, q)/S(\text{U}(p)\text{U}(q))$	$\min(p, q)$	$p + q - 1$
$\text{Sp}(p, q)/\text{Sp}(p)\text{Sp}(q)$	$\min(p, q)$	$p + q$
$\text{SO}(n, \mathbf{H})/\text{U}(n)$	$\lfloor \frac{n}{2} \rfloor$	n
$\text{SL}(n, \mathbf{H})/\text{Sp}(n)$	$n - 1$	$2n - 1$
$\text{SL}(n, \mathbf{R})/\text{SO}(n)$	$n - 1$	$n - 1$
$\text{Sp}(n, \mathbf{R})/\text{U}(n)$	n	n

We note that Helgason[H] refers to $\text{SO}(n, \mathbf{H})$ and $\text{SL}(n, \mathbf{H})$ as $\text{SO}^*(2n)$ and $\text{SU}^*(2n)$, respectively.

The Lie algebra $\mathfrak{so}(p, q)$ is a real form of $\mathfrak{so}(n, \mathbf{C})$ where $n = p + q$. Since $\text{SO}(p, q)$ is isomorphic to $\text{SO}(q, p)$, we may assume $p \leq q$. In case $q = p$ or $q = p + 1$, then $\mathfrak{so}(p, q)$ is the normal real form of $\mathfrak{so}(n, \mathbf{C})$. In all other cases, i.e., whenever $q - p \geq 2$, we will construct in Subsection 4.9 non-symmetric Einstein solvmanifolds associated with the symmetric space $M = \text{SO}(p, q)/\text{SO}(p)\text{SO}(q)$ as in Notation 4.2.

The Lie algebras $\mathfrak{su}(p, q)$ are real forms of $\mathfrak{sl}(p + q, \mathbf{C})$. None of these are normal forms; the normal real form of $\mathfrak{sl}(n, \mathbf{C})$ is $\mathfrak{sl}(n, \mathbf{R})$. However, we will obtain in Subsection 4.9 non-symmetric Einstein solvmanifolds associated with the symmetric space $M = \text{SU}(p, q)/S(\text{U}(p)\text{U}(q))$ only when $q \geq p + 2$. (As in the case of $\mathfrak{so}(p, q)$, we may take $q \geq p$.) Similarly, $\mathfrak{sp}(p, q)$ is a real form of $\mathfrak{sp}(n, \mathbf{C})$, while $\mathfrak{sp}(n, \mathbf{R})$ is the normal real form, but we will obtain new Einstein solvmanifolds only when $q \geq p + 2$.

Finally, we will obtain non-symmetric Einstein manifolds associated with $\text{SO}(n, \mathbf{H})/\text{U}(n)$ in Subsection 4.10 and with $\text{SL}(n, \mathbf{H})/\text{Sp}(n)$ in Subsection 4.11.

4.9. $\text{SO}(p, q)/\text{SO}(p)\text{SO}(q)$, $\text{SU}(p, q)/S(\text{U}(p)\text{U}(q))$, and $\text{Sp}(p, q)/\text{Sp}(p)\text{Sp}(q)$. We consider simultaneously the rank p non-compact irreducible symmetric spaces $\text{SO}(p, q)/S(\text{O}(p)\text{O}(q))$, $\text{SU}(p, q)/S(\text{U}(p)\text{U}(q))$ and $\text{Sp}(p, q)/\text{Sp}(p)\text{Sp}(q)$ dual to the real, complex and quaternionic Grassmannians. We assume that $q - p \geq 2$. The Lie algebras $\mathfrak{so}(p, q)$, $\mathfrak{su}(p, q)$ and $\mathfrak{sp}(p, q)$ are given by

$$\mathfrak{g}_0 = \{A \in \mathfrak{gl}(p + q, \mathbf{F}) \mid AM + M\bar{A}^t = 0\} \text{ where } M = \begin{pmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{pmatrix}$$

where $\mathbf{F} = \mathbf{R}$ in the case of $\mathfrak{so}(p, q)$, $\mathbf{F} = \mathbf{C}$ in the case of $\mathfrak{su}(p, q)$ and $\mathbf{F} = \mathbf{H}$ in the case of $\mathfrak{sp}(p, q)$.

