The Geometry and Integrability of the Suslov Problem

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Abstract

In this paper we discuss the integrability of a nonholonomic mechanical system—a generalized Klebsh–Tisserand case of the Suslov problem. Using the theory of Hamiltonization and the Poincaré–Hopf theorem we analyze the topology of the invariant manifolds and in particular describe their genus. We contrast the results with those for Hamiltonian systems.

Introduction

Consider a Hamiltonian system on a $2n$-dimensional symplectic manifold M with n independent smooth integrals $F_i : M \to \mathbb{R}$, $i = 1, \ldots, n$. Let

$$M_{\mathbf{a}} = \{x \in M \mid F_i(x) = a_i\}, \quad \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n.$$

If the integrals are also in involution (that is, with pairwise vanishing Poisson brackets) and the level set $M_{\mathbf{a}}$ is compact, then the Liouville–Arnold Theorem implies that each connected component of $M_{\mathbf{a}}$ is diffeomorphic to the torus $\mathbb{T}^n$ [1, Chapter 10.49]. These connected components are often called *Liouville tori*.

We now impose non-integrable velocity constraints on our mechanical system—forming a *nonholonomic system*. Despite the fact that these systems are not Hamiltonian [6], there are still many examples for which the associated invariant sets are toroidal [8, 9, 31, 47, 48].

A number of nonholonomic systems with toroidal invariant manifolds have the following properties:

1. They possess symmetry, leading to the natural construction of the reduced velocity phase space. As discussed in [7], the reduced velocity phase

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space of a nonholonomic system has the structure of a fiber bundle, with base space assumed to be a smooth manifold and denoted by Q. In certain special cases (that are of interest here) the dimensions of Q and the fiber coincide. In such cases, the phase space P is the dual of the aforementioned bundle, with base Q and fiber $\mathbb{R}^{n*}$. For instance, we may have $P = T^*Q$ or $P = Q \times \mathbb{R}^{n*}$; the latter is the setting of interest for this paper.

2. The vector field X_{nh} describing the nonholonomic dynamics can be written as
$$X_{\mathrm{nh}} = f(q)X, \tag{*}$$
where $f(q)$ is a nowhere vanishing smooth function on Q, $X_{\mathrm{nh}} \in \mathcal{X}(P)$, and X is in general the Hamilton–Poisson vector field for the Hamiltonian h of the nonholonomic system, i.e. the vector field defined by $X = \{\cdot, h\}$, where $\{\cdot,\cdot\}$ is a Poisson bracket on P (see [5, 9, 12, 16, 47]). In the special case that $P = T^*Q$, this is the Poisson bracket given by the canonical symplectic form on T^*Q.

The function $f(q)$ is often called the *Chaplygin reducing multiplier* (or simply the *multiplier*) after S.A. Chaplygin, who showed [10] that if a nonholonomic system on an n-dimensional configuration manifold Q can be transformed into a Hamiltonian system via the reparameterization of time $d\tau = f(q)\,dt$, $q \in Q$, then the original nonholonomic system has an invariant measure with density $N(q) = (f(q))^{n-1}$ with respect to Ω^n, where Ω is the canonical symplectic form on T^*Q [15, Thm 3.5]. For $n = 2$, Chaplygin also showed that the converse is true: For a nonholonomic system with two degrees of freedom possessing an invariant measure with density $N(q)$ the dynamics becomes Hamiltonian after the reparameterization of time $d\tau = N(q)\,dt$.

In the case of $P = T^*Q$, nonholonomic systems with X_{nh} of the form (*) are also called *conformally Hamiltonian* systems [9, 26], since X_{nh} becomes Hamiltonian after multiplication by $1/f$. If X is complete, one can show that X_{nh} is Hamiltonian in the time τ with respect to a modified Poisson bracket [47]. Time-invariant conformally Hamiltonian systems are also measure- and energy-preserving [9]. Due to these properties, conformally Hamiltonian systems are "practically indistinguishable from Hamiltonian systems" [9]. Accordingly, the integrability of conformally Hamiltonian systems can be investigated with the Liouville–Arnold theorem. Loosely speaking, after rescaling X_{nh} (or equivalently reparameterizing time), the resulting system is Hamiltonian, and its integrability can be determined with the help of the Liouville–Arnold theorem (we present an example of this in Section 1). In the case that $M_{\mathbf{a}}$ is two-dimensional, the following result, due to Kozlov and based on Jacobi's last multiplier theorem, is often used when analyzing the integrability of a conformally Hamiltonian nonholonomic system.

Theorem (Kozlov [2, 24])**.** *Suppose that the system*
$$\dot{x} = g(x), \tag{**}$$

where $x \in P$, an m-dimensional smooth manifold, possesses an invariant measure and has $m-2$ first integrals $F_1, \ldots, F_{m-2}$. Suppose that the differentials of the functions $F_1, \ldots, F_{m-2}$ are linearly independent on the associated invariant set $M_{\mathbf{a}}$. Then:

1. *The solutions of (**) lying on $M_{\mathbf{a}}$ can be found by quadratures.*
2. *If $N_{\mathbf{a}}$ is a compact connected component of the level set $M_{\mathbf{a}}$ and $g(x) \neq 0$ on $N_{\mathbf{a}}$, then $N_{\mathbf{a}}$ is a smooth manifold diffeomorphic to the two-dimensional torus $\mathbb{T}^2$.*
3. *There exist angle variables $\varphi_1, \varphi_2 \bmod 2\pi$ on $N_{\mathbf{a}}$ such that the system (**) on $N_{\mathbf{a}}$ takes the form*

$$\dot{\varphi}_1 = \frac{\alpha_1}{\Phi(\varphi_1, \varphi_2)}, \quad \dot{\varphi}_2 = \frac{\alpha_2}{\Phi(\varphi_1, \varphi_2)},$$

   where α_1, α_2 are constants, $|\alpha_1| + |\alpha_2| \neq 0$, and Φ is a smooth positive 2π-periodic function with respect to φ_1 and φ_2.

(In the first part of this theorem, integrability by quadratures "means that the complete solution can be found by algebraic operations (including inversion of functions) and calculating integrals of functions of one variable" [25, pg. 135].)

Aside from the aforementioned examples of nonholonomic systems that are conformally Hamiltonian, there are also examples in which the vector field X_{nh} still has the form (*) but where one or more of the assumptions characterizing the conformally Hamiltonian setting do not hold; namely, the assumptions that the phase space $P = T^*Q$, the f in (*) has no zeros, and the X in (*) is Hamiltonian with respect to a symplectic form on T^*Q. This is the case, for instance, in the examples considered in [14, 46]. In those examples f has zeros and the phase space P is not a cotangent bundle but instead a trivial fiber bundle $P = Q \times \mathbb{R}^n$ (where $n = \dim Q$) of even dimension, and X is Hamiltonian with respect to a Poisson bracket of the form $(1/f)\{\cdot,\cdot\}_{\mathrm{AP}}$, where the latter is an almost Poisson bracket on P.

