Bounded rigidity of manifolds and asymptotic dimension growth

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Let X be a metric space and define the category $\mathcal{B}(X)$ of *manifolds bounded over* X in the usual way [FW]. The objects of $\mathcal{B}(X)$ are manifolds M equipped with proper[2] maps $p\colon M \to X$. These maps p need not be continuous. A morphism $(M_1, p_1) \to (M_2, p_2)$ between objects of $\mathcal{B}(X)$ is a continuous map $f\colon M_1 \to M_2$ which is *bounded over* X in the sense that there exists $k > 0$ for which $d(p_1(m), p_2(f(m))) < k$ for all $m \in M_1$. We can additionally form the *bounded homotopy category over* X by insisting that all relevant maps and homotopies be bounded over the metric space X. Of course, the bounded category is interesting only when X is a space of infinite diameter.

If M is an n-dimensional noncompact manifold, one can construct a structure set $S^{bdd}(M)$ whose elements are homotopy equivalences $M' \to M$ bounded over M. As in the compact case, there are bounded L-groups $L_*^{bdd}(M)$ fitting into the exact sequence

$$H_{n+1}(M; \mathbb{L}(e)) \to L_{n+1}^{bdd}(M) \to S^{bdd}(M) \to H_n(M; \mathbb{L}(e)) \to L_n^{bdd}(M).$$

If $S^{bdd}(M)$ is trivial, we say that M is *boundedly rigid*; i.e. if $f\colon M' \to M$ is a bounded homotopy equivalence, then f is boundedly homotopic to a homeomorphism. Results of the so-called *bounded Borel type* can be found in [C], [PW], [FP]. See also [S], [PR], [Ra], [W] and [WW]. One recent theorem of Chang and Weinberger [CW] proves that arithmetic manifolds (those of the form $\Gamma\backslash G/K$ where Γ is an arithmetic lattice in a real connected linear Lie group G) are boundedly rigid, even though they are in general not properly rigid if the rational rank of Γ exceeds 2. One can view bounded rigidity as topological rigidity in the category of continuous coarsely Lipschitz maps.

Let M be a noncompact manifold. We say that M is *uniformly contractible* if there is a function $f\colon (0, \infty) \to (0, \infty)$ such that, for each $x \in X$ and $t > 0$, the ball $B(x, t)$ is contractible in the ball $B(x, f(t))$. This definition arises from the natural notion of the *bounded fundamental group* in this bounded context. See [W] or [CW] for details. For such uniformly contractible manifolds, Ferry and Pedersen [FP] have shown that two basic principles of surgery theory, the product theorem and the π-π theorem, still hold for bounded surgery problems. It is natural then to consider the rigidity properties of uniformly contractible spaces. In this paper we prove a rigidity theorem for such spaces that satisfy a particular asymptotic dimension condition.

The *asymptotic dimension* of a metric space X is the smallest integer n such that, for any $r > 0$, there exists a uniformly bounded cover $C = \{C_i\}_{i \in I}$ for which no ball of radius r in X intersects more than $n + 1$ members of C. This notion was introduced by Gromov in [G]. As mentioned in [Y1], the notion of asymptotic dimension is a coarse geometric analogue of the covering dimension in topology, invariant under quasi-isometries. It is well known that any finitely generated subgroup of Gromov's hyperbolic groups has finite asymptotic dimension as metric spaces with word-length metrics. For spaces whose asymptotic dimension is not necessarily finite, we can formulate a more precise notion.

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[2] Here we use the term *proper* loosely. The map p does not have to be continuous, but the inverse image of bounded sets should be bounded.

Definition: Let X be a metric space. For every $r > 0$, let $f(r)$ be the infimum over all $n \in \mathbb{N}$ for which there is a uniform bounded cover $C = \{C_i\}_{i \in I}$ such that, for every $x \in X$, the ball $B(x, r)$ intersects at most $n + 1$ members of C. We will call $f\colon (0, \infty) \to \mathbb{R}$ the *asymptotic dimension growth* of X.

It is clear that the asymptotic growth of any metric space X is nondecreasing in r and is eventually constant iff X has finite asymptotic dimension.

Theorem: Let X be a uniformly contractible manifold with bounded geometry whose asymptotic growth f satisfies $f(r) = o(\ln r)$. Then X is boundedly rigid.