Let $m = q - p$. Then

$$\mathfrak{g}_0 = \begin{pmatrix} A & B & C \\ \bar{B}^t & D & E \\ \bar{C}^t & -\bar{E}^t & F \end{pmatrix}$$

where A , B , and D are $p \times p$ matrices, C and E are $p \times m$ matrices and F is an $m \times m$ matrix. The matrix A satisfies $\bar{A}^t + A = 0$ and similarly for D and F ; the matrices B , C and E are arbitrary.

The Cartan subalgebra \mathfrak{a} consists of those matrices for which B is real and diagonal and all other blocks are zero. Letting $\{\omega_1, \dots, \omega_p\}$ be the basis of linear functionals on \mathfrak{a} dual to the standard basis of diagonal matrices, then the positive roots are ω_i for $1 \leq i \leq p$ and $\omega_j \pm \omega_i$ for $1 \leq i < j \leq p$, and in the case of $\mathfrak{su}(p, q)$ and $\mathfrak{sp}(p, q)$, $2\omega_i$ for $1 \leq i \leq p$.

The root spaces $\mathfrak{n}_{\omega_j - \omega_i}$, $\mathfrak{n}_{\omega_j + \omega_i}$ and $\mathfrak{n}_{2\omega_k}$ are all contained in the space of matrices for which C , E and F are all zero. The root spaces \mathfrak{n}_{ω_k} , $1 \leq k \leq p$, together span the space W of matrices for which A , B and D are zero and $C = E$. For each k , the root space \mathfrak{n}_{ω_k} is the set of such matrices for which all rows of C except the k th one are zero.

Fix an integer a with $1 \leq a < m$, set $b = m - a$, and decompose W into subspaces $W = W_a + W_b$ where W_a , respectively W_b , is the space of those matrices in W for which the last b columns, respectively first a columns, of C are zero. We modify the Lie algebra \mathfrak{s} to construct a new Lie algebra \mathfrak{s}_a by replacing all elements X in W_b by X in $\mathfrak{s}^{\mathbf{C}}$, leaving the spaces W_a and the root spaces $2\omega_i$, $1 \leq i \leq p$, and $\omega_j \pm \omega_i$, $1 \leq i < j \leq p$ unchanged. The fact that \mathfrak{s}_a is closed under bracket in $\mathfrak{s}^{\mathbf{C}}$ follows from the observations that (i) W commutes with all root spaces of the form $\mathfrak{n}_{\omega_j + \omega_i}$ or $\mathfrak{n}_{2\omega_k}$, (ii) W_b commutes with W_a , and (iii) \mathfrak{a} and the root spaces of the form $\mathfrak{n}_{\omega_j - \omega_i}$ normalize each of W_a and W_b .

The simply-connected solvmanifold S_a corresponding to \mathfrak{s}_a is Einstein by Theorem 4.3. S_a has some 2-planes of positive curvature and thus is not isometric to the symmetric space S . Indeed, let i and j be any two positive integers with $i \leq a$ and $a < j \leq a + b$. For $1 \leq k \leq m$, let X_k be the element of \mathfrak{n}_{ω_k} for which C has the (k, i) and (k, j) entries equal to $\frac{1}{\sqrt{2}}$ and all other entries zero. Let X'_k be the corresponding unit vectors in the algebra \mathfrak{s}_a . Then for $k < l$, we have $[X'_k, X'_l] = 0$. Moreover, $\langle U(X'_k, X'_k), U(X'_l, X'_l) \rangle = 0$, but $U(X'_k, X'_l)$ is easily seen to be a non-zero element of $\mathfrak{n}_{\omega_k - \omega_l}$. Thus by equation (4.2), $K(X'_k, Y'_l) > 0$.

