1 Integration Basics

1. Using high-school geometry, compute the following:

   a) $\displaystyle\int_0^2 x\,dx$

   b) $\displaystyle\int_1^4 (1+3x)\,dx$

   c) $\displaystyle\int_{-1}^3 (1-x)\,dx$

   d) $\displaystyle\int_0^1 (1-x^2)^{1/2}\,dx$

   e) $\displaystyle\int_0^{1/2} (1-x^2)^{1/2}\,dx$

2. For this problem:

   (a) Give the definition of a definite integral.

   (b) State the Fundamental Theorem of Calculus.

   (c) What is wrong with the following argument?

$$\int_{-1}^2 \frac{1}{t^2}\,dt = \left.\frac{-1}{t}\right|_{-1}^2 = \frac{-1}{2} - \left(-\frac{1}{-1}\right) = -\frac{1}{2} - 1 = -\frac{3}{2}.$$

2 Integration Techniques

1. Let f be a continuous function.

   (a) Suppose that $\displaystyle\int_0^4 f(x)\,dx = 10$. Find $\displaystyle\int_0^2 f(2x)\,dx$.

   (b) Suppose that $\displaystyle\int_0^9 f(x)\,dx = 4$. Find $\displaystyle\int_0^3 xf(x^2)\,dx$.

2. Suppose that $\displaystyle\int_{-a}^a f(x)\,dx = c\int_0^a f(x)\,dx$.

   (a) If f is an odd function, then what is the value of c?

   (b) If f is an even function, then what is the value of c?

3. Compute the following integrals

   (a) $\displaystyle\int x\sin x\,dx$

   (b) $\displaystyle\int x\ln x\,dx$

   (c) $\displaystyle\int x\cos(3x)\,dx$

(d) $\displaystyle\int \frac{\ln x}{x^5}\,dx$

(e) $\displaystyle\int \ln x\,dx$

(f) $\displaystyle\int x\sqrt{x+3}\,dx$

(g) $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$

(h) $\displaystyle\int \tan^3 x\,dx$

(i) $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

3 Trigonometric Substitution

1. Integrate:

   (a) $\displaystyle\int x\sqrt{1-x^4}\,dx$

   (b) $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

4 Area Between Curves

1. An ellipse has the equation
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$
   (a) Set up an integral which equals the area of the portion of the ellipse in the first quadrant. (Set it up to integrate with respect to x.)

   (b) Now use the substitution $u = x/a$ and simplify the integral as much as you can.

   (c) Use geometry to evaluate the resulting integral. (In other words, what figure does this integral represent the area of?)

   (d) What is the formula for the area of an ellipse? Describe this in terms of the equation above and also in terms of the geometry of an ellipse.

2. Find the area of the region between the given curves:
$$y = \left|\cos\left(\frac{\pi x}{2}\right)\right|, \qquad y = -x+1.$$

5 Volumes of Revolution

1. Let R be the region bounded by the y-axis, the line $y = 4$ and the curve $y = \sqrt{x}$. Revolve R around the y-axis. Compute the volume by

   (a) the disk method (and draw an a representative disk).

(b) the shell method (and draw a representative shell).

(Verify that both methods give the same answer: $\frac{4^5\pi}{5}$.)

2. Let R be the region bounded by the y-axis, the line $x = 16$ and the curve $y = \sqrt{x}$. Revolve R around the x-axis. Compute the volume by

   (a) the disk method (and draw an a representative disk).

   (b) the shell method (and draw a representative shell).

   (Verify that both methods give the same answer: $\frac{4^4\pi}{2}$.)

3. Set up and evaluate an integral for the volume of the solid that results when the region in the first quadrant bounded by $y = 3 + 2x - x^2$, the x-axis, and the y-axis is revolved about

   (a) the x-axis

   (b) the y-axis

   (c) the line $y = -1$

   (d) the line $x = 4$.

   In each case state what method (disk or shell) you are using, and draw representative disks/shells.

4. Let R be the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = 6$. Set up (but do not evaluate) the integrals for finding this volume using the disk and shell methods.

6 Volumes Using Cross-Sections

1. Let the base of a solid be the first quadrant region bounded by $y = 1 - x^2/4$, the x-axis, and the y-axis. Suppose that cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

7 Work

1. A spring has a natural length of 10 cm. A 20 N force is required to stretch (and hold the spring) to a length of 20 cm. How much work is done in stretching the spring from 25 cm to 30 cm? Remember that Hooke's law says that $F(x) = kx$, where k is the spring constant.