This paper is inspired by two results. In [Y1] Yu proves that the coarse Baum-Connes conjecture holds for proper metric spaces with finite asymptotic dimension. The coarse Baum-Connes conjecture for a space X states that, if $\{C_k\}$ is an anti-Cech system of locally finite and uniformly bounded covers for X and $N(C_k)$ is the associated nerve space endowed with the spherical metric for each k, then the induced map

$$\lim_{k \to \infty} K_i(N(C_k)) \to K_i(C^*(X))$$

is an isomorphism. Here $C^*(X)$ is the usual Roe algebra of locally compact operators on X with finite propagation speed. With the descent principle, one can prove that the (strong) Novikov conjecture holds for Γ if (1) Γ is a finitely generated group for which $B\Gamma$ has the homotopy type of a CW complex and (2) Γ has finite asymptotic dimension as a metric space with the word-length metric. The coarse Baum-Connes conjecture also implies the Novikov conjecture on homotopy invariance of higher signature, the Gromov-Lawson-Rosenberg conjecture for $K(\pi, 1)$ manifolds and the zero-in-the-spectrum for uniformly contractible Riemannian manifolds with bounded geometry and finite asymptotic dimension. Our paper is an extended L-theoretic analogue of Yu's theorem. We should mention that the Novikov conjecture in algebraic K-theory was proved for the groups with finite asymptotic dimension in [B] and [CG]. In [B] the L-theory analogue was stated for the case of finite asymptotic dimension (without proof). Dranshnikov has recently proved that the polynomial asymptotic dimension growth condition implies property A [D] and hence the coarse Baum-Connes conjecture by a result in [Y2].

Our other source of inspiration was the development of "squeezing structures," i.e. an ε-δ surgery exact sequence for noncompact manifolds with bounded geometry. We recall that a connected locally finite polyhedron B with a metric bilipschitz equivalent to the barycentric metric is said to have *bounded geometry* if there is a finite bound on the number of vertices of B adjacent to a given vertex [F]. Classically, a smooth manifold M has bounded geometry if it has bounded sectional curvature and a lower bound on the injectivity radius. Under various conditions on the dimension of the manifold M and the control $p\colon M \to B$, the ε-structure set $S_\varepsilon(M \to B)$ is stable for all sufficiently small $\varepsilon > 0$. Using such notions of squeezing, we shall prove the bounded rigidity of uniformly contractible manifolds with slow asymptotic growth by establishing that the assembly map $A\colon H_n(M; \mathbb{L}(e)) \to L_n^{bdd}(M)$ is an isomorphism. We will also exploit a Mayer-Vietoris sequence of [RY] for these controlled surgery groups.

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I. Bounded surgery

Let M be a metric space and let X be a CW-complex. Consider a map $p\colon X \to M$, not necessarily continuous, with the property that there is a positive integer k and a covering $\{U_\alpha\}$ of X such that $\operatorname{diam} p(U_\alpha) < k$ for all α. If k can be chosen so that $\operatorname{diam} p(C) < k$ for each cell of X, then we say that (X,p) is a *bounded CW complex* over M. We say that the pair (X,p) is *0-connected* if, for every $d > 0$, there exists k so that, if $x, y \in X$ and $d(p(x), p(y)) \leq d$, then x and y may be joined by a path in X whose image in M has diameter less than k. It is *(-1)-connected* if there is k such that, for each $m \in M$, there is $x \in X$ such that $d(p(x), m) < k$. The following definitions may be found in [FP].

Definition: Let (X,p) be a bounded CW complex over the metric space M, and let $p\colon X \to M$ be 0-connected.

1. We say that (X,p) has *trivial bounded fundamental group* if, for each $d > 0$, there is k so that for every loop $\alpha\colon S^1 \to X$ with $\operatorname{diam} p \circ \alpha(S^1) < d$, there is a map $\overline{\alpha}\colon D^2 \to X$ so that $\operatorname{diam} p \circ \overline{\alpha}(D^2) < k$.
2. We say that (X,p) has *bounded fundamental group* π if there is a π-cover $\widetilde{X}$ such that the map $\widetilde{\pi}\colon \widetilde{X} \to M$ gives rise to a bounded CW complex $(\widetilde{X}, \widetilde{\pi})$ over M which has trivial bounded fundamental group.

One of the main results of Ferry and Pederson is a surgery result of the following sort. Consider a bounded surgery problem

$$(M^n, \partial M) \to (X, \partial X) \to Z,$$

where $\partial M \to \partial X$ is a bounded (simple) homotopy equivalence, where X is 0- and (-1)-connected with bounded fundamental group π and $n \geq 5$. There is then a geometrically defined L-group $L_n^{bdd}(M, \mathbb{Z}\pi)$ such that one can do surgery relative boundary to produce a bounded (simple) homotopy equivalence iff an invariant in $L_n^{bdd}(M, \mathbb{Z}\pi)$ vanishes. Moreover every element in $L_n^{bdd}(M, \mathbb{Z}\pi)$ is realized as the obstruction problem with target $N \times I$ and homotopy equivalence on the boundary for an arbitrary $(n-1)$-dimensional manifold $N \to Z$ which is 0- and (-1)-connected with bounded fundamental group π.