For these nonholonomic systems, since the multiplier f now has zeros, item 2 of Kozlov's theorem cannot tell us about the topology of $M_{\mathbf{a}}$. In this paper we study the problem of determining the topology of the invariant sets for such systems. To parallel Kozlov's theorem, we restrict our attention to the case when $M_{\mathbf{a}}$ is two-dimensional. We discuss why the Poincaré–Hopf theorem is a natural tool to use, and utilize various other tools from topology and the theory of ordinary differential equations that are useful for analyzing the problem and that we hope will be helpful in generalizing the analysis here to other integrable nonholonomic systems.

We illustrate our approach by studying an integrable case of the *Suslov problem* [24, 44], which describes the rotational dynamics of a three-dimensional rigid body subject to the nonholonomic constraint that forces the angular velocity component along a given direction in the body to vanish (we describe this system in detail in Section 1.2). The reduced dynamics of this system take place on the space $P = S^2 \times \mathbb{R}^2$, where S^2 is the reduced configuration space, and the vector field X_{nh} still has the form (*), where X is Hamilton–Poisson as we will see in Section 3, but where f has zeros. As we will see, the invariant sets $M_{\mathbf{a}}$ of this system are not tori; they are surfaces of genus between zero and five. This seems to suggest that the presence of zeros of f affect the topology of the invariant sets. In Section 2 we describe precisely how, by way of the Poincaré–Hopf theorem. We relate our approach to that of Tatarinov in [45, 46], who used a surgery approach to arrive at the same conclusion.

The paper is organized as follows. In Section 1 we consider nonholonomic systems verifying (*), which we term "Chaplygin Hamiltonizable." We then present an example that illustrates the fact that when f has no zeros the topology of the manifolds $M_{\mathbf{a}}$ of integrable Chaplygin Hamiltonizable systems can be determined via the Liouville–Arnold theorem. In §1.2 we present an overview of the Suslov problem and its integrability in the Klebsh–Tisserand case and then review Tatarinov's results. In Section 2 we review a generalization of the Poincaré–Hopf Theorem that we will need in the discussion of our approach to determining the topology of $M_{\mathbf{a}}$ when this set is a two-dimensional compact manifold. We illustrate this approach in the following sections, beginning in Section 3, where we review a generalized Klebsh–Tisserand potential and show that the vector field of the Suslov problem with this potential can be written as in (*), but where the underlying space is not a cotangent bundle. Then, in Section 4 we study how the zeros of f contribute to the zeros of the vector field X_{nh}, and count the indices at all zeros. Then, we prove via the Poincaré–Hopf Theorem that, under certain additional assumptions, the total sum of the zeros of X_{nh} determines the genus of the compact surfaces $M_{\mathbf{a}}$ to be $g = 1 + n$, where $0 \leq n \leq 4$ is the number of connected components of the set of zeros of the multiplier f. The similarities and differences between this result (Theorem 4.1 below), the Liouville–Arnold theorem, and Kozlov's theorem above are discussed in Section 5.

1 Integrable Chaplygin Hamiltonizable Systems

Dynamics of the form (1.1) often arises as a result of *Chaplygin reduction*. Briefly, in these cases one deals with a nonholonomic system with symmetry whose velocity space (i.e., the tangent space to the configuration manifold) at each point is the direct sum of the tangent to the group orbit and the constraint distribution. See [7] for details. Assuming that the system's Lagrangian and constraint distribution are invariant under the group action, one can then perform the Chaplygin reduction to reduce the dynamics (see [6, 7, 12] for a thorough exposition of Chaplygin reduction).

1.1 Hamiltonizable Systems

As mentioned in the Introduction, the nonholonomic dynamics we will study in this paper take place not on a cotangent bundle but on a trivial fiber bundle. The following definition describes the cases of interest to us, motivated in [9].

Definition 1.1. *Let Q be an n-dimensional smooth manifold* (*the reduced configuration space*) *and P be a smooth fiber bundle with base space Q and fiber*

$\mathbb{R}^{n*}$ *(with* $n = \dim Q$*). Denote by* $X_{\mathrm{nh}} \in \mathcal{X}(P)$ *the vector field describing the nonholonomic dynamics on* P *and associated to the Hamiltonian* $h : P \to \mathbb{R}$*. Further, suppose that*

$$X_{\mathrm{nh}} = f(q) X_h \tag{1.1}$$

for some smooth function f *on* Q*, where* $X_h = \{\cdot, h\}$ *is the Hamiltonian–Poisson vector field defined by the Hamiltonian* h *and Poisson bracket* $\{\cdot,\cdot\} : \mathcal{F}(P) \times \mathcal{F}(P) \to \mathbb{R}$ *on* P*. Then*

1. *If* f *is nowhere zero we will call the system* ***Chaplygin Hamiltonizable****.*
2. *If* f *has zeros we will call the system* ***Chaplygin Hamiltonizable almost everywhere****.*

In both cases the function f *will be called the* ***conformal factor*** *or* ***multiplier****.*

In the context of this definition, the conformally Hamiltonian systems discussed in the Introduction are Chaplygin Hamiltonizable and are characterized by $P = T^*Q$, f nowhere zero, and $\{\cdot,\cdot\}$ the Poisson bracket associated with a symplectic form on T^*Q; under certain conditions a multiplier f may be found by solving a coupled set of first-order partial differential equations (see [16] and [35]). By contrast, as we will see in Section 3 the problem we study in this paper is characterized by P a trivial fiber bundle, f having zeros, and $\{\cdot,\cdot\}$ a Poisson bracket of the form $(1/f)\{\cdot,\cdot\}_{\mathrm{AP}}$, where the latter bracket is an almost Poisson bracket on P. (Almost Poisson brackets on P that become Poisson after multiplication by a nowhere zero function on P were called *conformally Poisson* in [3].)

As mentioned in the Introduction, Chaplygin Hamiltonizable systems (in the sense of the first part of Definition 1.1) always possess an invariant measure, and in some cases their integrability can be investigated using the Liouville–Arnold theorem. Let us illustrate this with an example (paper [9] lists many other examples of Chaplygin Hamiltonizable systems).

Example: The Vertical Rolling Disk. Following [6], consider a vertical disk of mass m rolling on a flat surface. Label its center of mass by (x, y) and let θ be the angle between a fixed radius of the disk and the vertical direction and φ be the angle measured counterclockwise from the positive x-axis to the plane of the disk.

The configuration space of this system is $S^1 \times S^1 \times \mathbb{R}^2$, with $G = \mathbb{R}^2$ a symmetry group satisfying the condition for Chaplygin reduction discussed at the beginning of this section [6, Sect. 5.6.1]. After Chaplygin reduction we obtain the reduced configuration space $Q = S^1 \times S^1$, coordinatized by (θ, φ). The reduced Hamiltonian $h : T^*Q \to \mathbb{R}$ reads

$$h = \frac{1}{2}\left((I + mR^2)p_\theta^2 + J p_\varphi^2\right),$$

where I is the moment of inertia of the disk about the axis through the center of and orthogonal to the disk, and J is the moment of inertia about a diameter of the disk.

