

Hamiltonization and geometric integration of nonholonomic mechanical systems

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Abstract— In this paper we study a Hamiltonization procedure for mechanical systems with velocity-dependent (nonholonomic) constraints. We first rewrite the nonholonomic equations of motion as Euler-Lagrange equations, with a Lagrangian that follows from rephrasing the issue in terms of the inverse problem of Lagrangian mechanics. Second, the Legendre transformation transforms the Lagrangian in the sought-for Hamiltonian. As an application, we compare some variational integrators for the new Lagrangians with some known nonholonomic integrators.

Keywords— nonholonomic systems, Lagrangian and Hamiltonian formalism, inverse problem, geometric integration.

I. INTRODUCTION

MANY interesting mechanical systems are subject to additional velocity-dependent (i.e. nonholonomic) constraints. Typical engineering problems that involve such constraints arise for example in robotics, where the wheels of a mobile robot are often required to roll without slipping, or where one is interested in guiding the motion of a cutting tool.

The direct motivation for our paper [2] was to be found in interesting results that appeared in [3], where the authors propose a way to quantize some of the well-known classical examples of nonholonomic systems. On the way to quantization, the authors propose an alternative Hamiltonian representation for those nonholonomic systems. However, the “Hamiltonization” method introduced in [3] can only be applied to systems for which the solutions are already known explicitly.

Nonholonomic systems have a more natural description in the Lagrangian framework. In [2], we explained how one can associate to the nonholonomic equations of motion a family of systems of second-order ordinary differential equations and we applied the conditions of the inverse problem of the calculus of variations on those associated systems to search for the existence of a regular Lagrangian. (The inverse problem of the calculus of variations deals with the question of whether or not a given system of second-order differential equations is equivalent with the Euler-Lagrange equations of a yet to be determined regular Lagrangian, see e.g. [12]). If such an unconstrained regular Lagrangian exists for one of the associated systems, we can always find an associated Hamiltonian by means of the Legendre transformation. Since our method only made use of the equations of motion of the system it did not depend on the knowledge of its explicit solutions.

A system for which no exact solutions are known can only be integrated by means of numerical methods. In addition to the above mentioned application to quantization, our Hamiltonization method may also be useful from this point of view. Numerical integrators that preserve the underlying geometric structure of a system are called geometric integrators. A geometric integrator of a Lagrangian system uses a discrete Lagrangian that resembles as much as possible the continuous Lagrangian (see e.g. [11]). On the other hand, the success of a so-called nonholomic

integrator (see e.g. [4], [7]) relies not only on the choice of a discrete Lagrangian but also on the choice of a discrete version of the constraint manifold. It seems therefore reasonable that if a free Lagrangian for the nonholonomic system exists, the Lagrangian integrator may perform better than a nonholonomic integrator with badly chosen discrete constraints.

In the next section we recall the set-up and the main results of our paper [2]. In section III we compare some nonholonomic and variational geometric integrators for a few of the classical nonholonomic systems. In section IV, we indicate some ideas on how we wish to extend the results of this paper.

II. A CLASS OF NONHOLONOMIC SYSTEMS

We will consider only a certain class of nonholonomic systems on \mathbb{R}^n : We will assume that the configuration space of the system is a space with coordinates (r_1, r_2, s_α) , that the Lagrangian of the system is given by the function

$$L = \frac{1}{2}(I_1 \dot{r}_1^2 + I_2 \dot{r}_2^2 + \sum_{\alpha} I_{\alpha} \dot{s}_{\alpha}^2) \quad (1)$$

and that the nonholonomic constraints are all of the form

$$\dot{s}_{\alpha} = -A_{\alpha}(r_1) \dot{r}_2. \quad (2)$$

The nonholonomic equations of motion follow from d’Alembert’s principle (see e.g. [1]). For systems in our class they are given by the equations

$$\begin{cases} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}_1} \right) - \frac{\partial L}{\partial r_1} = 0, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}_2} \right) - \frac{\partial L}{\partial r_2} = \lambda_{\alpha} A_{\alpha}, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{s}_{\alpha}} \right) - \frac{\partial L}{\partial s_{\alpha}} = \lambda_{\alpha}, \end{cases}$$

together with the constraint equations (2). After eliminating the Lagrange multipliers by means of the constraints, one gets

$$\begin{cases} \ddot{r}_1 = 0, \\ \ddot{r}_2 = -N^2 K \dot{r}_1 \dot{r}_2, \\ \dot{s}_{\alpha} = -A_{\alpha} \dot{r}_2, \end{cases} \quad (3)$$

where $N(r_1) = (I_2 + \sum_{\alpha} I_{\alpha} A_{\alpha}^2)^{-\frac{1}{2}}$ is related to the invariant measure of the system and $K = \sum_{\beta} I_{\beta} A_{\beta} A'_{\beta}$, with $A'_{\beta} = \partial_{r_1} A_{\beta}$.