Let M be a uniformly contractible noncompact manifold with asymptotic growth f and bounded geometry. Let the identity $p\colon M \to M$ be the control map. We want to show that, under various conditions on f, the manifold M is boundedly rigid, i.e. the bounded structure set $S^{bdd}(M \to M)$ contains only one element. Notice that uniform contractibility implies that the bounded fundamental group is trivial. The bounded surgery exact sequence is given by

$$\cdots \to L_{n+1}^{bdd}(M, \mathbb{Z}) \to S^{bdd}(M \to M) \to H_n(M, \mathbb{L}(e)) \to L_n^{bdd}(M, \mathbb{Z}).$$

We say that M is *boundedly rigid* if its bounded structure set $S^{bdd}(M \to M)$ contains only one element; that is, if $f\colon M' \to M$ is a bounded homotopy equivalence, then f is boundedly homotopic to a homeomorphism. The following sections are devoted to validate the bounded rigidity, i.e. the bijectivity of the assembly map $H_n(M, \mathbb{L}(e)) \to L_n^{bdd}(M, \mathbb{Z})$, for manifolds M whose asymptotic dimension growth f satisfies $f(r) = o(\ln r)$.

II. The nerve space

Consider a locally finite and uniformly bounded cover $C = \{\mathcal{O}_\alpha\}$ for a proper metric space X. Construct a simplicial complex $N(C)$ whose vertex set is C and for which a finite subset $\{\mathcal{O}_{k_1}, \ldots, \mathcal{O}_{k_n}\} \subseteq C$ spans an n-simplex in $N(C)$ if and only if the intersection $\bigcap_{i=1}^n \mathcal{O}_{k_i}$ is nonempty. This complex $N(C)$, called the *nerve space* of the cover C, can be endowed with the so-called *spherical metric*, whereby every simplex $\{\mathcal{O}_{k_1}, \ldots, \mathcal{O}_{k_n}\}$ in $N(C)$ is identified with the upper n-hemisphere in $\mathbb{R}^{n+1}$ by the correspondence

$$\sum_{i=1}^n t_i \mathcal{O}_{k_i} \longmapsto \left(\sum_{i=1}^n t_i^2\right)^{-1/2} (t_1, \ldots, t_n),$$

where $\sum_{i=1}^n t_i = 1$ and $t_i \geq 0$ for all i. The standard spherical metric on each simplex of $N(C)$ is defined to be the metric induced from the standard spherical metric on S^n, and the spherical metric on $N(C)$ is defined as the maximal metric whose restriction to each simplex is the standard spherical metric. The distance between two points is infinite if and only if they lie in different connected components of $N(C)$.

Let M be a manifold with bounded geometry. By Lemma 3.7 of [HR], the asymptotic dimension growth f of such a manifold is always defined, so that there is a sequence of open covers $\{C_k\}_{k=1}^\infty$ such that each $C_k = \{\mathcal{O}_{k,\ell}\}_{\ell=1}^\infty$ is an open cover of M satisfying the following conditions (these conditions may be somewhat relaxed):

1. For each positive integer k, there is a positive real number R_k such that $\sup_\ell \operatorname{diam} \mathcal{O}_{k,\ell} < R_k$ with R_k increasing and tending to infinity.

2. If $k' > k$, each member of C_k is contained in some member of $C_{k'}$.

3. For each $k \in \mathbb{Z}_{\geq 1}$, if $x \in M$, then $B(x, k)$ intersects with at most $d_k + 1$ members of C_k, where $d_k = f(k)$ and f is the asymptotic dimension growth.

4. For all $x \in M$ and $r > 0$, there is K such that, for each $k \geq K$, there is ℓ such that $B(x, r) \subseteq \mathcal{O}_{k,\ell}$ (i.e. every ball in M is eventually wholly contained in some open set for all sufficiently large covers).

Properties 1 and 2 imply that the sequence $\{C_k\}_{k=1}^\infty$ is an anti-Cech system of M. See [R] and [Y1]. As a result, for every pair $k_2 > k_1$, there is a simplicial map $i_{k_1,k_2} \colon N(C_{k_1}) \to N(C_{k_2})$ which maps a simplex $\{\mathcal{O}_1, \ldots, \mathcal{O}_n\} \in N(C_{k_1})$ to a simplex $\{\mathcal{O}'_1, \ldots, \mathcal{O}'_n\} \in N(C_{k_2})$ satisfying $\mathcal{O}_i \subseteq \mathcal{O}'_i$ for all i. Our theorem reduces to the following claim.

Claim: Let $N(C_k)$ be the nerve space associated to C_k endowed with the spherical metric. Note that $\dim N(C_k) \leq d_k$. If $d_k = o(\ln k)$, then

$$\lim_{k\to\infty} H_n(N(C_k), \mathbb{L}(e)) \to \lim_{k\to\infty} L_n^{bdd}(N(C_k))$$

is an isomorphism.