Finally we verify that if $1 \leq a', a'' < m$, then $S'_{a'}$ and $S''_{a''}$ are not isometric unless either $a'' = a'$ or $a'' = m - a'$. For simplicity, we will just work with the case that $\mathbf{F} = \mathbf{R}$, although similar arguments work in the other cases. In the original Lie algebra \mathfrak{s} , consider two root spaces of the form \mathfrak{n}_{ω_k} , say $k = 1$ and $k = 2$. Then the subspace \mathfrak{h} of \mathfrak{s} spanned by \mathfrak{n}_{ω_1} , \mathfrak{n}_{ω_2} , and $\mathfrak{n}_{\omega_1 + \omega_2}$ is closed under brackets and is isomorphic to the $2m + 1$ -dimensional Heisenberg algebra. (See Example 2.3.) Indeed, letting X_i be the element of \mathfrak{n}_{ω_1} for which the only non-zero entry of C is the $(1, i)$ -entry, letting Y_i be the analogous element of \mathfrak{n}_{ω_2} , and letting Z be an appropriately chosen element of the one-dimensional space $\mathfrak{n}_{\omega_1 + \omega_2}$, then the basis $\{X_1, \dots, X_m, Y_1, \dots, Y_m, Z\}$ of \mathfrak{h} satisfies the Heisenberg bracket relations $[X_i, Y_i] = Z$, with all other brackets of basis vectors being zero. In the modified algebra $\mathfrak{s}_{a'}$, we replace X_i by X'_i and Y_i by Y'_i with the new bracket relations $[X'_i, Y'_i] = \pm Z$ where the sign is positive when $i \leq a'$ and negative when $i > a'$. The subspace \mathfrak{h}' of $\mathfrak{s}_{a'}$ is still isomorphic to the Heisenberg algebra, although the particular basis $\{X'_1, \dots, X'_m, Y'_1, \dots, Y'_m, Z\}$ does not satisfy the canonical Heisenberg bracket relations due to the introduction of negative signs. Similar statements hold in $\mathfrak{s}_{a''}$.

The eigenspace $\mathfrak{n}_{\omega_2 - \omega_1}$ in \mathfrak{s} is one-dimensional and a choice of basis vector U satisfies $[U, X_i] = Y_i$ for all $i = 1, \dots, m$; i.e., $\text{ad}(U) : \mathfrak{n}_{\omega_1} \rightarrow \mathfrak{n}_{\omega_2}$ is the canonical isomorphism σ given by $\sigma(X_i) = Y_i$. Now suppose that $\tau : \mathfrak{s}_{a'} \rightarrow \mathfrak{s}_{a''}$ is an isometry of metric Lie algebras. By Proposition 4.4, τ induces an isomorphism of the root system Σ . However, as one can see by a direct computation or by referring to the Satake diagrams BI and DI on pages 532-533 of [H], the only isomorphism of Σ is the identity. It follows that τ restricts to the identity on \mathfrak{a} and carries each root space \mathfrak{n}'_{α} in \mathfrak{n}' to the corresponding root space \mathfrak{n}''_{α} in \mathfrak{n}'' . Applying this fact to $\alpha = \omega_1 - \omega_2$, we see that $\tau(U) = \pm U$ since τ is an isometry. Consequently, the restrictions

$\tau : \mathfrak{n}'_{\omega_1} \rightarrow \mathfrak{n}''_{\omega_1}$ and $\tau : \mathfrak{n}'_{\omega_2} \rightarrow \mathfrak{n}''_{\omega_2}$ satisfy $\tau \circ \sigma = \pm \sigma \circ \tau$. Finally, τ must restrict to an isomorphism between the Heisenberg algebras \mathfrak{h}' and \mathfrak{h}'' . It is easy to see that this is impossible unless $a'' = a'$ or $a'' = m - a'$.

4.10. $\mathbf{SO}(\mathbf{n}, \mathbf{H})/\mathbf{U}(\mathbf{n})$. Our next non-compact symmetric space is dual to $\mathbf{SO}(2n)/\mathbf{U}(n)$, the compact symmetric space of special orthogonal complex structures on \mathbf{R}^{2n} , with rank $m = \lfloor \frac{n}{2} \rfloor$. The dimension is $n^2 - n$. On the Lie algebra level,

$$\mathfrak{so}(n, \mathbf{H}) = \left\{ \begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix} \in \mathfrak{gl}(2n, \mathbf{C}) \mid X = -X^t, Y = \bar{Y}^t \right\}.$$