The vector field X_{nh} on T^*Q describing the reduced nonholonomic dynamics is

$$X_{\mathrm{nh}} = (I + mR^2)p_\theta \frac{\partial}{\partial\theta} + Jp_\varphi \frac{\partial}{\partial\varphi}.$$

This vector field is of the form (1.1), with $f(\theta, \varphi) = 1$ (meaning that the dynamics is already Hamiltonian) and $\{\cdot, \cdot\}$ is the canonical Poisson bracket associated with the canonical 2-form $d\theta \wedge dp_\theta + d\varphi \wedge dp_\varphi$ on T^*Q. Thus, from Definition 1.1 we conclude that the system is Chaplygin Hamiltonizable. In addition, the vector field X_{nh} is complete and the integrals of motion $F_1 = (I+mR^2)p_\theta^2$ and $F_2 = Jp_\varphi^2$ are independent. Since these add up to h, it follows that $M_{\mathbf{a}}$, given by

$$M_{\mathbf{a}} = \{(\theta, \varphi, p_\theta, p_\varphi) \in T^*Q \mid F_1 = a_1, F_2 = a_2\},$$

is compact. Clearly $M_{\mathbf{a}} \cong \mathbb{T}^2$, consistent with the conclusion of the Liouville–Arnold theorem. We note in passing that unlike the *Hamiltonian* dynamics of the reduced system, the *full* dynamics of the vertical rolling disk, describing motion in all four variables (θ, φ, x, y), is *not* Hamiltonian.

1.2 Integrability of the Suslov Problem with a Klebsh–Tisserand Potential

We will now introduce a classic example of a system that is Chaplygin Hamiltonizable almost everywhere (meaning that the system's multiplier f has zeros).

Following [45, 46], and [18, Section 6.1.1] let us first describe the Euler–Poincaré equations. Consider a rigid body with inertia tensor

$$A = \mathrm{diag}(A_1, A_2, A_3)$$

rotating about the origin in $\mathbb{R}^3$. Denote by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the standard orthonormal frame in $\mathbb{R}^3$ (the "fixed spatial frame") and by $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ an orthonormal frame moving with the body (the "body frame", here the principal inertia frame). Then the configuration of the body is specified by a rotation matrix $g \in SO(3)$ relating the moving body frame to the fixed spatial frame: $\mathbf{E}_i = g\mathbf{e}_i$. The vertical unit vector $\mathbf{e}_3$, viewed from the body frame, is called the *Poisson vector* and denoted $\boldsymbol{\gamma}$. This vector determines the attitude of the body up to rotations around a vertical line [1, Appendix 5].

Let $\boldsymbol{\Omega} = \Omega_1\mathbf{E}_1 + \Omega_2\mathbf{E}_2 + \Omega_3\mathbf{E}_3 \in \mathfrak{so}(3) \cong \mathbb{R}^3$ be the *body angular velocity* of the rigid body. Suppose further that the rigid body moves in an axisymmetric potential field $V(\boldsymbol{\gamma})$. Then the dynamics on the velocity phase space $\mathbb{R}^3\{\boldsymbol{\gamma}\} \times \mathbb{R}^3\{\boldsymbol{\Omega}\}$ is given by the Euler–Poincaré equations [18] (see also [27, 41] for the related semidirect product approach):

$$A\dot{\boldsymbol{\Omega}} = (A\boldsymbol{\Omega}) \times \boldsymbol{\Omega} + \frac{\partial V}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\Omega}.$$

In Suslov's problem [44] we now impose the nonholonomic constraint

$$\langle \mathbf{a}, \boldsymbol{\Omega} \rangle = 0,$$

where the vector $\mathbf{a}$ is fixed in the body. Following [24, Section 5], let us consider the case $\mathbf{a} = \mathbf{E}_3$, so that the nonholonomic constraint is simply $\Omega_3 = 0$. Then the equations of motion turn out to be the Euler–Poisson equations above with $\Omega_3 = 0$ [17, 24, 44]:

$$A_1\dot{\Omega}_1 = \gamma_2\frac{\partial V}{\partial \gamma_3} - \gamma_3\frac{\partial V}{\partial \gamma_2}, \quad A_2\dot{\Omega}_2 = \gamma_3\frac{\partial V}{\partial \gamma_1} - \gamma_1\frac{\partial V}{\partial \gamma_3}, \tag{1.2a}$$

$$\dot{\gamma}_1 = -\gamma_3\Omega_2, \qquad \dot{\gamma}_2 = \gamma_3\Omega_1, \qquad \dot{\gamma}_3 = \gamma_1\Omega_2 - \gamma_2\Omega_1. \tag{1.2b}$$

This system preserves the measure $d^3\boldsymbol{\gamma} \wedge d^3\boldsymbol{\Omega}$ on $\mathbb{R}^3\{\boldsymbol{\gamma}\} \times \mathbb{R}^3\{\boldsymbol{\Omega}\}$, as can easily be verified by taking the divergence of (1.2). It is also $SO(2)$-invariant, so that the reduced configuration space $Q = SO(3)/SO(2) \cong S^2$ [14, Lemma 1]. Thus, the velocity phase space of the system is [14]:

$$\mathcal{V} = \{(\boldsymbol{\gamma}, \boldsymbol{\Omega}) \in \mathbb{R}^3\{\boldsymbol{\gamma}\} \times \mathbb{R}^3\{\boldsymbol{\Omega}\} \mid \Omega_3 = 0, \langle \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle = 1\} = S^2 \times \mathbb{R}^2. \tag{1.3}$$

Clearly, $F_1 = \langle \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle$ is an integral of motion of both the Euler–Poisson equations and the system (1.2).

The space $\mathcal{V}$ is of the form $Q \times \mathbb{R}^n$ discussed in Definition 1.1, and is four-dimensional, so we need two more integrals of motion to conclude integrability via Kozlov's theorem. These integrals have been well studied [11, 14]; one integral is

$$F_2 = \frac{1}{2}\langle A\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle + V(\boldsymbol{\gamma}),$$

and several classes of potentials are known to generate the last needed integral (see [11, 14] for more details). Among these is the *Klebsh–Tisserand case*, where

$$V(\boldsymbol{\gamma}) = \frac{b}{2}\langle A\boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle, \quad F_3 = \frac{1}{2}\langle A\boldsymbol{\Omega}, A\boldsymbol{\Omega}\rangle - \frac{1}{2}\langle D\boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle, \quad \text{where} \quad D = bA^{-1}\det A.$$

With this potential, the Suslov problem is known to be integrable by quadratures, and the dynamics can be written in the form [24]

$$\dot{\varphi}_1 = \lambda_1/\Phi, \quad \dot{\varphi}_2 = \lambda_2/\Phi, \quad \Phi = \left(1 - c_1^2\sin^2\varphi_1 - c_2^2\sin^2\varphi_2\right)^{-1/2}, \tag{1.4}$$

where c_1 and c_2 depend on the values of the integrals F_1 and F_3, and (φ_1, φ_2) are angle variables. If $c_1^2 + c_2^2 < 1$ then item 2 of Kozlov's theorem guarantees that (1.4) determines a flow on two-dimensional tori. However, for $c_1^2 + c_2^2 \geq 1$ the vector field X_{nh} has zeros (as we discuss in the next paragraph), and so the hypotheses of item 2 of Kozlov's theorem are not met. For this reason, the authors of [36, 37, 47] studied the flow (1.4) in the case when $c_1^2 + c_2^2 \geq 1$ and determined the topology via the surgery approach used by Tatarinov and alluded to in the Introduction. We will now review this approach.