Let M be a uniformly contractible manifold with bounded geometry. Let $C = \{\mathcal{O}_k\}_k$ be a uniformly bounded open cover that is uniformly locally finite, i.e. $\sup_k \operatorname{diam} \mathcal{O}_k < \infty$. We wish to construct maps $\phi \colon M \to N(C)$ and $\psi \colon N(C) \to M$ which are "nearly" homotopy inverses.

1. Let $\{\chi_k\}$ be a partition of unity subordinate to $\{\mathcal{O}_k\}$. Define, for all $x \in M$, the quantity $\phi(x) = \sum_k \chi_k(x)\mathcal{O}_k \in N(C)$.

2. We now use the uniformly contractibility of M to construct a map $\psi\colon N(C) \to M$ by the following process. Let $f\colon C \to M$ be a map for which $f(\mathcal{O}_k) \in \mathcal{O}_k$. Define ψ inductively on the dimension i of the skeleton $N(C)^{(i)}$. When $i = 0$, let $\psi = f$. Assume that we have defined $\psi\colon N(C)^{(i-1)} \to M$. Let $[\mathcal{O}_{k_1}, \dots, \mathcal{O}_{k_i}]$ be an i-simplex. Extend ψ to $[\mathcal{O}_{k_1}, \dots, \mathcal{O}_{k_i}]$ from $\psi|_{boundary}$ such that, for all $j = 1, \dots, i$ and $x \in [\mathcal{O}_{k_1}, \dots, \mathcal{O}_{k_i}]$, we have $d(\psi(x), \psi(\mathcal{O}_{k_j})) < R$ for some uniform R.

Note that $\psi \circ \phi\colon M \to M$ is homotopic to the identity function on M but $\phi \circ \psi\colon N(C) \to N(C)$ may not be homotopic to the identity on $N(C)$, although it moves points at most a distance R. Notice in particular that the map $\psi\colon N(C) \to M$ is a homotopy domination.[3] If the cover C' is very small, then $\psi \circ \phi\colon M \to M$ is very close to the identity and may be homotopically connected to the identity geodesically. For larger covers C, we consider a sufficiently large cover C' subordinating C and the composition $\phi\colon M \to N(C) \to N(C')$ so that $\psi \circ \phi$ is still homotopic to the identity on M.

III. Controlled L-theory

At this point, we must appeal to controlled ε-δ surgery theory, particularly surgery over the ring $\mathbb{Z}$. In this section we will offer a brief overview and important theorems taken from [F] and [RY], where a more complete treatment of the material may be found.

Definition: For an integer $n \geq 0$, a pair of nonnegative numbers $\delta \geq \varepsilon \geq 0$, and a ring with involution R, the set $L_n^{\delta,\varepsilon}(X, Y, p_X, R)$ contains the equivalence classes of finitely generated n-dimensional ε-connected ε-quadratic complexes on p_X that are ε-Poincaré over $X - Y$, where X is a fixed metric space and Y is a subset of X. The equivalence relation is generated by finitely generated δ-connected δ-cobordisms that are δ-Poincaré over $X - Y$. Here p_X denotes a fixed control map from a space M to the metric space X. For the precise definitions of these δ-ε concepts, please refer to [RY, pages 3–5].

Notation: We use the following abbreviations proposed by [RY page 9]:

1. $L_n^{\delta,\varepsilon}(X, p_X, R) = L_n^{\delta,\varepsilon}(X, \emptyset, p_X, R)$

2. $L_n^{\varepsilon}(X, Y, p_X, R) = L_n^{\varepsilon,\varepsilon}(X, Y, p_X, R)$

3. $L_n^{\varepsilon}(X, p_X, R) = L_n^{\varepsilon,\varepsilon}(X, p_X, R)$

[3] A map $d\colon X \to Y$ is a *homotopy domination* if there is a map $u\colon Y \to X$ so that $d \circ u$ is homotopic to the identity. Wall's papers on "Finiteness conditions for CW complexes" show that, when X is a finite polyhedron, there is a K_0 obstruction to modifying a homotopy domination to give a homotopy equivalence from a finite complex to Y. The main technical result of [Q] is that the epsilon K_0 group is trivial and that "controlled dominations" can be modified to give controlled equivalences. What we have at this stage of the construction is a sort of bounded domination. We have a bit of extra information about distances in the nerve's metric on the domain side, so it is not exactly a bounded domination in the sense of Ferry-Pedersen, since in [FP] we do not pay attention to the metric in the domain.