Some basic examples of nonholonomic systems that lie in this class are the following ones. The classic example of a nonholonomically constrained free particle has a Lagrangian and constraint given by

$$L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \text{and} \quad \dot{z} + xy = 0.$$

A knife edge on a horizontal plane corresponds physically to a blade with mass m moving in the xy plane at an angle ϕ to the x -axis. Its Lagrangian and constraint are given by

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\phi}^2 \quad \text{and} \quad x \sin(\phi) - y \cos(\phi) = 0.$$

Also the vertically rolling disk is an example in our class. The assumption that the disk rolls without slipping over the plane gives rise to nonholonomic constraints. Let R be the radius of the disk. If the triple $(x, y, z = R)$ stands for the coordinates of its centre of mass, φ for its angle with the (x, z) -plane and θ for the angle of a fixed line on the disk and a vertical line, then the nonholomic constraints are of the form

$$\dot{x} = R \cos \varphi \dot{\theta} \quad \text{and} \quad \dot{y} = R \sin \varphi \dot{\theta}.$$

The Lagrangian of the disk is

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2,$$

where $I = \frac{1}{2}MR^2$ and $J = \frac{1}{4}MR^2$ are the moments of inertia and M is the total mass of the disk. For the vertically rolling disk N is a constant and $K = 0$.

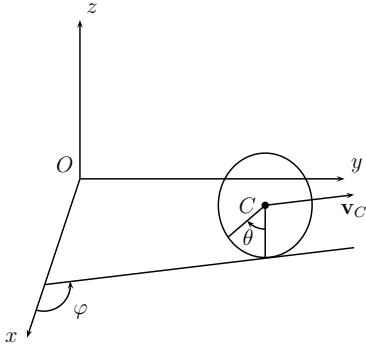


Fig. 1. The vertically rolling disk

Finally, also the examples of the mobile robot with fixed orientation and the two-wheeled carriage (see e.g. [8]) lie within our class.

The equations of motion (3) are a mixed set of first- and second-order differential equations. On the other hand, the Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial \dot{q}^i} \right) - \frac{\partial \tilde{L}}{\partial q^i} = 0$$

of a regular Lagrangian \tilde{L} are second-order differential equations (only) [The tilde in \tilde{L} will always denote that the Lagrangian is free, and that it should not be confused with the original Lagrangian L of the nonholonomic system.]. We therefore need a way to associate a second-order system to our nonholonomic system. One possible choice of doing so is the system

$$\begin{cases} \dot{r}_1 = 0, \\ \dot{r}_2 = -N^2 K \dot{r}_1 \dot{r}_2, \\ \dot{s}_\alpha = - \left(A'_\alpha - N^2 K A_\alpha \right) \dot{r}_1 \dot{r}_2. \end{cases} \quad (4)$$

The above second-order system has the property that its solution set contains, among other, also the solutions of the nonholonomic dynamics (3) when restricted to the constraints. Another choice for an ‘associated system’ with the same property is e.g.

$$\begin{cases} \dot{r}_1 = 0, \\ \dot{r}_2 = -N^2 K \dot{r}_1 \dot{r}_2, \\ \dot{s}_\alpha = (A'_\alpha - N^2 K A_\alpha) \dot{r}_1 \left(\frac{\dot{s}_\alpha}{A_\alpha} \right) \end{cases} \quad (5)$$

(no sum over α). It is clear, that there are in fact an infinite number of such associated second-order systems, but we will concentrate in this paper on the above two. For some other possible choices, see [2].

Proposition 1: 1. There does not exist a regular Lagrangian whose Euler-Lagrange equations are equivalent with the second-order system (4) (for the classical examples cited above).

2. The Euler-Lagrange equations of the Lagrangian

$$\tilde{L} = \frac{1}{2}I_1 \dot{r}_1^2 + \frac{1}{2N} \left(C_2 \frac{\dot{r}_2^2}{\dot{r}_1} + \sum_{\beta} C_{\beta} \frac{\dot{s}_{\beta}^2}{A_{\beta} \dot{r}_1} \right) \quad (C_{\alpha} \neq 0) \quad (6)$$

are equivalent with the second-order system (5). If the invariant measure density N is a constant, then also

$$\tilde{L} = \frac{1}{2}I_1 \dot{r}_1^2 + \frac{1}{2}I_2 \dot{r}_2^2 + \frac{1}{2N} \sum_{\beta} a_{\beta} \frac{\dot{s}_{\beta}^2}{A_{\beta} \dot{r}_1} \quad (C_{\alpha} \neq 0) \quad (7)$$

is a regular Lagrangian for the system (5).

