Putnam Seminar, Week 2: Generating Functions, Part I

Brian Munson

Disclosure: This document has been heavily influenced by, and borrowed heavily from, those prepared by Ravi Vakil of Stanford. Many of the problems are the same. The same can be said of the general format of the seminar as well as the list of topics.
Instructions: Work on problems you don't already know how to solve! Have fun, and work together! Eat snacks! And remember: I don't necessarily know how to solve these problems myself!

Generating Functions

This is a topic I really can't do justice to in one session. It's likely we'll come back to it at some point, so I'm optimistically calling this "Part 1". First, I'd like to point you to a wonderful and very readable reference, a book titled "Generatingfunctionology", written by Herbert Wilf.

Not only is it a great book, but it's available as a download at his website: http://www.math.upenn.edu/ wilf/DownldGF.html (it's a clickable link in this .pdf, so you really have no excuse not to own this book).

OK, onto the idea. Suppose you had a sequence of numbers $a_0, a_1, a_2, \dots$ you wanted to know about. Maybe you want to know its limit, or an explicit formula for the n^{th} term. The idea is to make a power series $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ whose coefficients are the sequence in question. For instance, if you have the constant sequence $1 = a_0 = a_1 = a_2 = \cdots$, its generating function is $f(x) = 1 + x + x^2 + \cdots$, which equals $\frac{1}{1-x}$, at least as long as $|x| < 1$. This is an important formula which you will (ab)use often when employing generating functions. This latter point, the question of convergence of the series, can be bothersome, but the good news is that you don't need to worry about it. One way this can be used (and I will show you how) is if you have a recursive formula for your sequence and you want an explicit formula. For instance, the Fibonacci numbers are defined by $f_0 = f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$. You can use generating functions to help you find an explicit formula for f_n for all n. The problems:

1. Find an explicit formula for the f_n, where f_n is the n^{th} Fibonacci number, defined by $f_0 = f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$.

2. Show that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

3. Find a formula for $\sum_{i=1}^{n} i\binom{n}{i}$.

4. (Vandermonde Identity) Prove that for any positive integers $k < m, n$,

$$\sum_{j=0}^{k} \binom{n}{j} \binom{m}{n-j} = \binom{n+m}{k}.$$

5. (Putnam 1992) For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j}.$$

(Zeitz, 4.3.13)

6. The function $(1-x-x^2-x^3-x^4-x^5-x^6)^{-1}$ is the generating function of what sequence? (Zeitz, 4.3.20).

7. Show that $\sum_{i=0}^{s} \binom{r+i}{r} \binom{t-r-i}{t-r-s} = \binom{t+1}{t-s+1}$.

8. Find $\frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \cdots$.

9. Find $\frac{1}{1} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{11} - \cdots$.

10. Suppose that in base p, $n = n_0 + n_1 p + \cdots + n_k p^k$, and $a = a_0 + a_1 p + \cdots + a_k p^k$. Show that

$$\binom{n}{a} = \prod_{i=1}^{k} \binom{n_i}{a_i} (\text{ mod } p.$$

11. (Putnam B4, 2005) For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \le m$. Show that $f(m,n) = f(n,m)$.