Purely group-theoretic generalizations of the ham sandwich theorem (any $d$ masses on $\mathbb{R}^d$ can be simultaneously bisected by a single hyperplane) where obtained, with measures “balanced” by linear representations of a finite group. For finite subgroups $G$ of the spheres of the algebras $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we showed that for any $d$-tuple $\mu$ of $F$-valued measures on $F^d$ and any free $d$-dimensional $F$-unitary representation $\rho$, there exists a canonical $G$-Voronoi decomposition $\{V_g\}_{g \in G}$ with vanishing “$(\rho, G)$-average” $\sum_{g \in G} \rho^{-1}(\mu(V_g)) \in F^d$. As a corollary, any $d$ measures on $\mathbb{R}^d(p-1)$ could be equipartitioned by a single regular $p$-fan with center a complex hyperplane, $p$ an odd prime.

Harmonic analysis on finite groups was introduced to mass partition theory, with equipartitions obtained as the annihilation of prescribed Fourier transforms. Famous problems considered by various authors (e.g., Grünbaum hyperplane equipartitions [6]) were shown to be particular cases of this perspective, which was in turn shown to be a special case of the “$G$-balancing” of [1]. While full equipartitions obtained by equivariant topology have held only for a prime-power number of regions, a careful selection of transforms here produced optimal non-prime-power results, such as those of a “modulo-type” for collections of complex regular fans $F_{q_1}, \ldots, F_{q_k}$ with arbitrary $q_j$.

We extended the “Fourier perspective” of [2] to the compact Lie group setting. For the circle group, the annihilation of transforms produced measure center-transversality similar in spirit to the classical “center-point” theorem (for any mass on $\mathbb{R}^d$, there exists a point $p$ such that any half-space containing it must hold at least $1/(d + 1)$-th of the total mass): for any $q \geq 2$, there exists a complex hyperplane whose surrounding regular $q$-fans are close – in the $L^2$-norm in general and uniformly under further assumptions – to equipartitioning a given set of masses. This result represented the first use of continuous instead of finite group actions in topological combinatorics.

The long-standing topological Tverberg conjecture claimed, for any continuous map from the boundary of a $N := (q - 1)(d + 1)$-simplex to $d$-dimensional Euclidian space, the existence of $q$ pairwise disjoint faces whose images have non-empty $q$-fold intersection. Though true for all prime powers, counterexamples in non-prime-power cases were shown to exist in 2015. Considering maps below the tight dimension $N$, we showed that for arbitrary $q$ one can nonetheless guarantee collections of $q$ pairwise disjoint faces which satisfy a variety of optimal average-value coincidences stemming from the annihilation of carefully chosen finite Fourier transforms.
A wide variety of results in topological combinatorics are ultimately guaranteed by the preclusion of sections of fiber bundles arising naturally in the context of a geometric partitioning problem. This brief note established measure partitions which arise as a direct consequence of the existence of equivariant sections of real or complex Stiefel Bundles. For instance, the triviality of the circle bundle $S^1 \hookrightarrow U(2)/\mathbb{Z}_q \rightarrow L^3(q)$ over Lens spaces implied that any mass on $\mathbb{C}^2$ could be equipartitioned by each of a pair of complex orthogonal $q$-fans, $q = 3, 4$.


A famous problem, dating back to Grünbaum, seeks the minimum dimension $d := \Delta(m, k)$ such that any $m$ masses on $\mathbb{R}^d$ can be equipartitioned by $k$ affinely independent hyperplanes. Our main result shows that the best general upper bound thus far $- \Delta(2^q + r, k) \leq 2^{k-r} + r, 0 \leq r < 2^q$ holds for orthogonal hyperplanes as well if $2^{k-2}r < 2^q - 1$. We now aim for upper bounds to a “Makeev” generalization – finding the minimal dimension $d := \Delta^\perp(m, k, \ell)$ for which equipartition occurs by any $\ell \leq k$ of some collection of $k$ pairwise orthogonal hyperplanes – which would interpolate between our $\ell = k$ result and our $\ell = 2$ bound $\Delta^\perp(2q + r, k, 2) \leq k \cdot 2q + r, r < 2q - 1$.


This paper consists of the measure partition results obtained by my research students during the 2015 Summer Research Program at Wellesley College. Using the “Fourier method” of [2], they obtained upper bounds of a “Makeev” variety for the minimum complex dimension $d := \Delta(p; m, k, 2)$ for which any $m$ masses on $\mathbb{C}^d$ can be equipartitioned by any two of $k$ affinely independent complex regular $p$-fans if $p$ is an odd prime. Further upper bounds are expected over the course of this academic year.

Average-value and $L^2$-analogues of the colored Tverberg–Vrećica conjecture, in preparation.

A well-known transversality result of Vrećica and Živaljević – for any $d - k + 1$ masses on $\mathbb{R}^d$, there exists a $(d - k)$ flat $A$ such that any half-space containing $A$ must hold at least $1/(k + 1)$ of each total mass – links the ham sandwich theorem and the center-point theorem when $k = 1$ and $k = d$, respectively. Perhaps the most famous conjecture of topological combinatorics seeks to unify (colored) Tverberg-type partitions with this affine transversality. We are currently working on extensions of our $L^2$ center-transversality results for complex hyperplanes [3] to all complex flats – i.e., a $(d - k)$-dimensional flat $A$ such that any complex regular $q$-fan with center containing $A$ must be $L^2$-close to an equipartition, along with connections between our “average-value” Tverberg results [4] and their colored analogues in the case of point collections.