Proof: We give here only an outline of the method we’ve used to prove the statements. For full details, see [2]. Assume we are given a system of second-order ordinary differential equations

$$\ddot{q}^i = f^i(q, \dot{q}).$$

The search for a regular Lagrangian is known in the literature as ‘the inverse problem of the calculus of variations’, and has a long history (for a recent survey on this history, see e.g. [10] and the long list of references therein). In order for a regular Lagrangian $\tilde{L}(q, \dot{q})$ to exist we must be able to find functions $g_{ij}(q, \dot{q})$, so-called multipliers, such that

$$g_{ij}(\ddot{q}^j - f^j) = \frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial \dot{q}^i} \right) - \frac{\partial \tilde{L}}{\partial q^i}.$$

It can be shown [6], [12] that the multipliers must satisfy

$$\begin{aligned} \det(g_{ij}) &\neq 0, & g_{ji} &= g_{ij}, & \frac{\partial g_{ij}}{\partial \dot{q}^k} &= \frac{\partial g_{ik}}{\partial \dot{q}^j}, \\ \Gamma(g_{ij}) - \nabla_j^k g_{ik} - \nabla_i^k g_{kj} &= 0, \\ g_{ik} \Phi_j^k &= g_{jk} \Phi_i^k; \end{aligned}$$

where

$$\nabla_j^i = -\frac{1}{2} \partial_{\dot{q}^j} f^i$$

and

$$\Phi_j^k = \Gamma \left(\partial_{\dot{q}^j} f^k \right) - 2 \partial_{\dot{q}^i} f^k - \frac{1}{2} \partial_{\dot{q}^j} f^l \partial_{\dot{q}^i} f^k.$$

The symbol Γ stands for the vector field $\dot{q}^i \partial_{q^i} + f^i \partial_{\dot{q}^i}$ that can naturally be associated to the system $\ddot{q}^i = f^i(q, \dot{q})$. Conversely, if one can find functions g_{ij} satisfying these conditions then the equations $\ddot{q}^i = f^i$ are derivable from a regular Lagrangian. Moreover, if a regular Lagrangian \tilde{L} can be found, then its Hessian $\frac{\partial^2 \tilde{L}}{\partial \dot{q}^i \partial \dot{q}^j}$ is a multiplier.

The above conditions are generally referred to as the Helmholtz conditions. They are a mixed set of coupled algebraic and PDE conditions in (g_{ij}) . We will refer to the penultimate condition as the ‘ ∇ -condition,’ and to the last one as the ‘ Φ -condition.’ The algebraic Φ -conditions are of course the most interesting to start from. In fact, we can easily derive more algebraic conditions (see e.g. [5]). For example, by taking a Γ -derivative of the Φ -condition, and by replacing $\Gamma(g_{ij})$ everywhere by means of the ∇ -condition, we arrive at a new algebraic condition of the form

$$g_{ik}(\nabla\Phi)_j^k = g_{jk}(\nabla\Phi)_i^k,$$

where

$$(\nabla\Phi)_j^i = \Gamma(\Phi_j^i) - \nabla_m^i \Phi_j^m - \nabla_j^m \Phi_m^i.$$

This $(\nabla\Phi)$ -condition will, of course, only give new information as long as it is independent from the Φ -condition (this will not be the case, for example, if the commutator of matrices $[\Phi, \nabla\Phi]$ vanishes). One can repeat the above process on the $(\nabla\Phi)$ -condition, and so on to obtain possibly independent $(\nabla \dots \nabla\Phi)$ -conditions. A second route to additional algebraic conditions arises from the derivatives of the Φ -equation in \dot{q} -directions. One can sum up those derived relations in such a way that the terms in $\partial_{\dot{q}^k} g_{ij}$ disappear on account of the symmetry in all their indices. The new algebraic relation in g_{ij} is then of the form

$$g_{ij} R_{kl}^j + g_{lj} R_{ik}^j + g_{kj} R_{li}^j = 0,$$

where

$$R_{kl}^j = \partial_{\dot{q}^j}(\Phi_k^l) - \partial_{\dot{q}^k}(\Phi_l^j).$$

As before, this process can be continued to obtain more algebraic conditions. Also, any mixture of the above mentioned two processes leads to possibly new and independent algebraic conditions. Once we have used up all the information that we can obtain from this infinite series of algebraic conditions, we can start looking at the partial differential equations in the ∇ -conditions.