Tatarinov's method begins by considering the $c_1^2 + c_2^2 < 1$ case of the flow (1.4), and its associated Liouville tori in the phase space $\mathcal{V}$. (Strictly speaking, the Liouville tori are situated in the momentum phase space. Here, however, they are viewed as submanifolds of the velocity phase space.) As described in [37], one then defines the *region of possible motions* P_{fh} as the projection $(\Omega_1, \Omega_2, \gamma_1, \gamma_2) \to (\gamma_1, \gamma_2)$ of these Liouville tori on the plane of independent coordinates (γ_1, γ_2), where $\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1$; we note in passing that in general, the "region of possible motions" is defined as that part of the configuration space where the trajectories belong to (for a given energy level), and does not require integrability (see [2]). For the Klebsh–Tisserand case, the trajectories on $\mathcal{V}$ project onto curves in the $\gamma_1\gamma_2$-plane called *Lissajous figures* [17, 46].

These curves are bounded by rectangles we will denote by $R(\gamma_0)$, with $\gamma_0 = (\gamma_1^0, \gamma_2^0)$ denoting the center of the rectangle (see Figure 1). These are the regions of possible motions in the Klebsh–Tisserand case. However, the actual

Figure 1: Some examples of the region of possible motions R and its intersection with ∂D, the boundary of the unit disk $\gamma_1^2 + \gamma_2^2 \leq 1$. Note that in (c) three different possible intersections are presented.

dynamics described by (1.2) require that $\gamma_1^2 + \gamma_2^2 \leq 1$. Therefore, the actual motion in the $\gamma_1\gamma_2$-plane occurs on the set $R(\boldsymbol{\gamma}_0) \cap \mathrm{int}(D)$, where $\mathrm{int}(D)$ denotes the interior of the closed unit disk (Figure 1 illustrates several possibilities for $R \cap \mathrm{int}(D)$). Then, as [37] writes:

> "We first intersect the domains P_{fh} with the circle ∂D, remove parts corresponding to $P_{fh} \backslash (P_{fh} \cap \partial D)$ from each of the Liouville tori projected on P_{fh}, and duplicate (because of the symmetry $t \to -t$,

$\gamma_3 \to -\gamma_3$) the two-dimensional manifolds thus obtained. Thus, the topological type of integral manifolds is determined by the shape of the domains $P_{fh} \backslash \partial D$."

Removing the parts corresponding to $R \backslash (R \cap \mathrm{int}(D))$ from R is tantamount to removing the edges on and outside of D in Figure 1. If the result is not the entire disk D, then the two-dimensional manifold obtained will be a torus with n holes, where n is the number of edges of the rectangle R that were removed. Gluing the two-dimensional manifolds obtained along these holes produces the higher genus surfaces reported in [45].

2 The Topology of M_{a} via the Poincaré–Hopf Theorem

Before we present our alternative to the surgery method discussed in the previous section, let us first review the main tool we will use, the Poincaré–Hopf theorem. (This theorem is a generalization of an earlier result by Poincaré [39], who proved the two-dimensional version. The history of these developments is discussed in the excellent book [38], particularly Chapter I.3.)

Theorem (Poincaré–Hopf [22])**.** *Let M be a compact orientable manifold of dimension n without boundary, and let W be a smooth vector field on M with finitely many isolated zeros x_i, $i = 1, \ldots, m$. Then the sum of the indices of the vector field at these zeros equals the Euler characteristic $\chi(M)$ of M. That is,*

$$\sum_{i=1}^{m} \mathrm{ind}_{x_i}(W) = \chi(M). \tag{2.1}$$

In [30] (see also [20, 40]) the Poincaré–Hopf theorem was extended to the setting of *compact manifolds with boundary*. Specifically, let M now be a compact manifold with boundary ∂M comprised of k components, and W a smooth vector field on M with finitely many isolated zeros x_i, $i = 1, \ldots, m$, and without zeros on the boundary. Then

$$\sum_{i=1}^{m} \mathrm{ind}_{x_i}(W) + \sum_{i=1}^{k} \mathrm{ind}_i(\partial_- W) = \chi(M), \tag{2.2}$$

where $\mathrm{ind}_i(\partial_- W)$ is defined as follows.

For each boundary component, denote by ∂W the restriction of $W|_{\partial M}$ to the component field tangent to ∂M. Next, denote by $\partial_- M$ the subset of ∂M defined by

$$\partial_- M := \big\{ x \in M \mid W(x) \text{ points inward (away from the boundary component)} \big\}.$$

Then $\partial_- W = \partial W|_{\partial_- M}$; figure 3 illustrates this construction for the particular vector fields used in the proof of our main result in Section 4. We will refer to the quantity $\mathrm{ind}_i(\partial_- W)$ as the *boundary index* of the i-th boundary component.

Now, in the two-dimensional case the classification theorem for closed orientable surfaces says that a compact orientable surface M without boundary is determined, up to homeomorphism, by its genus g. And since the Euler characteristic of such a surface $\chi(M) = 2-2g$ [23, Section 5.4], the Poincaré–Hopf theorem allows us to determine the topology of M by counting the indices of the zeros of a smooth vector field on M.

We can now use the preceding results to describe our approach to determining the topology of $M_{\mathbf{a}}$. First, suppose that $M_{\mathbf{a}}$ is the invariant set of a nonholonomic system satisfying either case of Definition 1.1. To parallel Kozlov's theorem, suppose further that $M_{\mathbf{a}}$ is two-dimensional, compact, and orientable. Then, since X_{nh} is a vector field on $M_{\mathbf{a}}$ (we will assume it is smooth), we may use it as the vector field W used in the Poincaré–Hopf theorem. Two cases may arise:

1. If $M_{\mathbf{a}}$ is without boundary then we can count the indices of the isolated zeros of X_{nh} and use (2.1) to evaluate the genus of $M_{\mathbf{a}}$.

2. If $M_{\mathbf{a}}$ has boundary then we must compute the boundary indices

$$\mathrm{ind}_i(\partial_- X_{\mathrm{nh}})$$

for each boundary component and then use (2.2) to determine the Euler characteristic. We now know the Euler characteristic and the number k of boundary components. To find the genus, note that from the classification theorem for closed orientable surfaces it follows that a compact orientable surface M with genus g and k boundary components has Euler characteristic $\chi(M) = 2-2g-k$. Therefore, using the Euler characteristic and the known number k of boundary components we can again determine the genus g in this case.

We note that in both cases, the set of zeros of the multiplier f directly affect the topology of $M_{\mathbf{a}}$ through the indices the zeros of f contribute to either (2.1) or (2.2). We will see this illustrated in Section 4. Let us now return to the Suslov problem.

