1 Review

1. (a) Use a right triangle to compute the distance between (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$.
   (b) Use this formula to compute the equation of a circle centered at (a, b) with radius r.
   (c) Extend the above to compute the distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^3$.
   (d) Use this formula to compute the equation of a sphere centered at (a, b, c) with radius r.
2. (a) Find the equation of largest circle that lies in the square created by the lines $y = 1$, $y = 5$, $x = -3$ and $x = 7$.
   (b) Find the equation of largest sphere that lies in the box created by the planes $y = 1$, $y = 5$, $x = -3$, $z = 7$, $z = 0$, $z = 4$.
3. Graph the locus of points (x, y) satisfying the equations in $\mathbb{R}^2$. Do not solve for y. Instead find a few points that satisfy the formula and try to find the two-dimensional shape.
   (a) $xy = 0$
   (b) $x^2 + y^2 = 1$
   (c) $x^2 + y^2 = -1$
   (d) $x^2 + \frac{y^2}{4} = 1$
   (e) $x^2 - y^2 = 1$
   (f) $y^2 - x^2 = 1$
   (g) $|x| + |y| = 1$
   (h) $xy = 1$
   (i) $\max\{|x|, |y|\} = 1$
4. Write an equation in the variables x, y and z such that, if y is set to 0, the remaining formula gives a circle AND, if z is set to 0, the reamining formula gives a hyperbola.

2 Graphing in 3D

1. Identify these three dimensional surfaces
   (a) $x^2 + 3y^2 + z^2 = 12$
   (b) $x^2 + 2x + y^2 + 2y = 5$
   (c) $z^2 = x^2 + 3y^2$
   (d) $z = x^2 - 3y^2$
   (e) $z = x^2 + 5y^2$
   (f) $x^2 - y^2 + z^2 = 3$
   (g) $x^2 - 3y^2 + 3z^2 = -5$
   (h) $z = 3|x| + |y|^2$

2. Sketch contour diagrams (i.e. level sets) for the following functions with at least four labelled contours. Draw the intersection of these surfaces with the xz-plane and the yz-plane.
   (a) $z = -x^2 - y^2 + 1$
   (b) $z = \sqrt{x^2 + 2y^2}$
3. (a) Find a function $z = f(x, y)$ whose level surface at $z = 1$ is given by $y = \ln(x^2 + 3)$.
   (b) Find a function $w = f(x, y, z)$ whose level surface at $w = 3$ is a sphere centered at $(2, 4, 3)$ with radius 5.
4. Describe and sketch the graph of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \max\{|x|, |y|\}$ by drawing some level sets.
5. Let $v = (1, 5, 2)$ and $w = (1, 2, -4)$. Compute the following quantities or say why they cannot be computed.

3 Vectors

1. (a) Find a unit vector from the point $P = (1, 2)$ and toward the point $Q = (4, 6)$.
   (b) Find a vector of length 10 pointing in the same direction.
   (c) Find all vectors v in 2 dimensions having $||v|| = 5$ and for which the i-component is 3.
2. Which statements are true for vectors v and w in $\mathbb{R}^2$?
   (a) $||v + w|| = ||v|| + ||w||$
   (b) $||v - w|| = ||v|| - ||w||$
   (c) $||v - w|| = ||v|| + ||w||$
   (d) $||v + w||^2 = ||v||^2 + 2||v||||w|| + ||w||^2$
   (e) $||v - w||^2 = ||v||^2 - 2||v||||w|| + ||w||^2$
   (f) $v \cdot v = v^2$
3. If possible, correct any false equations in Problem 2.
4. (a) If v and w are non-parallel vectors both originating from the origin, what is the vector originating at the tip of v and ending at the tip of w?
   (b) Write down the law of cosines for a triangle with side lengths a, b and c and angle θ opposite the side c.
   (c) Apply the law of cosines to the triangle in (a) to derive the equation $v \cdot w = ||v||||w|| \cos\theta$.
5. Let x and y be vectors in $\mathbb{R}^n$. Show the following identities.
   (a) $||x + y||^2 - ||x - y||^2 = 4x \cdot y$.
   (b) $||x - y||||x + y|| \leq ||x||^2 + ||y||^2$. Hint: square both sides and prove this new inequality.
6. (a) Let a, b, c be vectors in $\mathbb{R}^n$. Show that the vector $(b \cdot c)\, a - (a \cdot c)\, b$ is perpendicular to c.
   (b) Let a and b be nonzero vectors in $\mathbb{R}^n$. Suppose that the vector $v = ||a||b + ||b||a$ is nonzero. Prove that v bisects the angle between a and b. Hint: Show that the angle between a and v is the same as the angle between b and v.