Let us now come back to the second-order systems (4) and (5) at hand. The proof of the proposition relies on the fact that for the first systems (4), the only matrices (g_{ij}) that satisfy the first few algebraic conditions must be non-singular. On the other hand, the two Lagrangians for the system (5) follow from an analysis of the Helmholtz conditions with carefully chosen ansatzes. For more details, see [2]. ■

Remark that the Lagrangians are not defined for $\dot{r}_1 = 0$, and we will in general exclude the solutions with that property from the further considerations in this paper. Any regular Lagrangian system with Lagrangian \tilde{L} can be transformed into a Hamiltonian one, by making use of the Legendre transformation

$$(q^i, \dot{q}^i) \mapsto (q^i, p_i = \frac{\partial \tilde{L}}{\partial \dot{q}^i}).$$

The corresponding Hamiltonian is then

$$\tilde{H} = p_i \dot{q}^i - \tilde{L}.$$

Similarly, the Legendre transformation maps the constraints, viewed as a submanifold of the tangent manifold, onto a submanifold in the cotangent manifold.

Proposition 2: Using the Legendre transformation, the Hamiltonian that corresponds to the Lagrangian (6) is given by

$$\tilde{H} = \frac{1}{2I_1} \left(p_1 + \frac{1}{2} N \left(\frac{p_2^2}{C_2} + \sum_{\beta} A_{\beta} \frac{p_{\beta}^2}{C_{\beta}} \right) \right)^2.$$

The corresponding constraints are

$$C_2 p_{\alpha} = -C_{\alpha} p_2.$$

If N is constant, the Hamiltonian that corresponds to the Lagrangian (7) is

$$\tilde{H} = \frac{1}{2I_2} p_2^2 + \frac{1}{2I_1} \left(p_1 + \frac{1}{2} N \left(\sum_{\beta} \frac{A_{\beta}}{a_{\beta}} p_{\beta}^2 \right) \right)^2,$$

and the constraints transform into

$$I_2 N \dot{r}_1 p_{\alpha} + a_{\alpha} p_2 = 0,$$

where $\dot{r}_1 = (p_1 + \frac{1}{2} N \sum_{\alpha} A_{\alpha} p_{\alpha}^2 / a_{\alpha}) / I_1$.

In [2], [9] we explain how the above Hamiltonians can be directly derived from Pontryagin’s Maximum principle.

III. GEOMETRIC INTEGRATORS

A. Set-up

As we explained in the introduction, there are now two ways to compute a numeric approximation of a solution of a system in our class: we can use either a nonholonomic integrator for the original Lagrangian (1) and constraints (2), or we can use a variational integrator for one of the Lagrangians (6) and (7) we have found in Proposition 1. Let us come to some details.

Geometric integrators are integrators that preserve the underlying structure of the system. In particular, variational integrators are integrators that are derived from a discrete version of Hamilton’s principle. From this discrete variational principle one obtains the so-called discrete Euler-Lagrange equations as follows. For a mechanical system with Lagrangian L , one needs to choose a discrete Lagrangian $L_d(q_1, q_2)$ (a function on $Q \times Q$ which resembles as close as possible the continuous Lagrangian). A solution $q(t)$ is then discretised by an array q_k which are the solutions of the so-called discrete Euler-Lagrange equations

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0. \quad (8)$$

These integrators preserve the symplectic and conservative nature of the algorithms. It is important to realize that a different choice for the discrete Lagrangian may lead to a different geometric integrator. The presence of additional holonomic constraints (i.e. ‘integrable’ nonholonomic constraints) can be included by introducing Lagrange’s multipliers.

On the other hand, for a nonholonomic integrator of a non-holonomic system with Lagrangian L and constraints $\omega^a(q) \dot{q}^a =$

0, we need to choose both a discrete Lagrangian L_d and discrete constraint functions ω_d^a on $Q \times Q$. The nonholonomic discrete equations are then

$$\begin{cases} D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = (\lambda_k)_a \omega^a(q_k), \\ \omega_d^a(q_k, q_{k+1}) = 0. \end{cases} \quad (9)$$

Usually, if Q is a vector space, one takes the discretization in one of the following ways (for certain α and certain h):

$$L_d(q_1, q_2) = L\left(q = (1 - \alpha)q_1 + \alpha q_2, \dot{q} = \frac{q_2 - q_1}{h}\right) \quad (10)$$

$$\omega_d^a(q_1, q_2) = \omega_i^a\left(q = (1 - \alpha)q_1 + \alpha q_2\right) \frac{q_2^i - q_1^i}{h}. \quad (11)$$

For the rest of the paper, we will concentrate on this discretization procedure. There are, however, many more possibilities to obtain a discrete Lagrangian and discrete constraints. For example, one could take a symmetrized version of the above procedure and use discrete Lagrangians and discrete constraints of the form

$$L_d(q_1, q_2) = \frac{1}{2}L\left(q = (1 - \alpha)q_1 + \alpha q_2, \dot{q} = \frac{q_2 - q_1}{h}\right) + \frac{1}{2}L\left(q = \alpha q_1 + (1 - \alpha)q_2, \dot{q} = \frac{q_2 - q_1}{h}\right),$$

$$\omega_d^a(q_1, q_2) = \frac{1}{2}\omega_i^a\left(q = (1 - \alpha)q_1 + \alpha q_2\right) \frac{q_2^i - q_1^i}{h} + \frac{1}{2}\omega_i^a\left(q = \alpha q_1 + (1 - \alpha)q_2\right) \frac{q_2^i - q_1^i}{h}.$$

Also, if the system is invariant under a symmetry group, it is advantageous to construct the integrator in such a way that the discrete system inherits as many as possible of those symmetry properties, see e.g. [4].