Let E_{ij} denote the skew-symmetric matrix with 1 in the ij^{th} entry and -1 in the ji^{th} entry, and zeros elsewhere. The Cartan subalgebra is

$$\mathfrak{a} = \text{span} \left\{ H_j = \frac{i}{\sqrt{2}}(E_{2j-1, 2j} - E_{n+2j-1, n+2j}) \mid 1 \leq j \leq m \right\}.$$

Define $\omega_j \in \mathfrak{a}^*$ by $\omega_j(H) = a_j$ for $H = i \sum_{j=1}^m a_j H_j$, an arbitrary element of \mathfrak{a} . If n is even ($m = \frac{n}{2}$), the positive roots of \mathfrak{a} are $\omega_j \pm \omega_k$ for $1 \leq j < k \leq m$, and $2\omega_j$ for $1 \leq j \leq m$. If n is odd ($m = \frac{n-1}{2}$), we also have the positive roots ω_j for $1 \leq j \leq m$. We describe the root spaces \mathfrak{n}_α :

$$\begin{aligned} \mathfrak{n}_{\omega_j \pm \omega_k} &= \text{span} \{ A_{jk}^\pm = \frac{1}{2}((E_{2j-1, 2k-1} \mp E_{2j, 2k} + E_{n+2j-1, n+2k-1} \mp E_{n+2j, n+2k}) \\ &\quad + i(\mp E_{2j-1, 2k} - E_{2j, 2k-1} \pm E_{n+2j-1, n+2k} + E_{n+2j, n+2k-1})), \\ B_{jk}^\pm &= \frac{1}{2}((E_{2j-1, 2k} \pm E_{2j, 2k-1} + E_{n+2j-1, n+2k} \pm E_{n+2j, n+2k-1}) \\ &\quad + i(\pm E_{2j-1, 2k-1} - E_{2j, 2k} \mp E_{n+2j-1, n+2k-1} + E_{n+2j, n+2k})), \\ C_{jk}^\pm &= \frac{1}{2}((E_{2j-1, n+2k} \mp E_{2j, n+2k-1} \mp E_{2k-1, n+2j} + E_{2k, n+2j-1}) \\ &\quad + i(\mp E_{2j-1, n+2k-1} - E_{2j, n+2k} \pm E_{2k-1, n+2j-1} + E_{2k, n+2j})), \\ D_{jk}^\pm &= \frac{1}{2}((E_{2j-1, n+2k-1} + E_{2k-1, n+2j-1} \pm E_{2j, n+2k} \pm E_{2k, n+2j}) \\ &\quad + i(\pm E_{2j-1, n+2k} - E_{2j, n+2k-1} + E_{2k-1, n+2j} \mp E_{2k, n+2j-1})); \\ \mathfrak{n}_{2\omega_k} &= \text{span} \{ G_k = \frac{1}{\sqrt{2}}((E_{2k-1, n+2k-1} + E_{2k, n+2k}) + i(E_{2k-1, n+2k} - E_{2k, n+2k-1})) \}; \\ (\text{If } n \text{ odd}) \quad \mathfrak{n}_{\omega_k} &= \text{span} \{ X_k = \frac{1}{\sqrt{2}}((E_{2k, n} + E_{n+2k, 2n}) + i(E_{2k-1, n} - E_{n+2k-1, 2n})), \\ &\quad Y_k = \frac{1}{\sqrt{2}}((E_{2k-1, n} + E_{n+2k-1, 2n}) - i(E_{2k, n} - E_{n+2k, 2n})), \\ &\quad Z_k = \frac{1}{\sqrt{2}}((E_{2k, 2n} + E_{n, n+2k}) + i(E_{2k-1, 2n} - E_{n, n+2k-1})), \\ &\quad W_k = \frac{1}{\sqrt{2}}((E_{2k-1, 2n} + E_{n, n+2k-1}) - i(E_{2k, 2n} - E_{n, n+2k})) \}. \end{aligned}$$

These bases of the root spaces together with the basis above of \mathfrak{a} give an orthonormal basis for the solvable subalgebra \mathfrak{s} of $\mathfrak{so}(n, \mathbf{H})$.