2. We have a cable that weighs 10 lbs/ft attached to a bucket filled with coal that weighs 1000 lbs. The bucket is initially at the bottom of a 600 ft mine shaft. Answer each of the following about this.

   (a) Determine the amount of work required to lift the bucket to the midpoint of the shaft.

   (b) Determine the amount of work required to lift the bucket from the midpoint of the shaft to the top of the shaft.

   (c) Determine the amount of work required to lift the bucket all the way up the shaft.

8 Average Value of a Function

1. Let $f(x)$ be the distance from $(0,0)$ to the point (x,y) on the graph of $y=x^2$. What is the average value of $f(x)$ on the interval $[0,\sqrt{15}]$?

9 Improper Integrals

1. Determine whether the following improper integrals converge or diverge.

   (a) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx.$

   (b) $\displaystyle\int_1^3 \frac{1}{x-1}\,dx.$

   (c) $\displaystyle\int_1^e \frac{1}{x\ln x}\,dx.$

   Evaluate or establish the divergence or convergence of the following improper integrals:

   (a) $\displaystyle\int_{-\infty}^{-1} \frac{e^{-x}}{x}\,dx$

   (b) $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$

   (c) $\displaystyle\int_0^1 \left(1-x^2\right)^{-1/2}\,dx$

   (d) $\displaystyle\int_1^\infty \left(\frac{1}{x}-\frac{1}{x+1}\right)\,dx$

10 Sequences

1. Find the limits of the following sequences:

   (a) $a_n = \dfrac{n^2-2n+1}{2n^2+5}$

   (b) $b_n = \dfrac{n^3+5n}{3n^4-6}$

   (c) $c_n = \dfrac{n^2-5}{n+1}$

   (d) $d_n = \dfrac{\sin n}{n}$

2. Let $\{a_n\}$ be a sequence with limit -3. If seven is added to each of the first 5 billion terms of this sequence, what is the new limit?

3. Give examples of sequences $\{a_n\}$ that satisfy the following. If there is no such sequence, explain why.

   (a) $la_n = 0$ and $l1/a_n = \infty$;

(b) $la_n = 0$ and $l1/a_n$ does not exist;

(c) $la_n = \infty$ and $l1/a_n$ does not exist;

(d) la_n does not exist and $l1/a_n = 0$;

(e) la_n does not exist and $l1/a_n$ does not exist.

4. Consider the sequence $\{1, -1/2, 1/3, -1/4, \ldots\}$.

   (a) Find a formula for the general term a_n of the sequence.

   (b) Find the limit of this sequence as $n \to \infty$.

   (c) Do the same with the sequence $\{2/9, 6/27, 24/81, 120/243, \ldots\}$

5. Do the following sequences converge or diverge? Why? If they converge, what is the limit?

$$a_n = \frac{5^n}{2^n + 3^n} \qquad a_n = \frac{5^n}{2^n 3^n} \qquad a_n = \arctan\left(\frac{n!}{n^2}\right)$$

6. Find the limit of the sequence

$$\sqrt{2}, \quad \sqrt{2\sqrt{2}}, \quad \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

11 Summing a Series

1. Evaluate the following:

   (a) $\displaystyle\sum_{i=1}^{100} 2$

   (b) $\displaystyle\sum_{i=1}^{100} \left(\frac{1}{i+3} - \frac{1}{i+4}\right)$

   (c) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$ *Hint:* $\dfrac{1}{i} - \dfrac{1}{i+1} = ?$

   (d) $\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \ldots$

2. For what values of a, if any, does the sum

$$\sum_{n=1}^{\infty} \frac{a}{n+2} - \frac{1}{n+4}$$

converge?

12 Convergence of Series

1. Test each series for convergence or divergence.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n^2+n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} e^{-n^2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3-1}$

2. For what positive integers k does the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

converge?

3. Find the interval of convergence of the power series below.

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} n^n x^n.$$

13 Power Series

1. Recall that a power series centered around $x = a$ has the general form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

for some coefficients $c_n \in \mathbb{R}$. For each of the following power series, find the center a, the coefficients c_n, the radius of convergence R, and the interval of convergence. For the last part, don't forget to check the endpoints!

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{5^n}$

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x+4)^n}{n+2}$ (d) $\displaystyle\sum_{n=0}^{\infty} n!(x-1)^n$

2. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Find the intervals of convergence for f, f', and f''.

3. (a) Starting with the geometric series $\frac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, find the sum of the series

$$\sum_{n=1}^{\infty} n x^{n-1}, \quad |x| < 1$$

(b) Find the sum of the following series.

$$\sum_{n=1}^{\infty} n x^n, \ |x| < 1$$