The bottom line of the next sections is the following one. If a free Lagrangian for the nonholonomic system exists, it seems reasonable that the Lagrangian integrator may perform better than a nonholonomic integrator with badly chosen discrete constraints. In the next sections, we will test this conjecture on a few of the classical examples in our class: the vertically rolling disk, the knife edge and the nonholonomic particle. It will be convenient that for those systems an exact solution of the nonholonomic equations (3) is readily available.

B. The vertically rolling disk

For the vertically rolling disk, we have $(r_1, r_2, s_\alpha) = (\varphi, \theta, x, y)$. It is well-known that the solutions of the nonholonomic equations with initial conditions $u_\varphi = \dot{\varphi}(0) \neq 0$ and $u_\theta = \dot{\theta}(0)$ are all circles with radius $R(u_\theta/u_\varphi)$:

$$\begin{aligned} \theta(t) &= u_\theta t + \theta_0, & \varphi(t) &= u_\varphi t + \varphi_0, \\ x(t) &= \left(\frac{u_\theta}{u_\varphi}\right) R \sin(\varphi(t)) + x_0, \\ y(t) &= -\left(\frac{u_\theta}{u_\varphi}\right) R \cos(\varphi(t)) + y_0. \end{aligned} \quad (12)$$

Let us put for convenience $M = 1$ and $R = 1$ and therefore $I = \frac{1}{2}$ and $J = \frac{1}{4}$. With that the (nonholonomic) Lagrangian and constraints are simply

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{4}\dot{\theta}^2 + \frac{1}{8}\dot{\varphi}^2, \quad \dot{x} = \cos \varphi \dot{\theta}, \quad \dot{y} = \sin \varphi \dot{\theta}.$$

We will first compute the solution of the discrete nonholonomic equations (9) with the discrete Lagrangian (10) and the discrete constraints (11). Second, since the vertically rolling disk is one of those examples with a constant invariant measure density N , we can choose a Lagrangian from the second type (7). The simplest choice is probably

$$\tilde{L} = 1/2 \left(\dot{\varphi}^2 + \dot{\theta}^2 + \frac{\dot{x}^2}{\cos(\varphi)\dot{\varphi}} + \frac{\dot{y}^2}{\sin(\varphi)\dot{\varphi}} \right). \quad (13)$$

We now investigate the variational integrator of this Lagrangian, where the discrete Lagrangian is given by (10). We will fix h (changing it did not have a significant effect) and only concentrate on what happens if we keep α variable. In figure 2 we have

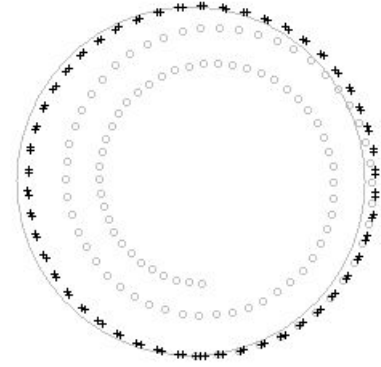


Fig. 2. Vertically rolling disk with $\alpha = 0$.

plotted the situation for $\alpha = 0$. For a given set of initial positions $(x_0, y_0, \theta_0, \varphi_0, \theta_1, \varphi_1)$ the other initial conditions were chosen in such a way that the solution lies initially on the discrete constraint manifold, i.e. in such a way that

$$x_1 = x_0 + \cos \varphi_0 (\theta_1 - \theta_0), \quad y_1 = y_0 + \sin \varphi_0 (\theta_1 - \theta_0).$$

Unlike the nonholonomic integrator (in grey with circle symbols) the variational integrator (in black with cross symbols) does not show a strong spiral-type solution but a circular path. It is true, however, that the variational solution deviates from the circle predicted by the initial conditions of the solution (12) (in grey in figure 2). However, since any circle is determined by 3 of its points, we can find a better match for the circle the variational discrete solution follows by considering the outcome (x_{k_i}, y_{k_i}) at three different times and by solving the three equations

$$(x_{k_i} - A)^2 + (x_{k_i} - B)^2 = C^2$$

for (A, B, C) . If we do so, we obtain the matching circle (in dots) in figure 3.