We first consider the case of n even. The non-zero bracket relations $[X, Y]$ in \mathfrak{n} are given in the following table:

$Y \setminus X$	A_{kl}^+	A_{kl}^-	B_{kl}^+	B_{kl}^-	C_{kl}^+	C_{kl}^-	D_{kl}^+	D_{kl}^-
A_{kl}^+	0	0	0	0	0	0	0	$\sqrt{2}G_k$
A_{jk}^-	A_{jl}^+	A_{jl}^-	B_{jl}^+	B_{jl}^-	C_{jl}^+	C_{jl}^-	D_{jl}^+	D_{jl}^-
B_{kl}^+	0	0	0	0	0	$\sqrt{2}G_k$	0	0
B_{jk}^-	$-B_{jl}^+$	B_{jl}^-	A_{jl}^+	$-A_{jl}^-$	$-D_{jl}^+$	D_{jl}^-	C_{jl}^+	$-C_{jl}^-$
C_{kl}^+	0	0	0	$-\sqrt{2}G_k$	0	0	0	0
C_{jk}^-	$-C_{jl}^+$	C_{jl}^-	D_{jl}^+	$-D_{jl}^-$	A_{jl}^+	$-A_{jl}^-$	$-B_{jl}^+$	B_{jl}^-
D_{kl}^+	0	$-\sqrt{2}G_k$	0	0	0	0	0	0
D_{jk}^-	D_{jl}^+	D_{jl}^-	C_{jl}^+	C_{jl}^-	$-B_{jl}^+$	$-B_{jl}^-$	$-A_{jl}^+$	$-A_{jl}^-$
G_l	0	$-D_{kl}^+$	0	$-C_{kl}^+$	0	B_{kl}^+	0	A_{kl}^+

We can modify \mathfrak{s} by replacing each of the basis elements B_{jk}^- , C_{jk}^- , A_{jk}^+ , D_{jk}^+ and G_k by their respective products with $\sqrt{-1}$, elements of $\mathfrak{s}^{\mathbf{C}}$, the complexified Lie algebra. With these changes, we obtain a new Lie algebra \mathfrak{s}' associated to \mathfrak{s} , and the corresponding simply-connected Riemannian solvmanifold S' is Einstein. S' has some positive sectional curvature: Let $X = \frac{1}{\sqrt{2}}((A_{j,j+1}^-)' + (B_{j,j+1}^-)')$ and let $Y = \frac{1}{\sqrt{2}}((C_{j+1,j+2}^-)' + (D_{j+1,j+2}^-)')$, so that $[X, Y] = 0$ and $U(X, Y) = 0$. Then $U(X, X) = \frac{1}{2}(H_j - H_{j+1})$ and $U(Y, Y) = \frac{1}{2}(H_{j+1} - H_{j+2})$. Hence $K(X, Y) = -\langle U(X, X), U(Y, Y) \rangle > 0$.

Next consider the case of n odd. In addition to the bracket relations above, we have these additional non-zero Lie bracket relations $[X, Y]$:

$Y \setminus X$	A_{kl}^-	B_{kl}^-	C_{kl}^-	D_{kl}^-	X_l	Y_l	Z_l	W_l
X_l	$-X_k$	$-Y_k$	$-W_k$	Z_k	0	0	$\sqrt{2}G_l$	0
Y_l	$-Y_k$	X_k	$-Z_k$	$-W_k$	0	0	0	$\sqrt{2}G_l$
Z_l	$-Z_k$	$-W_k$	Y_k	X_k	$-\sqrt{2}G_l$	0	0	0
W_l	$-W_k$	Z_k	X_k	Y_k	0	$-\sqrt{2}G_l$	0	0
X_j	0	0	0	0	A_{jl}^+	$-B_{jl}^+$	D_{jl}^+	$-C_{jl}^+$
Y_j	0	0	0	0	$-B_{jl}^+$	$-A_{jl}^+$	C_{jl}^+	D_{jl}^+
Z_j	0	0	0	0	$-D_{jl}^+$	C_{jl}^+	A_{jl}^+	$-B_{jl}^+$
W_j	0	0	0	0	$-C_{jl}^+$	$-D_{jl}^+$	$-B_{jl}^+$	$-B_{jl}^+$

In this case, we modify \mathfrak{s} by replacing basis elements X_k , Z_k , B_{jk}^+ , C_{jk}^+ , B_{jk}^- , C_{jk}^- by their respective products with $\sqrt{-1}$. With this basis, we obtain a new Lie algebra \mathfrak{s}' and thus an Einstein solvmanifold S' . Just as in the case of n even, S' has positively curved sections.

