1 Functions

1. Give an example of a function $f(x)$ (and sketch it) and constants b and c satisfying:

   (a) $f(b) + f(c) = f(b + c)$.

   (b) $f(b) + f(c) \neq f(b + c)$.

2. Let $f(x) = |x + 1| - 2$. Sketch

   i. $y = f(x)$, ii. $y = f(-x)$, iii. $y = -f(x)$, iv. $y = f(1/x)$

3. Sketch a graph *without using a calculator* for each of the following functions. Use intercepts and appropriate limits.

   (a) $y = \dfrac{2x}{x-1}$

   (b) $y = \dfrac{x^2}{x^2-9}$

   (c) $y = \dfrac{|x^2-4|}{x^2-4}$

4. Write each of the following functions f as a composition of two functions g and h so that $f(x) = (g \circ h)(x) = g(h(x))$. Express both g and h as functions of x.

   (a) $f(x) = (\sin x + 17)^{1/3}$

   (b) $f(x) = 1 + \sqrt{x^3 + \tan x}$

   (c) $f(x) = 5\cos(\pi \sec x)$

2 Limits

1. Let $f(x) = (x^2 - 1)/(x + 1)$

   (a) Sketch the graph of f, and determine its domain and range.

   (b) Evaluate $f(-1)$

   (c) Evaluate $f(+1)$

   (d) Evaluate $\lim_{x \to -1} f(x)$

   (e) Evaluate $\lim_{x \to +1} f(x)$

2. (a) For each of the following, sketch the graph and find $\lim_{x \to 2} f(x)$.

   i. $f(x) = (2x - 4)/(x - 2)$

   ii. $f(x) = (x^2 + x - 6)/(x - 2)$

iii. $f(x) = (x-2)/(x^2-4)$

iv. $f(x) = (x-2)/\sqrt{(x-2)}$

(b) Find a rational function g whose domain is $\{x|x \neq 1, 2\}$ and such that $g(x) = x^2$ wherever $g(x)$ is defined.

3. Determine the following limits.

(a) $\displaystyle\lim_{x\to 1} \frac{x^2+2x-3}{x-1}$.

(b) $\displaystyle\lim_{x\to\infty} \frac{x^2-3x}{2x^2+x+7}$.

(c) $\displaystyle\lim_{x\to\infty} (1+\cos(x))$.

(d) $\displaystyle\lim_{x\to 0^+} \frac{x}{\sqrt{x^2+x}}$.

(e) $\displaystyle\lim_{x\to\infty} (\sqrt{x^2+1}-x)$.

4. Sketch the functions $g(x)$, $h(x)$, and $f(x)$ such that the following conditions hold.

a) $\displaystyle\lim_{x\to 5} g(x) = 10$ however $g(5)$ is not defined.

b) $\displaystyle\lim_{x\to 2} h(x) = 3$ however $h(2) \neq 3$.

c) $\displaystyle\lim_{x\to 0} f(x)$ does not exist, however $f(0) = 2$.

5. Sketch a graph of a function which satisfies all of the given properties. (Remember, not all functions are continuous.)

a) $\displaystyle\lim_{x\to 0} f(x) = 0,\ \ f(0) = 10,\ \ \lim_{x\to 1^+} f(x) = -1,\ \ \lim_{x\to 1^-} = 1,\ \ f(1) = 0.$

b) $\displaystyle\lim_{x\to 2^-} g(x) = \infty,\ \ \lim_{x\to 2^+} g(x) = -\infty.$

c) $\displaystyle\lim_{x\to n^-} h(x) = n,\ \ h(n) = n+1,$ for every integer n.

6. Pretend that you are trying to guess $\displaystyle\lim_{x\to 0} f(x)$ for some function $f(x)$. You plug in $x = 0.1, 0.01, 0.001, \ldots$ and get $f(x) = 0$ for all of these values. In fact, you're told that for all $n = 1, 2, \ldots$ the function satisfies $f(\frac{1}{10^n}) = 0$. **True** or **False**: Since the sequence $0.1, 0.01, 0.001, \ldots$ goes to 0, we know $\displaystyle\lim_{x\to 0} f(x) = 0$.