Direct sum $(C,\psi)\oplus(C',\psi')=(C\oplus C',\psi\oplus\psi')$ of free R-module chain complexes C on $p_X\colon M\to X$ endowed with an n-dimensional ε-quadratic structure ψ induces an abelian group structure on $L_n^{\delta,\varepsilon}(X,Y,p_X,R)$. If $[C,\psi]=[C',\psi']\in L_n^{\delta,\varepsilon}(X,Y,p_X,R)$, then there is a finitely generated 100δ-connected 2δ-cobordism between (C,ψ) and (C',ψ') that is 100δ-Poincaré over $X-Y^{100\delta}$. Here Y^μ means the μ-neighborhood of Y in X. If $\delta'\geq\delta$ and $\varepsilon'\geq\varepsilon$, then there is a homomorphism

$$L_n^{\delta,\varepsilon}(X,Y,p_X,R)\to L_n^{\delta',\varepsilon'}(X,Y,p_X,R)$$

which sends $[C,\psi]$ to $[C,\psi]$. This is called the *relax-control map.*

More generally, [RY page 13] consider the following construction. Fix a subset Y of X and let $\mathcal{F}$ be a family of subsets of X such that $Z\supset Y$ for each $Z\in\mathcal{F}$. Let $n\geq 0$ and $\delta\geq\varepsilon\geq 0$. Define $L_n^{\mathcal{F},\delta,\varepsilon}(Y,p_X,R)$ to be the equivalence classes of finitely generated n-dimensional ε-Poincaré ε-projective quadratic complexes $((C,p),\psi)$ such that $[C,p]=0$ in $\widetilde{K}_0^{n,\varepsilon}(Z,p_Z,R)$ for each $Z\in\mathcal{F}$. The equivalence relation is generated by finitely generated δ-Poincaré δ-cobordisms on p_Y such that $[D,q]=0$ in $\widetilde{K}_0^{n+1,\delta}(Z,p_Z,R)$. Direct sum induces an abelian group structure on $L_n^{\mathcal{F},\delta,\varepsilon}(Y,p_X,R)$ and we abbreviate $L_n^{\mathcal{F},\varepsilon}(Y,p_X,R)=L_n^{\mathcal{F},\varepsilon,\varepsilon}(Y,p_X,R)$. There is an obvious map $i\colon L_n^{\delta,\varepsilon}(Y,p_Y,R)\to L_n^{\mathcal{F},\delta,\varepsilon}(Y,p_X,R)$ given by $[C,\psi]\mapsto[(C,1),\psi]$.

Construction: Let X be the union of two closed subsets A and B with intersection $N=A\cap B$. Set $\alpha=20000$ and $\mu_n=160000(n+5)$. There are three maps defined as follows.

1. When $\delta'\geq\alpha\delta$ and $\varepsilon'\geq\alpha\varepsilon$, then we have a map

$$i_*\colon L_n^{\{A,B\},\delta,\varepsilon}(N,p_X,R)\to L_n^{\delta',\varepsilon'}(A,p_A,R)\oplus L_n^{\delta',\varepsilon'}(B,p_B,R)$$

defined by $i_*(x)=(i_A(x),-i_B(x))$.

2. When $\delta'\geq\delta$ and $\varepsilon'\geq\varepsilon$, then we have a map

$$j_*\colon L_n^{\delta,\varepsilon}(A,p_A,R)\oplus L_n^{\delta,\varepsilon}(B,p_B,R)\to L_n^{\delta',\varepsilon'}(X,p_X,R)$$

given by $j_*(x,y)=j_{A*}+j_{B*}$, where $j_A\colon A\to X$ and $j_B\colon B\to X$ are inclusion maps.

3. When $W\supset N^{\mu_n\delta}$, $\delta'\geq\mu_n\delta$ and $\varepsilon'\geq\mu_n\varepsilon$, then there is the boundary map

$$\partial\colon L_n^{\delta,\varepsilon}(X,p_X,R)\to L_{n-1}^{\{A\cup W,B\cup W\},\delta',\varepsilon'}(W,p_X,R).$$

With these maps, one can construct a Mayer-Vietoris sequence for the quadruple (N,A,B,X) at the expense of enlarging some of the relevant subsets.

Theorem A1: [RY pages 25–26] For any integer $n\geq 2$, let μ_n and α be as given above. Then the following holds true for any control map p_X and two closed subsets A and B of X with $X=A\cup B$. Let $N=A\cap B$.

1. Suppose that $\delta'\geq\alpha\delta$, $\varepsilon'\geq\alpha\varepsilon$, $\delta''\geq\delta'$ and $\varepsilon''\geq\varepsilon'$ so that the following maps are defined:

$$L_n^{\{A,B\},\delta,\varepsilon}(N)\to L_n^{\delta',\varepsilon'}(A)\oplus L_n^{\delta',\varepsilon'}(B)\to L_n^{\delta'',\varepsilon''}(N).$$

Then the composition is zero.