It is well-known that the energy

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{4}\dot{\theta}^2 + \frac{1}{8}\dot{\varphi}^2,$$

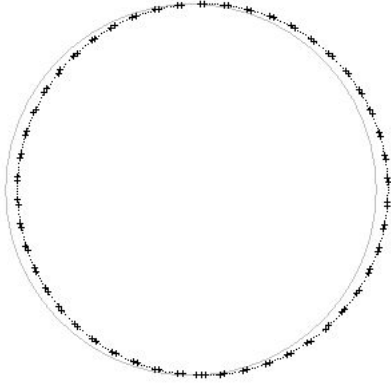


Fig. 3. Vertically rolling disk: circular path.

is conserved along the solutions (12) of the nonholonomic equations of motion. In figure 4 we investigate the performance of the two integrators on the energy function. The discrete version of the energy is the function we get by substituting, as usual, $1/h(q_k - q_{k-1})$ for \dot{q} in the function E above. The straight line in figure 4 is the energy level predicted by the initial conditions. It is clear that the variational integrator (with crosses) does a better job than the nonholonomic one (with circles).

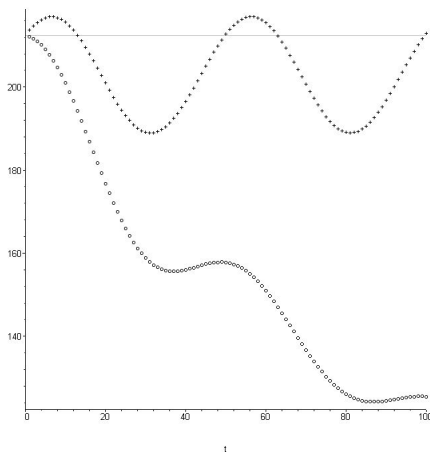


Fig. 4. Vertically rolling disk: the energy.

By construction the nonholonomic integrator conserves the constraints and the variational integrator does not. Indeed, in figure 6 we have plotted the constraint $\dot{x} - R \cos(\varphi) \dot{\theta} = 0$. Positive is that, although the variational integrator does not conserve this constraint, it reasonably oscillates around the zero level. Moreover, there is a method to fix this problem. We can introduce a ‘modified’ variational integrator which does conserve the constraints. This integrator considers the constraints as a constant along the (nonholonomic) motion. That is, it will use the variational discrete Lagrange equations (8) for the variables θ and φ (for the free Lagrangian \tilde{L} given in (13)), but not the corresponding equations for x and y . To get a full system of equations, we supplemented this with the discrete constraints $\omega_d^\alpha(q_k, q_{k+1}) = 0$ which can be written in terms of x_{k+1} and y_{k+1} . Figure 5 shows the modified integrator for $\alpha = 0$ (with box symbols). The circle in that figure is the one we had before, i.e. the one that matches the variational integrator. It shows that the modified integrator

has the same circular behaviour as the variational integrator, and on top, it keeps the constraints conserved, see the box symbols on the zero level in figure 6.

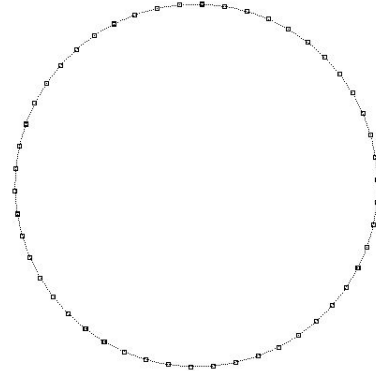


Fig. 5. Vertically rolling disk: the modified integrator.

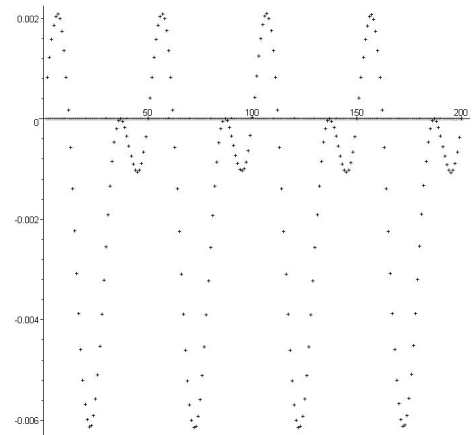


Fig. 6. Vertically rolling disk: the constraints.

Finally, figure 7 shows the effect of changing the parameter α . The results for the variational integrator (in black with cross symbols) remain accurate and more or less unchanged. For the nonholonomic integrator (in grey with circle symbols) the effect of changing α is that the inward spiral becomes an outward spiral. At some point (here $\alpha = 1/2$) the variational and nonholonomic integrator have the same accuracy.