7. Let $v = (1, 5, 2)$ and $w = (1, 2, -4)$. Compute the following quantities or say why they cannot be computed.

   (a) $||v||$
   (b) $||v - w||$
   (c) $v \cdot w \cdot v$
   (d) $(v \cdot w)v$
   (e) $(v \cdot w)(w \cdot w)$
   (f) $v \times w$
   (g) $(v \cdot w) + (v \times w)$

8. Let $v = (1, 2, 3)$. What is the vector pointing in the opposite direction of v? Find three vectors that are perpendicular to v.

4 Limits

1. Compute the following limits.

   (a) $\displaystyle\lim_{(x,y,z)\to(0,1,0)} \frac{\tan^{-1}(x+y+z)}{\sin^{-1}(x+y+z)}$
   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos(x^2+y^2)-1}{x^2+y^2}$
   (c) $\displaystyle\lim_{(x,y)\to(0,0)} (\sin(x^2+y^2))^{x^2+y^2}$

2. Compute the following limits or show that they do not exist.

   (a) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{2x^2+y^2}$
   (b) $\lim_{(x,y)\to(0,0)} \frac{xy^3)}{x^2+y^2}$

3. Construct a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ which is discontinuous at only $(1, 2)$ with a removable singularity there.

4. Consider the function given by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Show that $f(0, y)$ and $f(x, 0)$ are both continuous functions of one variable.
   (b) Determine with argument whether f is continuous at $(0, 0)$. If not, determine whether the discontinuity is removable.

5 Partial Derivatives

1. Find the first-order partial derivatives of the following functions.
   (a) $f(x, y) = x^4 + 6\sqrt{y} - 10$
   (b) $f(x, y) = \cos\left(\frac{4}{x}\right) e^{x^2 y - 5y^3}$

2. Find all second-order partial derivatives for $f(x, y) = \cos(2x) - x^2 e^{5y} + 3y^2$.

3. The wind-chill index is modeled by the function
$$W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16},$$
where T is the temperature (in °C) and v is the wind speed (in km/hr). When $T = -15$ and $v = 30$, by how much would you expect the apparent temperature to drop if the actual temperature decreases by 1 °C? What if the wind speed increases by 1 km/hr?

4. Compute $\dfrac{dz}{dx}$ for $z = x\ln(xy) + y^3$, $y = \cos(x^2 + 1)$.

6 Gradients and Directional Derivatives

1. Let $f(x, y) = x^2 + y^2$.
   (a) Sketch the level curves of f.
   (b) Find the gradient vector field $\nabla f(x, y)$.
   (c) Sketch the gradient vector field on the same picture that you drew the level curves. You will notice that the gradient vector field is perpendicular to the level curves of f.

2. The perpendicular property in the previous problem is true in general. Here's a proof. Let $w = f(x, y)$ and $P = (x_0, y_0)$ be a point on the level curve $f(x, y) = c$ (so that $f(x_0, y_0) = c$)).
   (a) Let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ be a curve on the level set $f(x, y) = c$ with $\mathbf{r}(t_0) = \langle x_0, y_0\rangle$, and let $g(t) = f(x(t), y(t))$. Explain why $g(t) = c$.
   (b) Find $\frac{dg}{dt}$ using the chain rule, and then explain why the expression you obtain must be equal to zero.
   (c) Use the equation from part (b) to prove that
$$\nabla f|_P \cdot \mathbf{r}'(t_0) = 0.$$
   It follows that the gradient vector is perpendicular to any tangent vector to the level curve $f(x, y) = c$.

3. Find the directional derivatives:
   (a) $D_{\mathbf{u}} f(x, y)$ for $f(x, y) = xy + y^2$ and in the direction of $\mathbf{v} = \langle 1, 1\rangle$
   (b) $D_{\mathbf{u}} f(x, y)$ for $f(x, y) = x\cos(y)$ and in the direction of $\mathbf{v} = \langle 2, 1\rangle$

4. Prove the following theorem:

   Theorem. The maximum value of $D_{\mathbf{u}} f(x, y)$ is $||\nabla f(x, y)||$ and occurs in the direction given by $\nabla f(x, y)$.

   (HINT: Use $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos\theta$.)