2. Suppose that $\delta' \geq \delta$, $\varepsilon' \geq \varepsilon$, $W \supset N^{\mu_n \delta'}$, $\delta'' \geq \mu_n \delta'$ and $\varepsilon'' \geq \mu_n \varepsilon'$ so that the following maps are defined:
$$L_n^{\delta,\varepsilon}(A) \oplus L_n^{\delta,\varepsilon}(B) \to L_n^{\delta',\varepsilon'}(X) \to L_{n-1}^{\{A\cup W, B\cup W\},\delta'',\varepsilon''}(W).$$
Then the composition is zero.

3. Suppose that $W \supset N^{\mu_n \delta}$, $\delta' \geq \mu_n \delta$, $\varepsilon' \geq \mu_n \varepsilon$, $\delta'' \geq \alpha\delta'$ and $\varepsilon'' \geq \alpha\varepsilon'$ so that the following maps are defined:
$$L_n^{\delta,\varepsilon}(X) \to L_{n-1}^{\{A\cup W, B\cup W\},\delta',\varepsilon'}(W) \to L_{n-1}^{\delta'',\varepsilon''}(A \cup W) \oplus L_{n-1}^{\delta'',\varepsilon''}(B \cup W).$$
Then the composition is zero.

Theorem A2: For any integers $n \geq 0$, there is a constant $\lambda_n > 1$ which is linear in n (one can take $\lambda_n = 100000(4n + 50)$) such that the following statements hold true for any control map p_X and any two closed subsets A and B of X with $A \cup B = X$. Let $N = A \cap B$.

1. Suppose that $\delta'' \geq \delta'$ and $\varepsilon'' \geq \varepsilon'$ so that the map $j_*\colon L_n^{\delta',\varepsilon'}(A) \oplus L_n^{\delta',\varepsilon'}(B) \to L_n^{\delta'',\varepsilon''}(X)$ is defined. If $\delta \geq \lambda_n \delta''$ and $W \supset N^{\lambda_n \delta''}$, then the relax-control image of $\ker j_*$ in $L_n^{\alpha\delta}(A \cup W) \oplus L_n^{\alpha\delta}(B \cup W)$ is contained in $\operatorname{im} i_*$ below:
$$\begin{array}{ccc} & & L_n^{\delta',\varepsilon'}(A) \oplus L_n^{\delta',\varepsilon'}(B) \xrightarrow{\ j_*\ } L_n^{\delta'',\varepsilon''}(X) \\ & & \downarrow \\ L_n^{\{A\cup W, B\cup W\},\delta}(W) & \xrightarrow{\ i_*\ } & L_n^{\alpha\delta}(A \cup W) \oplus L_n^{\alpha\delta}(B \cup W) \end{array}$$

2. Suppose that $W \supset N^{\kappa_n \delta'}$, $\delta'' \geq \kappa_n \delta'$ and $\varepsilon'' \geq \kappa_n \varepsilon'$, so that the map $\partial\colon L_n^{\delta',\varepsilon'}(X) \to L_{n-1}^{\{A\cup W, B\cup W\},\delta'',\varepsilon''}(W)$ is defined. If $\delta \geq \lambda_n \delta''$, then the relax-control image of $\ker \partial$ in $L_n^{\delta}(X)$ is contained in $\operatorname{im} j_*$ below:
$$\begin{array}{ccc} & & L_n^{\delta',\varepsilon'}(X) \xrightarrow{\ \partial\ } L_{n-1}^{\{A\cup W, B\cup W\},\delta'',\varepsilon''}(W) \\ & & \downarrow \\ L_n^{\delta}(A \cup W) \oplus L_n^{\delta}(B \cup W) & \xrightarrow{\ j_*\ } & L_n^{\delta}(X) \end{array}$$

3. Suppose that $\delta'' \geq \alpha\delta'$ and $\varepsilon'' \geq \alpha\varepsilon'$ so that $i_*\colon L_{n-1}^{\{A,B\},\delta',\varepsilon'}(N) \to L_{n-1}^{\delta'',\varepsilon''}(A) \oplus L_{n-1}^{\delta'',\varepsilon''}(B)$ is defined. If $\delta \geq \lambda_n \delta''$, $M \supset N^{\lambda_n \delta''}$ and $W = M^{\kappa_n \delta}$, then the relax-control image of $\ker i_*$ in $L_{n-1}^{\{A\cup W, B\cup W\},\delta}(W)$ is contained in $\ker \partial$ associated with the two closed subsets $A \cup M$ and $B \cup M$.
$$\begin{array}{ccc} & & L_{n-1}^{\{A,B\},\delta',\varepsilon'}(N) \xrightarrow{\ i_*\ } L_{n-1}^{\delta'',\varepsilon''}(A) \oplus L_{n-1}^{\delta'',\varepsilon''}(B) \\ & & \downarrow \\ L_n^{\delta}(X) & \xrightarrow{\ \partial\ } & L_{n-1}^{\{A\cup W, B\cup W\},\delta}(W) \end{array}$$