7. Consider the function

$$f(x) = \begin{cases} x^2, & x \text{ is irrational, } x \neq 0 \\ -x^2, & \alpha \text{ is rational,} \\ \text{undefined,} & x = 0. \end{cases}$$

Then:

(a) There is no a for which $\displaystyle\lim_{x\to a} f(x)$ exists.

(b) There may be some a for which $\lim_{x\to a} f(x)$ exists, but it is impossible to say without more information.

(c) $\lim_{x\to a} f(x)$ exists only when $a = 0$.

(d) $\lim_{x\to a} f(x)$ exists for infinitely many a.

8. Sketch the graph of a function satisfying the following conditions:

$$(a)\, f(0) = 10, \quad (b)\, f(2) = 0, \quad (c)\, \lim_{x\to 2^-} f(x) = -\infty, \quad (d)\, \lim_{x\to 3^-} f(x) = 1, \quad (e)\, \lim_{x\to 5^+} f(x) = 3,$$
$$(f)\, \lim_{x\to -2^+} f(x) = \infty.$$

3 Continuity

1. State and explain where the following functions are continuous:

(a) $f(x) = \dfrac{3x^5}{x^2 - 4x}$

(b) $h(x) = \left[(4x^3 - 1)^{20} + 2\right]^{10}$

(c) $y = \dfrac{3}{1 + \cos x}$

2. Determine the value of k, m so that the following functions are continuous, or state that it is impossible:

(a) $f(x) = kx^3 + 7$

(b) $f(x) = \begin{cases} x^2 & x \le 2, \\ e^{kx} & x > 2 \end{cases}$

(c) $g(x) = \begin{cases} x + k & x \le 2, \\ \sqrt{kx^2 + 8} & x > 2 \end{cases}$

(d) $h(x) = \begin{cases} \dfrac{x^3 - x}{x^2 - 1} & x \ne -1, 1, \\ k & x = 1, \\ m & x = -1 \end{cases}$

4 Intermediate Value Theorem

1. Verify with a careful and complete explanation that the equation

$$x^5 - x^2 + 17 = 2x$$

has at least one solution.

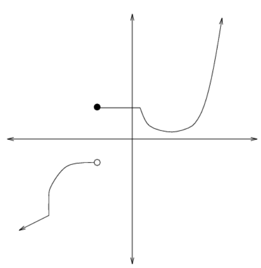
5 The Derivative

1. Use $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ to compute the derivative of the following functions:
   (a) $f(x) = x - x^2$;
   (b) $g(x) = \sqrt{x-5}$;
   (c) $f(x) = \cos(4x)$.

2. Use the limit definition of the derivative to compute $f'(a)$ when
   (a) $f(x) = 3x^2$
   (b) $f(x) = 5$
   (c) $f(x) = \frac{1}{x}$
   (d) $f(x) = \sqrt{x}$ (hint: multiply top and bottom by the conjugate)

3. The equation of the tangent line to the graph of a function f at $x = 4$ is $y = 3x - 17$.
   (a) What is $f(4)$? What is $f'(4)$?
   (b) Given that $f(x) = ax^3 + b$, find the constants a and b.

4. (a) In each of the following graphs label all points of discontinuity and non-differentiability: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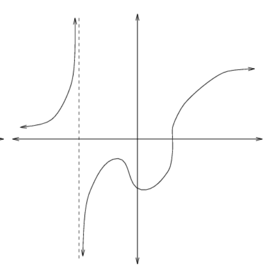
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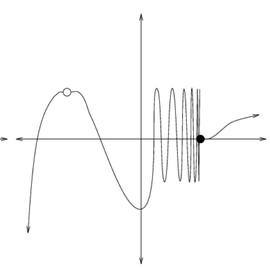
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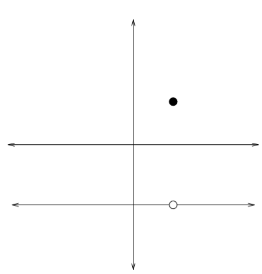
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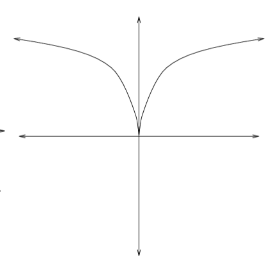
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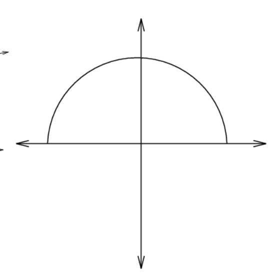

(f)

   (b) For each discontinuity, explain which parts of the definition of continuity are not satisfied.
   (c) For each point of non-differentiability, draw a blow-up of the graph at that point. Then draw a sequence (or two sequences if necessary) of secant lines which show that $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ does not exist. If it is not possible to draw the appropriate secant lines, explain why.