7 Tangent Planes

1. Find the equation of the tangent plane to $z = \ln(2x + y)$ at $(-1, 3)$.
2. Sketch the graph of $x^2 + y^2 + z^2 = 30$. Then, find the tangent plane and normal line (line perpendicular to) to $x^2 + y^2 + z^2 = 30$ at the point $(1, -2, 5)$.

8 Extrema of Multivariable Functions

1. Find and classify all critical points of $f(x, y) = 4 + x^3 + y^3 - 3xy$.
2. If $x + y + z = 1$, find the values of x, y, z so that $x^2 + 2y^2 + 3z^2$ is a maximum.
3. Find the absolute max and absolute min of $f(x, y) = x^2 + 4y^2 - 2x^2y + 4$ on the rectangle given by $-1 \le x \le 1$ and $-1 \le y \le 1$.

9 Optimization and Lagrange Multipliers

1. Determine the point on the plane $4x - 2y + z = 1$ that is closest to $(-2, -1, 5)$.
2. Find the maximum and minimum of the following functions subject to the given constraint.
   (a) $f(x, y) = 3x - 2y$ subject to $x^2 + 2y^2 = 44$
   (b) $f(x, y) = x + 3y$ subject to $x^2 + y^2 \le 2$
   (c) $f(x, y, z) = x^2 - 2y + 2z^2$ subject to $x^2 + y^2 + z^2 = 1$
3. Find the dimensions of the box with the largest volume if the total surface area is 64 ft^2.

10 Multiple Integrals

1. For the following, sketch the region of integration and evaluate the integral.
   (a) $\displaystyle\int_1^3 \int_0^4 e^{x+y}\, dy\, dx$
   (b) $\displaystyle\int_1^4 \int_{\sqrt{y}}^{y} x^2y^3\, dx\, dy$
   (c) $\displaystyle\int_0^3 \int_0^{2x} (x^2 + y^2)\, dy\, dx$
   (d) $\displaystyle\int_{-2}^0 \int_{-\sqrt{9-x^2}}^{0} 2xy\, dy\, dx$
2. Reverse the order of integration and evaluate $\displaystyle\int_0^1 \int_y^1 e^{x^2}\, dxdy$
3. Sketch the region of integration and evaluate the integrals by reversing the order.

(a) $\int_0^1 \int_y^1 \cos(x^2)\, dx\, dy$

(b) $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3}\, dx\, dy$

(c) $\int_0^1 \int_{e^y}^e \frac{x}{\ln x}\, dx\, dy$

4. Compute the integral $\int\int_R \sin(x^2+y^2)\, dA$ where R is the disk of radius a around the origin.

5. Compute the volume above the plane $z = 2$ and inside the sphere $x^2 + y^2 + z^2 = 9$.

6. Sketch the region of integration and evaluate the integrals.

   (a) $\iiint_E 2x\, dV$, where E is the region under the plane $2x+3y+z = 6$ that lies in the first octant.

   (b) Determine the volume of the region that lies behind the plane $x + y + z = 8$ and in front of the region in the yz-plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$.

   (c) Evaluate $\iiint_E \sqrt{3x^2+3z^2}\, dV$, where E is the solid bounded by $y = 2x^2 + 2z^2$ and the plane $y = 8$.

   (d) Evaluate $\iiint_E y\, dV$, where E is the region that lies below the plane $z = x + 2$ above the xy-plane and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

7. Convert these integrals to spherical or cylindrical coordinates and evaluate. Sketch the region of integration.

   (a) $\iiint_R \sin(x^2+y^2)\, dV$, where R is the solid cylinder with height 4 and with base of radius 1 centered on the z-axis at $z = -1$.

   (b) $\iiint_R (x^2+y^2+z^2)^{-1/2}\, dV$ over the bottom half of the sphere of radius 5 centered at the origin.

8. Evaluate $\iiint_B \sqrt{\frac{x^2+y^2+z^2}{2+x^2+y^2+z^2}}\, dx\, dy\, dz$ where B is the unit ball. You will have to conduct an integration by parts.

11 Vector Fields and Flows

1. Sketch the vector fields $\mathbf{F}(x, y) = (0, -y)$ and $\mathbf{F}(x, y) = (y, -x)$.

2. Check that the curve $c(t) = (ae^t, be^{-t})$ is a flow line for the vector fields $\mathbf{F}(x, y) = (x, -y)$. Sketch both the field and a few flow lines. Here a and b are constants.