Theorem B: [RY, page 10, Proposition 3.10] Let $F = (f, \overline{f})$ be a map from $p_X : M \to X$ to $p_Y : N \to Y$, and suppose that $\overline{f}$ is Lipschitz continuous with Lipschitz constant λ, i.e. there is a constant $\lambda > 0$ such that $d(\overline{f}(x_1), \overline{f}(x_2)) \leq \lambda d(x_1, x_2)$ for all $x_1, x_2 \in X$. Then F induces a homomorphism

$$F_* : L_n^{\delta,\varepsilon}(X, X', p_X, R) \to L_n^{\delta',\varepsilon'}(Y, Y', p_Y, R)$$

if $\delta' \geq \lambda\delta$, $\varepsilon' \geq \lambda\varepsilon$ and $\overline{f}(X') \subseteq Y$. If two maps $F = (f, \overline{f})$ and $G = (g, \overline{g})$ are homotopic through maps $H_t = (h_t, \overline{h}_t)$ such that each $\overline{h}_t$ is Lipschitz continuous with Lipschitz constant λ, $\delta' \geq \lambda\delta$, $\varepsilon' \geq \lambda\varepsilon$ and $\overline{h}_t(X') \subseteq Y'$, then F and G induce the same homomorphism

$$F_* = G_* : L_n^{\delta,\varepsilon}(X, X', p_X, R) \to L_n^{\delta',\varepsilon'}(Y, Y', p_Y, R).$$

IV. Main Theorem

Before proving our theorem, we review a construction given by the third author in [Y1, page 348]. Given a proper metric space X with asymptotic dimension growth f, we can arrange our covers from Section II so that $R_{k+1} > 4R_k$ for all k and $\operatorname{diam}(\mathcal{O}_{k,\ell}) < R_k/4$ for all $\mathcal{O}_{k,\ell} \in C_k$. For each k, let $C'_k = \{B(\mathcal{O}_{k+1,\ell}, R_k) : \mathcal{O}_{k+1,\ell} \in C_{k+1}\}$. Clearly the family $\{C'_k\}$ is an anti-Cech system for X. In addition we can arrange the covers $\{C_k\}$ so that no ball with radius R_k intersects more than $f(R_k) + 1$ members of C_{k+1}. Let m be a fixed positive integer. For each $n > m$, we can construct differentiable functions $\chi_n : [0, \infty) \to \mathbb{R}$ such that

1. $\chi_n(t) = 1$ for all $t \in [0, 1]$,
2. $\chi_n(t) = 0$ for all $t \geq R_n$,
3. $|\chi'_n(t)| \leq c/R_n$ for some uniform constant $c > 0$.

For such n and $U \in C_{n+1}$, define $U' = \{V \in N(C'_m) : V \in C'_m \text{ and } V \cap U \neq \emptyset\}$, where $V \in N(C'_m)$ is the vertex of $N(C'_m)$ corresponding to $V \in C'_m$. Given $x \in N(C'_m)$, we construct the map

$$g_{m,n}(x) = \frac{1}{A} \sum_{U \in C_{n+1}} \chi_n(d(x, U'))B(U, R_n) \quad \text{where } A = \sum_{W \in C_{n+1}} \chi_n(d(x, W')).$$

Furthermore, we can choose the map $i_{m,n} : N(C'_m) \to N(C'_n)$ in Section II so that, for each $V \in C_{m+1}$, we have $i_{m,n}(B(V, R_m)) = B(U, R_n)$ for some $U \in C_{n+1}$ satisfying $U \cap V \neq \emptyset$. Also, define $F(t, x) = tg_n(x) + (1 - t)i_{m,n}(x)$ for all $t \in [0, 1]$ and $x \in N(C'_m)$.

Lemma 1: Let X be a proper metric space with asymptotic dimension growth f. Assume that $g_{m,n}$, F and $i_{m,n}$ are as above. Assume that all nerve complexes are endowed with the spherical metric. We then have the following.

1. The assignment $g_{m,n}$ is a proper Lipschitz map from $N(C'_m)$ to $N(C'_n)$ with a Lipschitz constant depending only on f;
2. The assignment F is a Lipschitz straight-line homotopy $[0, 1] \times N(C'_m) \to N(C'_n)$ between $g_{m,n}$ and $i_{m,n}$ with a Lipschitz constant depending only on f;

3. For any $R > 0$, there is a constant $c(R) > 0$ such that $d(g_{m,n}(x), g_{m,n}(y)) < c(R)\sqrt{\frac{f(R_n)^3}{R_n}}$ for all $x, y \in N(C'_m)$ satisfying $d(x,y) \leq R$.

The proof of the lemma is a straightforward consequence of the properties of χ_n and the definition of the spherical metric. See [Y1, pages 347–348]. Note that (3) implies that $g_{m,n}$ is distance-shrinkning if X has slow asymptotic dimension growth.