5. Let
$$f(x) = \begin{cases} x^2, & \text{for } x > 0 \\ x, & \text{for } x \leq 0 \end{cases}$$
and $g(x) = [f(x)]^2$.
   (a) Sketch the graph of f;
   (b) Is f continuous at $x = 0$?
   (c) Is f differentiable at $x = 0$?
   (d) Sketch the graph of g;
   (e) Is g continuous at $x = 0$?
   (f) Is g differentiable at $x = 0$?

6. A giddily gleeful student, elated over passing a Calculus I examination, hurls a somewhat large calculus book directly upward from the ground. It moves according to the law $s(t) = 96t - 16t^2$, where t is the time in seconds after it is thrown and $s(t)$ is the height in feet above the ground at time t. Find:
   (a) the velocity of the book after 1.5 seconds;
   (b) the maximum height the book reaches;
   (c) the average speed of the book during its upward rise;
   (d) the acceleration of the book at its maximum height;
   (e) the rate of change of the acceleration of the book after 4 seconds;
   (f) the time it would take for the 6 ft. tall student to have the misfortune of being hit on the head by the book.

7. A spaceship is traveling away from Earth along the curve $y = x^2$. At some point along this path, a payload will be released from the spaceship and travel through space along a line tangent to the curve. Where should the payload be released in order to be intercepted at a space station positioned at the point $(3, 2)$?

8. Differentiate the following.
   (a) $f(x) = 3x \tan x$
   (b) $f(x) = \sin(\sqrt{x})$
   (c) $f(x) = \frac{\sin x}{1+\cos x}$
   (d) $f(x) = \sqrt{1 + \sqrt{1 + \sqrt{1 + x}}}$

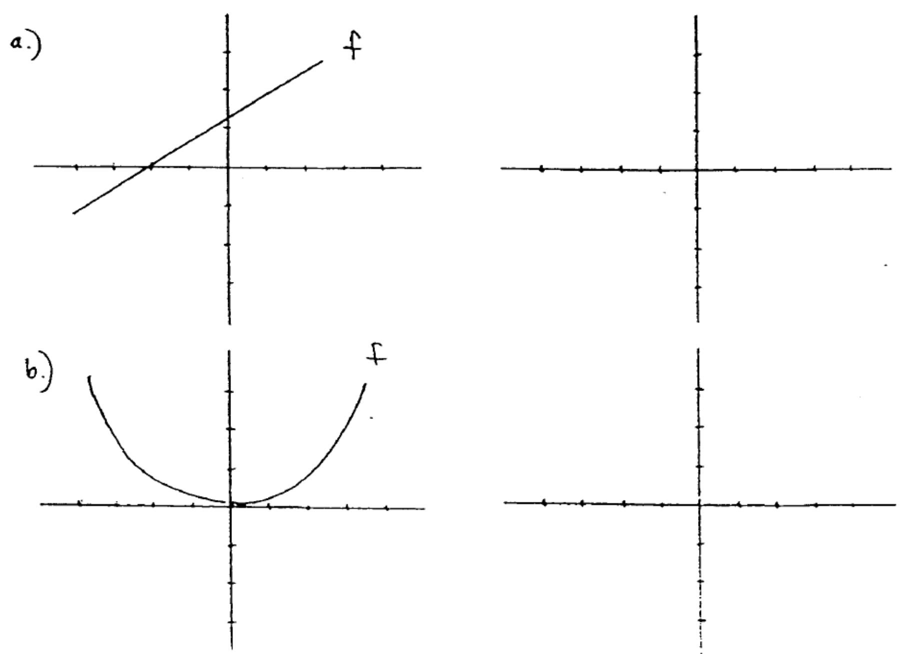
9. Which of the following are true and which are false?
   (a) $\frac{d}{dx}\frac{1}{x} = \ln x$
   (b) $\frac{d}{dx} f(x)g(x) = f'(x)g'(x)$
   (c) $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
   (d) $\frac{d}{dx} \cos x = \sin x$
   (e) $\frac{d}{dx} 2^x = x \cdot 2^{x-1}$