3 The Quadratic Potential Suslov Problem

Let $H^{\pm}$ be the upper and lower hemispheres,

$$H^{\pm} = \left\{\boldsymbol{\gamma} \in \mathbb{R}^3 \mid \gamma_1^2+\gamma_2^2 < 1, \gamma_3 = \pm\sqrt{1-(\gamma_1^2+\gamma_2^2)}\right\}.$$

Then the decomposition of the velocity phase space $\mathcal{V}$ from (1.3) reads

$$\mathcal{V} = \mathcal{V}^+ \cup \Gamma \cup \mathcal{V}^-,$$

where

$$\mathcal{V}^{\pm} = \left\{(\gamma_1,\gamma_2,\gamma_3,\Omega_1,\Omega_2) \in \mathcal{V} \mid \boldsymbol{\gamma} \in H^{\pm}\right\}, \tag{3.1}$$

$$\Gamma = \{(\gamma_1,\gamma_2,\gamma_3,\Omega_1,\Omega_2) \in \mathcal{V} \mid \gamma_1^2+\gamma_2^2 = 1\}. \tag{3.2}$$

Thus, we can coordinatize $\mathcal{V}^{\pm}$ by $(\gamma_1,\gamma_2,\Omega_1,\Omega_2)$, so that $\mathcal{V}^{\pm} \cong TH^{\pm}$.

Consider now the (constrained) Lagrangian $l : \mathcal{V} \to \mathbb{R}$ given by

$$l(\gamma, \Omega) = \frac{1}{2}\left(A_1\Omega_1^2 + A_2\Omega_2^2\right) - \frac{1}{2}\left(B_1\left(\gamma_1 - \gamma_1^0\right)^2 + B_2\left(\gamma_2 - \gamma_2^0\right)^2\right), \tag{3.3}$$

where $A_i, B_i > 0$, and the second parenthetical term represents a class of quadratic potentials in which γ_i^0, where $|\gamma_i^0| < 1$, are arbitrary constants. The potential considered here is a natural extension of the Klebsh–Tisserand potential (see [14]), and hereafter we will refer to this case as the *quadratic potential Suslov problem.*

The Hamiltonian associated with (3.3) is

$$h(\gamma, m) = \frac{1}{2}\left(\frac{m_1^2}{A_1} + \frac{m_2^2}{A_2}\right) + \frac{1}{2}\left(B_1\left(\gamma_1 - \gamma_1^0\right)^2 + B_2\left(\gamma_2 - \gamma_2^0\right)^2\right),$$

where $m_i = A_i\Omega_i$ are the momenta. The associated momentum phase is $P = S^2 \times \mathbb{R}^2$, and we define $P^{\pm}$ analogously to $\mathcal{V}^{\pm}$ in (3.1), and also denote by Λ the subset of P defined by $\gamma_1^2 + \gamma_1^2 = 1$, in analogy with (3.2). In terms of (γ, m), the equations (1.2) become

$$\dot{m}_1 = -\gamma_3 B_2\left(\gamma_2 - \gamma_2^0\right), \quad \dot{m}_2 = \gamma_3 B_1\left(\gamma_1 - \gamma_1^0\right), \tag{3.4a}$$

$$\dot{\gamma}_1 = -\gamma_3\frac{m_2}{A_2}, \qquad \dot{\gamma}_2 = \gamma_3\frac{m_1}{A_1}, \tag{3.4b}$$

$$\dot{\gamma}_3 = \frac{\gamma_1 m_2}{A_2} - \frac{\gamma_2 m_1}{A_1}. \tag{3.4c}$$

As observed in [17], the subsystem (3.4a) and (3.4b) resembles the equations of motion of a two-dimensional harmonic oscillator. This motivates the change of coordinates

$$(\gamma_1, \gamma_2, \gamma_3, m_1, m_2) \mapsto (\gamma_1, \gamma_2, \gamma_3, -m_2, m_1) =: (q_1, q_2, q_3, p_1, p_2),$$

after which we have

$$h(q, p) = \frac{1}{2}\left(\frac{p_2^2}{A_1} + \frac{p_1^2}{A_2} + B_1\left(q_1 - q_1^0\right)^2 + B_2\left(q_2 - q_2^0\right)^2\right), \tag{3.5}$$

and the equations of motion become

$$\dot{p}_1 = -q_3 B_1\left(q_1 - q_1^0\right), \quad \dot{p}_2 = -q_3 B_2\left(q_2 - q_2^0\right), \tag{3.6a}$$

$$\dot{q}_1 = q_3\frac{p_1}{A_2}, \qquad \dot{q}_2 = q_3\frac{p_2}{A_1}, \tag{3.6b}$$

$$\dot{q}_3 = -\frac{q_1 p_1}{A_2} - \frac{q_2 p_2}{A_1}. \tag{3.6c}$$

The corresponding vector field X_{nh} describing the flow (3.6) is

$$X_{\mathrm{nh}} = q_3\left(\frac{p_1}{A_2}\frac{\partial}{\partial q_1} + \frac{p_2}{A_1}\frac{\partial}{\partial q_2} - B_1(q_1 - q_1^0)\frac{\partial}{\partial p_1} - B_2(q_2 - q_2^0)\frac{\partial}{\partial p_2}\right) - \left(\frac{q_1 p_1}{A_2} + \frac{q_2 p_2}{A_1}\right)\frac{\partial}{\partial q_3}. \tag{3.7}$$

Now, the dynamics (3.6) can also be written as $\dot{z} = \{z, h\}_{\mathrm{nh}}$, where $z = (q_1, q_2, q_3, p_1, p_2)$, and where $\{\cdot,\cdot\}_{\mathrm{nh}}$ is the almost-Poisson bracket

$$\{f, g\}_{\mathrm{nh}} = q_3\{f, g\}_{\mathrm{can}} + q_1 \left[\frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_3} - \frac{\partial f}{\partial q_3}\frac{\partial g}{\partial p_1}\right] + q_2 \left[\frac{\partial f}{\partial p_2}\frac{\partial g}{\partial q_3} - \frac{\partial f}{\partial q_3}\frac{\partial g}{\partial p_2}\right],$$

where $\{\cdot,\cdot\}_{\mathrm{can}}$ is the canonical Poisson bracket on $P^{\pm}$:

$$\{f, g\}_{\mathrm{can}} = \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1} + \frac{\partial f}{\partial q_2}\frac{\partial g}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial g}{\partial q_2}.$$

As can be easily verified, the bracket $(1/q_3)\{\cdot,\cdot\}_{\mathrm{nh}}$ (for $q_3 \neq 0$) is Poisson [19]. Thus, writing

$$\dot{z} = q_3 \left(\frac{1}{q_3}\{z, h\}_{\mathrm{nh}}\right),$$

we see that the vector field (3.7) can be written in the form (1.1), where $f(q) = q_3$ and X is the vector field that is Hamiltonian with respect to the Poisson bracket $(1/q_3)\{\cdot,\cdot\}_{\mathrm{nh}}$. (Equivalently, the dynamics (3.6) is Hamiltonian after the change of time $d\tau = q_3\, dt$.) But since f has zeros, in terms of Definition 1.1 the quadratic potential Suslov problem is Chaplygin Hamiltonizable almost everywhere on P. We note, however, that along $\{q \in S^2 \mid q_3 = 0\}$ the quadratic Suslov problem's constraint distribution and tangent space to the group orbit do not span the system's tangent space [14, Theorem 2]. Thus the system is not Chaplygin, but instead is an example of a *quasi-Chaplygin* system (see [14]).