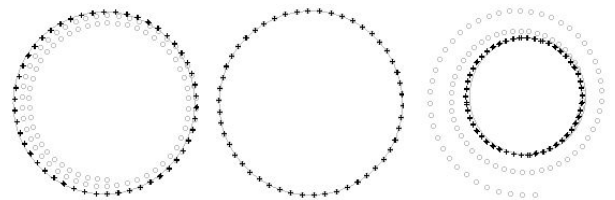


Fig. 7. Vertically rolling disk with $\alpha = 1/3, 1/2, 1$, respectively.

C. The knife edge

As was the case with the vertically rolling disk, also the solutions of the knife edge form a circular path in the (x, y) -plane.

Continuing the analogue with the previous example, the non-holonomic integration (in grey with circle symbols) results in a spiral, while the variational integration (in black with crosses) follows more closely the circular path, see figure 8.

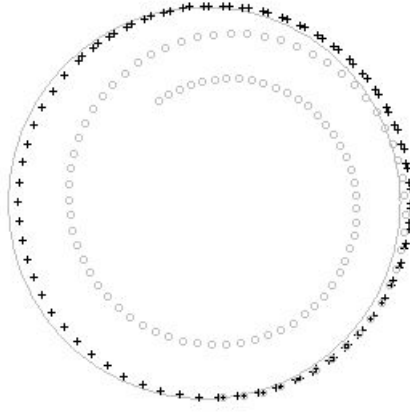


Fig. 8. The knife edge with $\alpha = 0$

D. The nonholonomic particle

The function

$$\tilde{L} = \frac{1}{2}x^2 + \frac{\sqrt{1+x^2}}{2} \left(\frac{\dot{y}^2}{\dot{x}} + \frac{\dot{z}^2}{x\dot{x}} \right)$$

is a free Lagrangian for the nonholonomic particle. In each of the figures 9 and 10 the dashed black curve represents the exact solution, the thick black the variational solution and the thick grey the nonholonomic solution. The figures show that both the variational method and the nonholonomic one do not give very accurate solutions. However, changing the parameter α does not seem to affect the variational solution as much as it does the nonholonomic one. Indeed, the variational solution remains more or less of the same accuracy for the different α -values. On the other hand, the nonholonomic solution can be made more or less accurate by changing α . It seems that the best accuracy is reached somewhere in the neighbourhood of $\alpha = 1/3$, but how could one have guessed this beforehand? Remark also that this value is not same as the the best choice we had found for the nonholonomic integrator of the vertically rolling disk (where $\alpha = 1/2$ gave the best accuracy).

E. Preliminary conclusion

In each of the discussed examples the variational integrator (with one of the Lagrangians (6) and (7) of proposition 1) seemed to give better results than the known nonholonomic integrators. Unlike the outcome for the nonholonomic integrator, the results for the variational integrator seemed to be independent of or, at least, stable under changing the parameter α . Needless to say, the results above are, of course, very partial and are they are only intended to motivate further investigation on this topic. For example, we need to check if more involved discretization procedures, such as the ones mentioned at the end of section IIIA demonstrate the same behaviour as the one we have encountered so far.

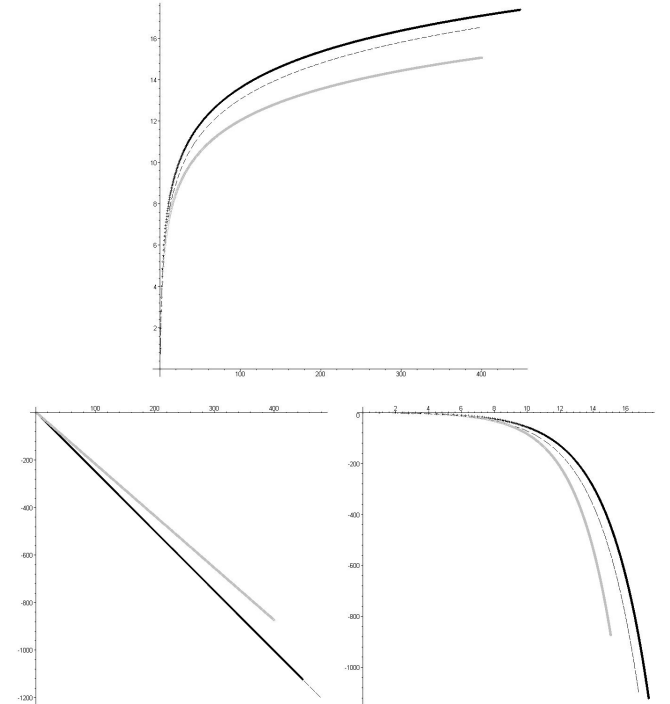


Fig. 9. The nonholonomic particle: xy -, xz - and yz -solution with $\alpha = 0$.