4.11. **SL(n, H)/Sp(n)**. Our final non-compact symmetric space has rank $n - 1$; it is the dual space to $SU(2n)/Sp(n)$, the compact symmetric space of special orthogonal quaternionic structures on \mathbf{C}^{2n} . This space has dimension $2n^2 - n - 1$. On the Lie algebra level,

$$\mathfrak{sl}(n, \mathbf{H}) = \left\{ \begin{pmatrix} X & -\bar{Y} \\ Y & X \end{pmatrix} \mid X, Y \in \mathfrak{gl}(n, \mathbf{C}), \quad \text{tr}(X + \bar{X}) = 0 \right\}.$$

We have

$$\mathfrak{a} = \{ \text{diag}(a_1, \dots, a_n, a_1, \dots, a_n) \in \mathfrak{gl}(2n, \mathbf{C}) \mid \sum_{i=1}^n a_i = 0 \}.$$

Let $H = \text{diag}(a_1, \dots, a_n, a_1, \dots, a_n)$ be an arbitrary element of \mathfrak{a} . Then define $\omega_j \in \mathfrak{a}^*$ by $\omega_j(H) = a_j$. The roots of \mathfrak{s} are all of the form $\omega_j - \omega_k$, where $1 \leq j < k \leq n$. Let F_{ij} denote the matrix with a 1 in the ij^{th} entry and zeros elsewhere. We describe the root spaces of \mathfrak{n} :

$$\mathfrak{n}_{\omega_j - \omega_k} = \{ A_{jk} = i\sqrt{2}(F_{jk} - F_{n+j, n+k}), B_{jk} = i\sqrt{2}(F_{j, n+k} + F_{n+j, k}), \}$$

$$C_{jk} = \sqrt{2}(F_{j,n+k} - F_{n+j,k}), \quad D_{jk} = \sqrt{2}(F_{jk} + F_{n+j,n+k}).$$

We can represent the Lie brackets $[X, Y]$ in \mathfrak{n} by the following table (notice that although we are shifting root spaces, one can think of the multiplication of A, B, C as quaternionic):

$Y \setminus X$	A_{ij}	B_{ij}	C_{ij}	D_{ij}
A_{jk}	$-\sqrt{2}D_{ik}$	$\sqrt{2}C_{ik}$	$-\sqrt{2}B_{ik}$	$\sqrt{2}A_{ik}$
B_{jk}	$-\sqrt{2}C_{ik}$	$-\sqrt{2}D_{ik}$	$\sqrt{2}A_{ik}$	$\sqrt{2}B_{ik}$
C_{jk}	$\sqrt{2}B_{ik}$	$-\sqrt{2}A_{ik}$	$-\sqrt{2}D_{ik}$	$\sqrt{2}C_{ik}$
D_{jk}	$\sqrt{2}A_{ik}$	$\sqrt{2}B_{ik}$	$\sqrt{2}C_{ik}$	$\sqrt{2}D_{ik}$

We modify the Lie algebra in the following way: For all $1 \leq j < k \leq n$, replace A_{jk} by $\sqrt{-1}A_{jk}$ and C_{jk} by $\sqrt{-1}C_{jk}$. This gives a new Lie algebra \mathfrak{s}' . The corresponding Einstein solvmanifold S' has some positive sectional curvature. Let $H_{ij} = X_{ii} - X_{jj} + X_{n+i,n+i} - X_{n+j,n+j}$. Let $X = \frac{1}{\sqrt{2}}(A'_{ij} + B'_{ij})$ and $Y = \frac{1}{\sqrt{2}}(C'_{jk} + D'_{jk})$. Then $[X, Y] = 0$, $U(X, Y) = 0$, and $U(X, X) = H_{ij}$, $U(Y, Y) = H_{jk}$ so that $K(X, Y) = -\langle U(X, X), U(Y, Y) \rangle = -\langle H_{ij}, H_{jk} \rangle > 0$.

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