4 Integrability of the Quadratic Potential Suslov Problem

Here we present a theorem that determines the topology of $M_{\mathbf{a}}$ under certain assumptions. The result is based on the approach described in Section 2.

The equations (3.6) have the two independent integrals

$$F_1(q, p) = \frac{1}{2}\left(\frac{p_1^2}{A_2} + B_1\left(q_1 - q_1^0\right)^2\right), \tag{4.1a}$$

$$F_2(q, p) = \frac{1}{2}\left(\frac{p_2^2}{A_1} + B_2\left(q_2 - q_2^0\right)^2\right). \tag{4.1b}$$

Define now the invariant set

$$M_{\mathbf{a}} = \{(q, p) \in P \mid F_i(q, p) = a_i, i = 1, 2\},$$

where $\mathbf{a} = (a_1, a_2)$, $a_i > 0$. It follows from (4.1) and (3.5) that $M_{\mathbf{a}}$ is two-dimensional and compact. Let

$$\Lambda_{\mathbf{a}} = \{(q, p) \in \Lambda \mid F_i(q, p) = a_i, i = 1, 2\}$$

be the set of points in $M_{\mathbf{a}}$ that satisfy $q_1^2 + q_2^2 = 1$, or equivalently $q_3 = 0$. Note that $\Lambda_{\mathbf{a}}$ is one-dimensional and consists of the set of zeros of f associated to the level $\mathbf{a}$.

Now, because $\mathbf{a} \neq 0$ and $h = F_1 + F_2$, from (3.7) it follows that the only zeros of X_{nh} are those of f. We can then prove the following:

Theorem 4.1. *Decompose $M_{\mathbf{a}}$ as*

$$M_{\mathbf{a}} = M_{\mathbf{a}}^{+} \cup \Lambda_{\mathbf{a}} \cup M_{\mathbf{a}}^{-}, \tag{4.2}$$

where

$$M_{\mathbf{a}}^{\pm} = \{(q,p) \in P^{\pm} \mid F_i(q,p) = a_i, i = 1,2\}.$$

Suppose that

$$\max_{i=1,2}\{q_i^0 + c_i, c_i - q_i^0\} < 1, \quad \text{where} \quad c_i = \sqrt{\frac{2a_i}{B_i}}, \quad i = 1,2. \tag{4.3}$$

Then for $N_{\mathbf{a}}$ a connected component of $M_{\mathbf{a}}$, $N_{\mathbf{a}}$ is diffeomorphic to a surface with genus

$$g = 1 + n, \quad \text{where} \quad 0 \leq n \leq 4. \tag{4.4}$$

The importance of assumption (4.3) will be discussed in Section 5 after we prove the theorem.

Proof. We first parameterize $M_{\mathbf{a}}^{\pm}$ and $\Lambda_{\mathbf{a}}$ as follows. Introduce the parameterizations

$$p_1 = \sqrt{2A_2a_1}\cos\varphi_1, \quad p_2 = \sqrt{2A_1a_2}\cos\varphi_2,$$
$$q_i = q_i^0 + c_i\sin\varphi_i, \quad c_i = \sqrt{\frac{2a_i}{B_i}}, \quad i = 1,2, \tag{4.5}$$

where the plus or minus signs refer to the choice of q_3:

$$q_3 = f(\varphi_1,\varphi_2) = \pm\sqrt{1 - \left[(q_1^0 + c_1\sin\varphi_1)^2 + (q_2^0 + c_2\sin\varphi_2)^2\right]}.$$

(This parameterization avoids using double–valued "functions" for p_i when enforcing $F_i(q,p) = a_i$.) Then, define the domains

$$\begin{aligned} T &= \{(\varphi_1,\varphi_2) \in \mathbb{T}^2 \mid \left(q_1^0 + c_1\sin\varphi_1\right)^2 + \left(q_2^0 + c_2\sin\varphi_2\right)^2 < 1\}, \\ S &= \{(\varphi_1,\varphi_2) \in \mathbb{T}^2 \mid \left(q_1^0 + c_1\sin\varphi_1\right)^2 + \left(q_2^0 + c_2\sin\varphi_2\right)^2 = 1\}, \end{aligned} \tag{4.6}$$

and let $\psi^{\pm} : T \to M_{\mathbf{a}}^{\pm}$ and $\rho : S \to \Lambda_{\mathbf{a}}$ be the charts for $M_{\mathbf{a}}^{\pm}$ and $\Lambda_{\mathbf{a}}$, respectively. The domain S, if nonempty, indicates that there are zeros of f. Lastly, from the vector fields $X_{\mathrm{nh}}|_{M_{\mathbf{a}}^{\pm}}(\mathbf{q})$ define the new vector fields $Y^{\pm}$ on T by

$$Y^{\pm} = \left(\psi^{\pm}\right)^{*} X_{\mathrm{nh}}|_{M_{\mathbf{a}}^{\pm}}.$$

Explicitly, we have

$$Y^{\pm} = \pm f(\varphi_1,\varphi_2)\left(\sqrt{\frac{B_1}{A_2}}\frac{\partial}{\partial\varphi_1} + \sqrt{\frac{B_2}{A_1}}\frac{\partial}{\partial\varphi_2}\right). \tag{4.7}$$

Our strategy will now be to use the Poincaré–Hopf theorem to determine $\chi(M_{\mathbf{a}})$. If $S = \emptyset$ then no additional singularities are introduced into the vector fields $Y^{\pm}$ and we may use (2.1). However, if $S \neq \emptyset$ then, as we will see, we will need to compute the boundary index of the boundaries for $M_{\mathbf{a}}^{\pm}$ that result. Thus, we have the following two cases.

Case 1: $\Lambda_{\mathbf{a}} = \emptyset$. We first note that this case is always possible given sufficiently small c_1, c_2 (corresponding to particular a_1, a_2). In this case, $S = \emptyset$ and $f(\varphi) \neq 0$. From (4.1) it follows that

$$|q_i - q_i^0| \leq c_i \iff q_i^0 - c_i \leq q_i \leq q_i^0 + c_i, \quad i = 1, 2. \tag{4.8}$$

Therefore, both $M_{\mathbf{a}}^{+}$ and $M_{\mathbf{a}}^{-}$ are closed subsets of the compact space $M_{\mathbf{a}}$ and hence each compact. From (4.7) we then see that $Y^{\pm}$ has no zeros. Therefore, by the Poincaré–Hopf theorem (2.1) the Euler characteristic of each $M_{\mathbf{a}}^{\pm}$ is zero. Thus,

$$\chi(M_{\mathbf{a}}^{\pm}) = 0 \implies 2 - 2g = 0 \implies g = 1.$$

Thus, in this case $M_{\mathbf{a}}$ is the disjoint union of two tori.

Case 2. $\Lambda_{\mathbf{a}} \neq \emptyset$. Then, as we now show, $S \neq \emptyset$ and has between one and four connected components, each homeomorphic to a circle.