10. Write each of these expressions as $e^{\text{something}}$ and compute the derivative.
    (a) x^x
    (b) $x^{\sin x}$
    (c) 3^{x^2}
    (d) $(\cos x)^{\ln x}$

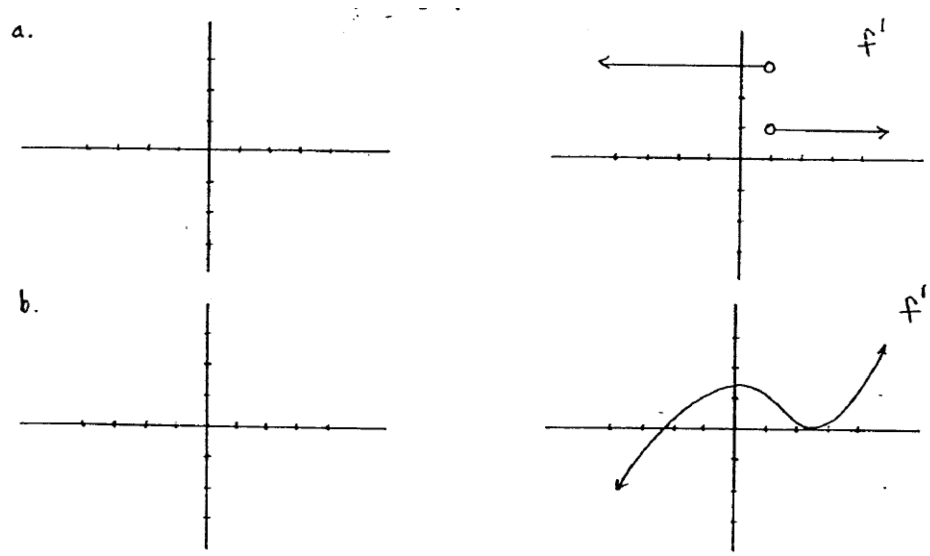
11. Consider the graph of $f(x) = x^2$.
    (a) Find the tangent line to the graph at $x = 3$.
    (b) Find, if possible, a point a such that the tangent line at $x = a$ is parallel to the line $3x+5y = 2$.
    (c) Find, if possible, a point a such that the tangent line at $x = a$ is perpendicular to the line $3x + 5y = 2$.
    (d) Find, if possible, a point a such that the tangent line at $x = a$ passes through the point $(5, 7)$.

12. Consider the graph of $f(x) = x^2$.
    (a) Find the tangent line to the graph at $x = 3$.
    (b) Find, if possible, a point a such that the tangent line at $x = a$ is parallel to the line $3x+5y = 2$.
    (c) Find, if possible, a point a such that the tangent line at $x = a$ is perpendicular to the line $3x + 5y = 2$.
    (d) Find, if possible, a point a such that the tangent line at $x = a$ passes through the point $(5, 7)$.

13. Sketch the graph of f' using the graph of f provided.



14. Sketch the graph of f using the graph of f' provided.



15. Suppose that $h(x) = g(x)u(x)$, where $u(x) = x^3 + 1$, and that $g'(1) = 2$, $g'(2) = 4$, $g'(9) = 16$, $g'(13) = 8$. Find $h'(2)$ and $h''(2)$.

16. Given that a spherical raindrop evaporates at a rate proportional to its surface area, how fast does the radius shrink?

17. For each function $y = f(x)$, solve $f'(x) = 0$ and $f''(x) = 0$ and set up the sign charts for f' and f''. The sign chart is given by the x-axis with a labelling that decides when the function is question is zero, positive or negative.

    (a) $f(x) = x(x-4)^3$

    (b) $f(x) = \frac{\sqrt{3}}{2}x + \cos x$

18. Compute $f'(0)$ if
$$f(x) = \begin{cases} \frac{g(x)}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$
with $g(0) = g'(0) = 0$ and $g''(0) = 17$. Use the formula $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

19. Let f be a function and let f^2 denote the square of the function.

    (a) True or false: If f is differentiable at a, then f^2 is differentiable at a.

    (b) True or false: If f^2 is differentiable at a, then f is differentiable at a.

    (c) True or false: If f has a critical point at a, then f^2 has a critical point at a.