3. (a) Sketch the vector fields $\mathbf{F}(x, y) = (0, -y)$ and $\mathbf{F}(x, y) = (y, -x)$.

(b) A vector field $\mathbf{F}$ in $\mathbb{R}^2$ has all vectors pointing at 45 degrees north of east. The magnitudes increase as x and y get larger. Give a possible equation for such a vector field.

4. In each of the following, check that the given curve is a flow line for the given vector fields. Sketch both the field and a few flow lines, including the direction of the flow. Here a and b are constants.

   (a) $\mathbf{F}(x, y) = (y, -x)$ and $c(t) = (a\cos t, -a\sin t)$

   (b) $\mathbf{F}(x, y) = (x, -y)$ and $c(t) = (ae^t, be^{-t})$

12 Line Integrals of Vector Fields

1. In the following, compute the line integral $\int_c \mathbf{F} \cdot d\mathbf{r}$ for the given $\mathbf{F}$ and c.

   (a) $\mathbf{F}(x, y) = (x, y)$ and c is the line from $(0, 0)$ to $(3, 3)$.

   (b) $\mathbf{F}(x, y) = (2y, -\sin y)$ and c traverses the unit circle counterclockwise starting at $(1, 0)$.

   (c) $\mathbf{F}(x, y) = (-y, x, 5)$ and c is the helix $c(t) = (\cos t, \sin t, t)$ for $0 \le t \le 4\pi$.

2. Let $\mathbf{F}(x, y, z) = (z^3 + 2xy, x^2, 3xz^2)$. Show that the line integral of $\mathbf{F}$ around the perimeter of the unit square with vertices $(\pm 1, \pm 1, 0)$ is zero. Traverse the square counterclockwise. Hint: Break up the path into four segments and calculate four integrals.

3. Let $\mathbf{F}(x, y) = (3x - y, x)$ and consider the two paths $c_1(t) = (t, t^2)$ and $c_2(t) = (t, t)$, both with $0 \le t \le 1$. Sketch these two paths, showing in particular that they have the same endpoints. Show however that the line integrals $\int_{c_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{c_2} \mathbf{F} \cdot d\mathbf{r}$ are different.

4. Determine which of the following vector fields $\mathbf{F}$ is the gradient of a scalar function f. If such an f exists, compute it.

   (a) $\mathbf{F}(x, y) = (xy, xy)$

   (b) $\mathbf{F}(x, y) = (2x\cos y + \cos y, -x^2\sin y - x\sin y)$

   (c) $\mathbf{F}(x, y, z) = (2xyz + \sin x, x^2z, x^2y)$

5. In each of the following problems, verify that the hypotheses of the Fundamental Theorem of Line Integrals are satisfied and use it to compute $\int_c \mathbf{F} \cdot d\mathbf{r}$.

   (a) $\mathbf{F}(x, y) = (x + 2, 2y + 3)$ and c is the line from $(1, 0)$ to $(3, 1)$ in the xy-plane

   (b) $\mathbf{F}(x, y) = (y\sin(xy), x\sin(xy))$ and c is the parabola $y = 2x^2$ from $(1, 2)$ to $(3, 18)$

6. A particle subject to a force $\mathbf{F}(x, y) = (y, -x)$ along some path beginning at $(-1, 0)$ and ending at $(0, 1)$.

   (a) Argue that $\mathbf{F}$ is not conservative.

   (b) Find two paths c_1 and c_2 from $(-1, 0)$ to $(0, 1)$ such that the work done in moving the particle along c_1 is different that that done in moving the particle along c_2.

7. Consider the vector field $\mathbf{F}(x, y) = (0, x)$.

(a) Decide with justification whether $\mathbf{F}$ is a gradient field.

(b) If possible, find paths c_1, c_2 and c_3 from $(-1, 0)$ to $(0, 1)$ such that (i) $\int_{c_1} \mathbf{F} \cdot d\mathbf{r} = 0$, (ii) $\int_{c_2} \mathbf{F} \cdot d\mathbf{r} > 0$, (iii) $\int_{c_3} \mathbf{F} \cdot d\mathbf{r} < 0$.

8. Let (a, b) be a fixed point in $\mathbb{R}^2$ and let $\mathbf{F}(x, y) = \nabla \log ||(x, y) - (a, b)||^2$. Let c be the circular path traversed counterclockwise with radius $r > 0$ and center (a, b). Show that, even though $\mathbf{F}$ is not defined at (a, b), the line integral $\int_c \mathbf{F} \cdot d\mathbf{r}$ is still zero.