We first note that (4.5) implies (4.8), and from there assumption (4.3) guarantees that $|q_i| < 1$. Therefore, the points (q_1, q_2) that satisfy $q_1^2 + q_2^2 = 1$ form closed arc segments of the unit circle in the q_1q_2-plane that do not cross the q_1 or q_2 axes (if they did at some point, we would have $|q_i| = 1$). Clearly there are between one and four such segments. We will denote by C_i the i-th closed arc segment. We note that it is possible that one or more of these arc segments is just a point. However, for now we will assume that all four arc segments have nonzero length and return to the zero-length issue later.

Now, from (4.8) the endpoints of the curve segments C_i occur at $q_i^0 \pm c_i$, and from (4.5) this occurs at the φ-coordinates of either $\pi/2$ or $3\pi/2$. Figure 2(a) illustrates this for the case when C_i is a curve segment in the first quadrant, where point A corresponds to $\varphi_2 = \pi/2$ and point B corresponds to $\varphi_1 = \pi/2$ (Figure 2(b)). Each point A and B has one φ-component fixed, and the other is now determined by solving (4.5) for the remaining φ. This produces *two* values in the interval $[0, 2\pi)$. For example, the point B in Figure 2(a) corresponds to the *two* points labeled b in Figure 2(b). Therefore, the endpoints of a given C_i are mapped to *two* points in the φ-coordinates.

For interior points of a given C_i, since each q_i corresponds to *two* φ-values, each interior point corresponds to *four* (φ_1, φ_2) coordinates. For example, the points C, D in Figure 2(a) correspond to the points c, d in Figure 2(b), respectively. Therefore, the set S from (4.6) can be expressed as

$$S = \bigcup_{i=1}^{n} C_i, \quad 1 \leq n \leq 4, \tag{4.9}$$

where the union is a disjoint one by our assumption that $|q_i| < 1$. Figure 2(c) shows an example of the case when S has four components (in addition to showing the vector field $Y^{\pm}$).

The manifolds $M_{\mathbf{a}}^{\pm}$ now have holes (the boundary of these holes is $\Lambda_{\mathbf{a}}$). This seems at first to make it not possible to use (2.2) (since the $M_{\mathbf{a}}^{\pm}$ are no

Figure 2: (a) A segment C_i with endpoints A, B and interior points C, D. (b) The parameterization (4.5) maps endpoint A to the points labeled a, and endpoint B to the points labeled b. The interior points C, D are mapped to the four points labeled c, d, respectively. (c) A MAPLE plot of the vector fields $Y^{\pm}$ for $\mathbf{q}^0 = (0, 0)$ and $c_1 = \sqrt{0.6}$, $c_2 = \sqrt{0.7}$. Here S is the union of the boundary of the four closed disks absent from the plot.

longer compact), but we can easily remedy this by adding in the boundary to form $M_{\mathbf{a}}^- \cup \Lambda_{\mathbf{a}} =: \widetilde{M}_{\mathbf{a}}^-$ and $M_{\mathbf{a}}^+ \cup \Lambda_{\mathbf{a}} =: \widetilde{M}_{\mathbf{a}}^+$, which are closed subsets of the compact set $M_{\mathbf{a}}$ and therefore compact. Moreover, since each component of $\Lambda_{\mathbf{a}}$ is contractible, we have

$$\chi\left(\widetilde{M}_{\mathbf{a}}^-\right) = \chi(M_{\mathbf{a}}^-) + \chi(\Lambda_{\mathbf{a}}) = \chi(M_{\mathbf{a}}^-), \quad \chi\left(\widetilde{M}_{\mathbf{a}}^+\right) = \chi(M_{\mathbf{a}}^+).$$

Finally, from (4.2) and (4.9) we have

$$\chi(M_{\mathbf{a}}) = \chi(M_{\mathbf{a}}^-) + \chi(\Lambda_{\mathbf{a}}) + \chi(M_{\mathbf{a}}^+) = \chi(M_{\mathbf{a}}^-) + \chi(M_{\mathbf{a}}^+) = \chi\left(\widetilde{M}_{\mathbf{a}}^-\right) + \chi\left(\widetilde{M}_{\mathbf{a}}^+\right).$$

Therefore, the topology of $M_{\mathbf{a}}$ is completely determined by the topologies of $\widetilde{M}_{\mathbf{a}}^{\pm}$. Moreover, since the genus g of the surfaces $M_{\mathbf{a}}$ is found from the Euler characteristics of the $\widetilde{M}_{\mathbf{a}}^{\pm}$:

$$2 - 2g = \chi(M_{\mathbf{a}}) \implies g = 1 - \frac{1}{2}\left[\chi\left(\widetilde{M}_{\mathbf{a}}^{-}\right) + \chi\left(\widetilde{M}_{\mathbf{a}}^{+}\right)\right],$$

we can now apply the extended Poincaré–Hopf theorem (2.2) to the manifolds $M_{\mathbf{a}}^{\pm} \cup \Lambda_{\mathbf{a}}$ to find g.

The setup for the calculation of the vector fields $\partial_{-}Y^{\pm}|_{\widetilde{M}^{\pm}}$ is illustrated in Figure 3. First, restrict $Y^{\pm}|_{\widetilde{M}^{\pm}}$ to the boundary of $M_{\mathbf{a}}^{\pm}$, illustrated by the circle in Figure 3(a). Then project onto the tangent vector field to form $\partial Y^{\pm}|_{\widetilde{M}^{\pm}}$, see Figure 3(b); the two black dots illustrate where this new vector field vanishes. Finally, restrict to the subset of the boundary component where $Y^{\pm}|_{\widetilde{M}^{\pm}}$ points inward, see Figure 3(c).

Figure 3: (a)–(c): An illustration of the construction of $\partial_{-}Y^{\pm}|_{\widetilde{M}^{\pm}}$.

Now, to calculate the boundary index $\mathrm{ind}(\partial_{-}Y^{\pm}|_{\widetilde{M}^{\pm}})$, we first note that $\partial_{-}Y^{\pm}|_{\widetilde{M}^{\pm}}$ is now a vector field on a one-dimension lower manifold, in this case the arc segment from a to b in Figure 3(c); this is homeomorphic to the closed interval $[a, b]$. Now, for a smooth vector field Z on a closed interval $[A, B]$ with no zeros at the boundary points, (2.2) gives [21, pg. 136]

$$\begin{aligned} \mathrm{ind}_{[A,B]}(Z) &= \chi([A, B]) - \mathrm{ind}(\partial_{-}Z) \\ &= 1 - (\text{number of points on the boundary where } Z \text{ points inside}). \end{aligned}$$

From this it follows that

$$\mathrm{ind}_{[a,b]}\, \partial_- Y^{\pm}|_{\widetilde{M}^{\pm}} = 1 - 2 = -1.$$

To complete the proof, suppose that S has n components (recall we have already shown that $1 \leq n \leq 4$ in the present case of $S \neq \emptyset$). Then, since the zeros of (4.7) are only those arising from f, we have

$$\begin{aligned} g &= 1 - \frac{1}{2}\left[\left(\mathrm{ind}(Y^-|_{\widetilde{M}^{\pm}}) + \mathrm{ind}(\partial_- Y^-|_{\widetilde{M}^{\pm}})\right) + \left(\mathrm{ind}(Y^+|_{\widetilde{M}^{\pm}}) + \mathrm{ind}(\partial_- Y^+|_{\widetilde{M}^{\pm}})\right)\right] \\ &= 1 - \frac{1}{2}\left[(0-n) + (0-n)\right] \\ &= 1 + n. \end{aligned} \tag{4.10}$$

We note that in the particular case when $n = 0$ (so that $\Lambda_{\mathbf{a}} = \emptyset$), this verifies our earlier conclusion. Therefore, we have established formula (4.4) for $0 \leq n \leq 4$.