    (d) True or false: If f^2 has a critical point at a, then f has a critical point at a.

20. Recall that, if f is the inverse function of g, then $f(g(x)) = x$.

    (a) Differentiate both sides and find an equation for $g'(x)$.

    (b) Suppose that $f(x) = \ln x$. Suppose that you know that $f'(x) = \frac{1}{x}$. Use the formula to show that $g'(x) = g(x)$.

21. Remember that $\tan(\arctan x) = x$.

(a) Use the quotient rule to show that the derivate of $\tan(x)$ is $\sec^2 x$.
(b) Manipulate the equation $\cos^2 x + \sin^2 x = 1$ to find a relationship between $\sec^2 x$ and $\tan^2 x$.
(c) Use these ideas to compute the derivative of $\arctan(x)$.

22. Find an equation for the line that is simultaneously tangent to both the graphs $y = x^2$ and $y = \frac{8}{x}$.

23. Show that no line tangent to the graph of $f(x) = 1 + \dfrac{1}{x}$ passes through the origin.

24. Compute $\lim_{x\to 1} f(x)$ for the following (i) using l'Hôpital's Rule and (ii) without l'Hôpital's rule.

    (a) $f(x) = \dfrac{x^3 - 1}{(x-1)^2}$

    (b) $f(x) = \dfrac{x^3 - 2x^2 + 2x + 1}{x^3 - 1}$

25. Recall that $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$. Use this equation to find the appropriate f and x and compute the limit simply by taking a derivative (in other words, do not use l'Hôpital's Rule).

    (a) $\lim_{h\to 0} \dfrac{(1+h)^{1000} - 1}{h}$

    (b) $\lim_{h\to 0} \dfrac{(8+h)^{4/3} - 16}{h}$

26. The graph of $y = x^3 + 3x^2 - 3x$ has two tangent lines parallel to $y = 6x + 100$. Find the equation of these two lines.

27. (a) By substituting $x = a + h$, show that $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$.

    (b) Use the above identification to compute $\lim_{x\to 8} \dfrac{x^{4/3} - 16}{x - 8}$ and $\lim_{x\to -1} \dfrac{x^{100} - 1}{x + 1}$ by identifying these limits as the derivatives of certain functions at certain points. In other words, do not use l'Hôpital's Rule!

28. Suppose that $f(x) = \frac{1}{(x-1)^n}$. Find a formula for $f^{(k)}(x)$, the k-th derivative of f.

6 Implicit Differentiation, Related Rates

1. (a) Sketch the graph of the relation $(y - x^2)(y - x^3) = 0$.
   (b) Find a formula for y' using implicit differentiation
   (c) Evaluate y' at (a, a^2), when $a \neq 0, 1$.
   (d) Evaluate y' at (a, a^3), when $a \neq 0, 1$.

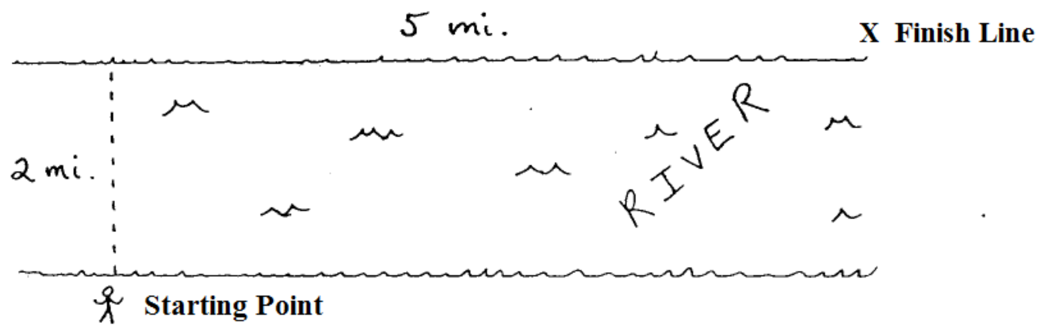
2. Two airplanes are flying north at the same height on parallel paths ten miles apart with speeds of 400 and 600 miles per hour. How fast is the distance between the planes changing when the slow plane is five miles further north than the fast one?