IV. FURTHER SYSTEMS

The class of nonholonomic systems treated above is very restricted. The reason is, of course, that the search for a solution of the inverse problem of the calculus of variations (in the proof of proposition 1) is too hard and too technical to be treated in the full generality of a nonholonomic systems with an arbitrary given Lagrangian and arbitrary given constraints. Also, since there are infinitely many possible choices for the associated systems, it is not clear from the outset which one of them will be variational, if any.

For these reasons, future extensions of the obtained results will strongly depend on well-chosen particular new examples. For example, we could try to find a free Lagrangian for a non-holonomic system with a potential of the form $V(r_2)$. Typical examples of such systems are the mobile robot with a fixed orientation

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{3}{2}J\dot{\psi}^2 - 10\sin\psi,$$

$$\dot{x} = R\cos\theta\dot{\psi}, \quad \dot{y} = R\sin\theta\dot{\psi},$$

(an example that also appears in the paper [4]) or the knife edge on an inclined plane, where

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\phi}^2 + mgx\sin\alpha, \quad \dot{x}\sin\phi = \dot{y}\cos\phi.$$

In more general terms, such systems have a Lagrangian of the form

$$L = \frac{1}{2}(I_1\dot{r}_1^2 + I_2\dot{r}_2^2 + I_3\dot{s}^2) - V(r_2)$$

and a constraint of the form

$$\dot{s} = -A(r_1)\dot{r}_2,$$

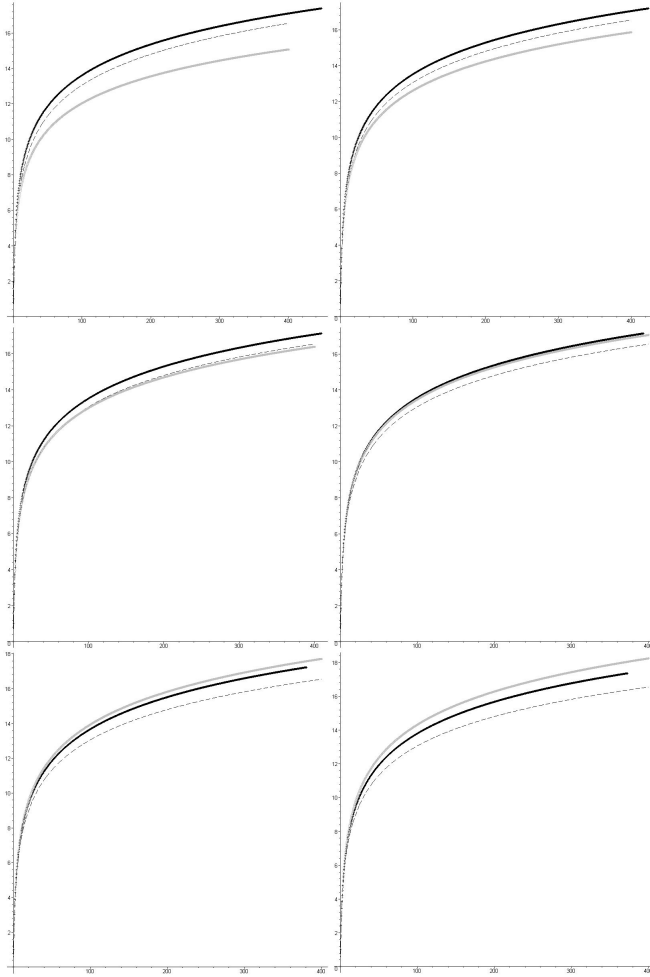


Fig. 10. The nonholonomic particle: xy -solution with $\alpha = 0, 1/5, 1/3, 1/2, 2/3, 4/5$, respectively.

and we can consider associated second-order equations, in a way that is analogous to the way we arrived at the second system (5) before: They are now of the form

$$\begin{cases} \ddot{r}_1 = 0, \\ \ddot{r}_2 = \Gamma_2(r_1)\dot{r}_1\dot{r}_2 + t_2(r_1, r_2), \\ \ddot{s} = \Gamma_3(r_1)\dot{r}_1\dot{s} + t_3(r_1, r_2). \end{cases} \quad (14)$$

Remark that compared to the equations (5), the presence of the extra potential brings the terms $t_i(r_1, r_2)$ into the picture. A first result is the following.

Proposition 3: There does not exist a regular Lagrangian for the second order systems (14).

Proof: As before, the proof follows from a careful analysis of the algebraic conditions which can be derived from the Helmholtz conditions. ■

For systems with more than one constraint, the result is still open. Remark that the proposition does not exclude the existence of an other variational ‘associated’ system.

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