Let us now return to the possibility that some of the sets C_i are isolated points. Let k be the number of such sets, where $0 \leq k \leq 4$. Then each of $\widetilde{M}^+_{\mathbf{a}}$ and $\widetilde{M}^-_{\mathbf{a}}$ has k isolated singularities and $4-k$ boundary components. Since each isolated singularity has index zero ($Y^-|_{\widetilde{M}^{\pm}}$ has zero circulation about these points), then

$$\begin{aligned} g &= 1 - \frac{1}{2}\left[\left(\mathrm{ind}(Y^-|_{\widetilde{M}^{\pm}}) + \mathrm{ind}(\partial_- Y^-|_{\widetilde{M}^{\pm}})\right) + \left(\mathrm{ind}(Y^+|_{\widetilde{M}^{\pm}}) + \mathrm{ind}(\partial_- Y^+|_{\widetilde{M}^{\pm}})\right)\right] \\ &= 1 - \frac{1}{2}\left[(0-(4-k)) + (0-(4-k))\right] \\ &= 1 + (4-k) = 1 + n, \end{aligned} \tag{4.11}$$

where $n = 4-k$ is again the number of boundary components. This completes the proof of Theorem 4.1. □

5 Conclusions and Remarks

We begin with a technical remark about assumption (4.3) in Theorem 4.1. More specifically, assumption (4.3) is not satisfied when either (i) S has a component that crosses either the q_1 or q_2 axes, or (ii) $S = \partial D$ (i.e., f has zeros at all points of the unit circle). In both cases the manifolds $M^{\pm}_{\mathbf{a}}$ are no longer compact (they are cylinders in the first case and unions of open disks in the second). In these cases the Poincaré–Hopf theorems discussed here do not apply. We note that there exist generalizations of the Poincaré–Hopf theorem to the setting of non-compact manifolds [28], but such investigations are outside the scope of this paper; our aim was to parallel the exposition of Kozlov's theorem (where compactness is assumed) for the case where f has zeros.

With assumption (4.3) in place, our approach to determining the topology of the components of the invariant manifolds, discussed in Section 2, relied on the Poincaré–Hopf theorem. In the proof of Theorem 4.1 we determine the topology of $M_{\mathbf{a}}$ by setting out to calculate its Euler characteristic $\chi(M_{\mathbf{a}})$, a topological invariant [29] that completely classifies compact orientable 2-manifolds according to their genus. We then use the Poincaré–Hopf theorem to relate $\chi(M_{\mathbf{a}})$ to the zeros of X_{nh}. Clearly, for a nonholonomic system verifying either part of Definition 1.1 these zeros include those introduced by the multiplier f (if any). Thus we use the information given to us by the zeros of the multiplier to determine the topology of the invariant manifolds. As we showed in (4.10) and (4.11), this amounts to applying a counting argument to the boundary indices of the vector field X_{nh} restricted to the invariant sets.

For the quadratic Suslov problem considered in Theorem 4.1, since the only zeros of X_{nh} on $M_{\mathbf{a}}$ are those of f, we conclude that for this system the topology of $M_{\mathbf{a}}$ is *completely determined* by the contribution of the zeros of f to the sums on the left hand side of (2.1) or (2.2). This is perhaps the best illustration of how our approach is different from that of Tatarinov (discussed in Section 1.2).

Theorem 4.1 also connects the study of invariant manifolds for nonholonomic systems with the vast and well-developed research on the Poincaré–Hopf theorem. As such, although this paper has focused on the quadratic potential Suslov problem, we hope that we have illustrated how the Poincaré–Hopf theorem and its focus on the zeros of vector fields might be helpful in studying the topology of the invariant sets of integrals of an integrable nonholonomic system.

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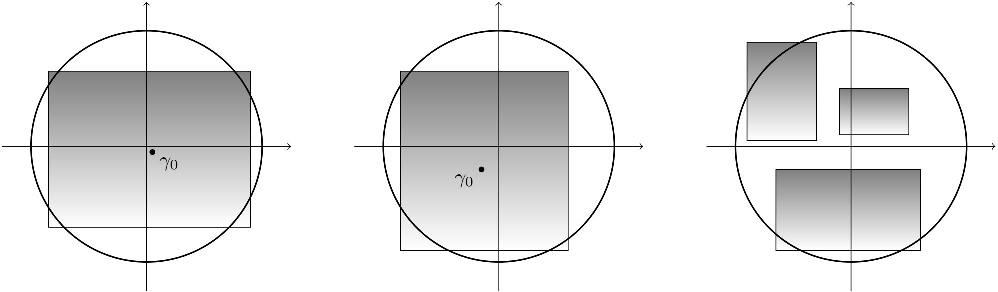

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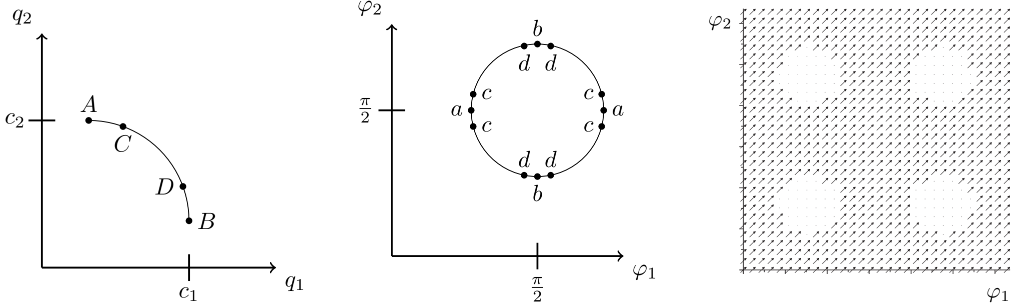

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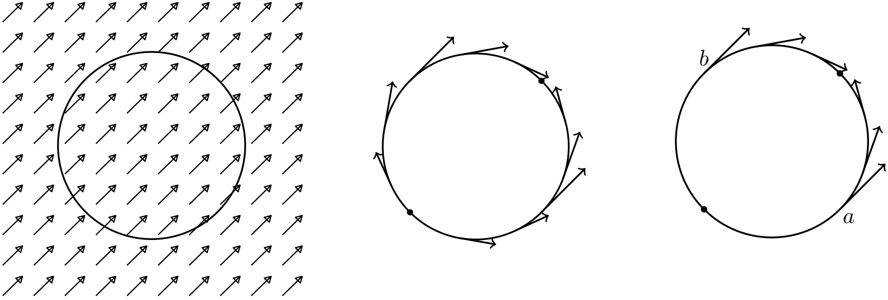

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