7 Maxima/Minima of Functions, Critical Points

1. Find a quadratic polynomial which is 0 at $x = 3$, is decreasing if $x < 1$ and is increasing if $x > 1$.
2. Suppose that a quadratic function f has roots r and s. Show that $f'(r) + f'(s) = 0$.
3. Show that the critical point of a quadratic occurs midway between its roots.
4. Find the critical points, relative maxima and minima of the function
$$f(x) = x^4 - 4x^3.$$
5. The function
$$y = Ax^3 + Bx^2 + Cx + D$$
has a relative maximum value at the point $(-1, 2)$ and a relative minimum value at the point $(1, -1)$. Determine the value of constants A, B, C and D.
6. Sketch the graph of $f(x) = \frac{x^2}{x^2-1}$. Find all critical points of f and determine whether function is (i) increasing, (ii) decreasing, (iii) concave up, (iv) concave down.
7. Find the equation of the cubic for which the origin is a point of inflection, and $(-2, 16)$ are the coordinates of the local maximum point.
8. Sketch the graph $f(x) = x^5 \ln x$ on the interval $(0, \infty)$.

8 Optimization

1. Suppose you are in charge of building an enclosed area for a concert. You decide that you'll make the area a rectangle, and you have 400 feet of fencing. What are the dimensions of the largest possible area?
2. Each day a 100-room hotel is filled to capacity at \$50 per room. For each increase of \$5 per room, the manager has estimated that 3 fewer rooms are rented. What price should be charged per room in order to maximize the hotel's daily revenue?
3. Find, among all right circular cylinders of fixed volume V, the one with the smallest surface area (counting the areas of the faces on the top and bottom).
4. A thin wire one meter long is to be cut into two pieces. One piece will be bent to form a square and the other a circle. How should the wire be cut in order that the sum of the areas of the circle and square be a maximum?
5. Kimberly has entered a run-swim race. In order to win the cash prize, she'll have to run *and* swim to the point marked "Finish Line," 5 miles away from the starting point, in the figure below. Her competitors aren't sure what they'll do first, run or swim, or for how long. But luckily Kimberly knows Calculus. She know that she can swim at the rate of 2 mph and jog at 4 mph. Help Kimberly determine the route she should take to finish the race the fastest.



6. The strength of a beam with rectangular cross section is proportional to the product of its width w and the square of its height h. Find the dimensions of the strongest beam that can be cut from a cylindrical log having a circular cross section with radius 12 inches.

7. What are the dimensions of the lightest cylindrical aluminum can with capacity $1000cm^3$?

9 Integration

1. A bus moves along the x-axis with velocity $v(t) = t^2 - 4t + 3$. First express each of these quantities as an integral, and then find its value:
   (a) the displacement of the bus between $t = 0$ and $t = 5$.
   (b) the actual distance the bus travels during this time.

2. Evaluate the following integrals
   (a) $\int_1^3 x/\sqrt{x^2+5}\, dx$ *Hint: Let* $f(x) = \sqrt{x^2+5}$. *What is* $f'(x)$*?*
   (b) $\int_1^2 (2x+1)/(x^2+x-1)^2\, dx$
   (c) $\int_1^3 (x-3/x)^5(1+3/x^2)\, dx$
   (d) $\int_{-1}^3 (x-3/x)^5(1+3/x^2)\, dx$

3. Calculate the following derivatives using the chain rule and the fundamental theorem of calculus.
   (a) $\frac{d}{dx}\int_0^x \frac{dt}{1+t^2}$
   (b) $\frac{d}{dx}\int_0^{x^2} \frac{dt}{1+t^2}$
   (c) $\frac{d}{dx}\int_{-x^2}^{x^2} \frac{dt}{1+t^2}$
   (d) $\frac{d^2}{dx^2}\int_0^x \frac{dt}{1+t^2}$

4. We are given a differentiable, odd function f defined on [-3, 3] which has zeros at $x = -2, 0, 2$ (and nowhere else) and critical points at $x = -1, 1$ (and nowhere else). Also we know that $f(-1) = 1$. Define a new function F on [-3, 3] by the formula

$$F(x) = \int_{-2}^{x} f(t)dt$$

   (a) Sketch a rough graph of f.
   (b) Find the value of $F(-2)$ and $F(2)$, and an upper and lower bound on $F(0)$.
   (c) Find the critical points and inflection points of F on [-3,3].
   (d) Sketch a rough graph of F on [-3,3].