Lemma 2: For each r, there are constants $c_0 > 0$ and $\varepsilon_0 > 0$ such that if Y is an ℓ-dimensional simplicial polyhedral complex and if $\delta_r < \varepsilon_0/c_0^\ell$, then we have a bijection $H_r(Y, \mathbb{L}(e)) \to L_r^{\delta_r}(Y)$.

Proof: The result certainly holds if $\ell = 0$. Assume by induction that the result holds when dim $Y = \ell - 1$ for some positive integer ℓ and for all r. We shall prove the theorem when dim $Y = \ell$. For each simplex Δ of dimension ℓ in Y, we define

$$\Delta_1 = \{x \in \Delta \colon d(x, c(\Delta)) \leq 1/100\} \text{ and } \Delta_2 = \{x \in \Delta \colon d(x, c(\Delta)) \geq 1/100\},$$

where $c(\Delta)$ is the center of Δ. Let $\mathcal{D}_\ell$ be the collection of all ℓ-dimensional simplices in Y and set

$$A_\ell = \bigcup_{\Delta \in \mathcal{D}_\ell} \Delta_1 \text{ and } B_\ell = \bigcup_{\Delta \in \mathcal{D}_\ell} \Delta_2.$$

Notice that A_ℓ and B_ℓ have the following properties:

1. A_ℓ is Lipschitz homotopy equivalent to the zero-dimensional simplex $\{c(\Delta) \colon \Delta \in \mathcal{D}_\ell\}$;

2. B_ℓ is Lipschitz homotopy equivalent to the $(\ell - 1)$-skeleton $Y^{(\ell-1)}$ of Y;

3. the ℓ-skeleton $Y^{(\ell)}$ is equal to $A_\ell \cup B_\ell$ and $A_\ell \cap B_\ell$ is the disjoint union of the boundaries of all Δ_1, where Δ is an ℓ-dimensional simplex in $Y^{(\ell)}$.

Statements (1) and (2), along with Lemma 1 and the induction hypothesis, imply that the result holds for A and B. By (3) and the induction hypothesis, the desired result also holds for $A_\ell \cap B_\ell$. Using the five-lemma, the controlled Mayer-Vietoris sequence (Theorems A1 and A2) and the Lipschitz homotopy equivalence result of [RY] (Theorem B), we conclude that there exist constants $c_0 > 0$ and $\varepsilon_0 > 0$ such that, if $\delta_r < \varepsilon_0/c_0^\ell$, then the assembly map $H_r(Y, \mathbb{L}(e)) \to L_r^{\delta_r}(Y)$ is a bijection. Here ε_0 depends on the control required to execute the base case, and c_0 depends on the control constants in the controlled Mayer-Vietoris sequence (which in turn depend on r) and in the Lipschitz result from Section III.

Theorem: Let M be an r-dimensional uniformly contractible manifold with bounded geometry whose asymptotic growth f satisfies $f(s) = o(\ln s)$. Then M is boundedly rigid.

Proof: Recall that it suffices to prove the claim in Section II. Let $\{C_k\}$ be a sequence of covers of M as given in Section II. Condition (3) gives an inclusion map $j_{m,n} \colon C_m \to C_n$ whenever $n > m$. By the discussion above, this map induces an inclusion map $i_{m,n} \colon N(C'_m) \to N(C'_n)$, where $\{C'_k\}$ is a sequence of covers of M satisfying the same conditions as $\{C_k\}$. Lemma 1 allows us to construct, for such pairs (m,n), a map $g_{m,n} \colon N(C'_m) \to N(C'_n)$ such that (a) $g_{m,n}$ is boundedly homotopic to $i_{m,n}$, and (b) $g_{m,n}$ is distance-shrinking in the sense of part 3 of Lemma 1. If r

is the dimension of M, let $g^r_{m,n}\colon L_r^{bdd}(N(C'_m)) \to L_r^{bdd}(N(C'_n))$ be the map of bounded L-theory and $i^r_{m,n}\colon H_r(N(C'_m), \mathbb{L}(e)) \to H_r(N(C'_n), \mathbb{L}(e))$ be the map of homology induced by $g_{m,n}$. Let $g^r_{m,n}\colon L_r^{bdd}(N(C'_m)) \to L_r^{bdd}(N(C'_n))$ be the composition $g^r_{n-1,n} \circ g^r_{n-2,n-1} \circ \cdots \circ g^r_{m,m+1}$.

For any $m \in \mathbb{Z}_{\geq 1}$, consider the diagram

$$\begin{array}{ccccccc}
H_r(N(C'_m), \mathbb{L}(e)) & \xrightarrow{i^r_{m,m+1}} & H_r(N(C'_{m+1}), \mathbb{L}(e)) & \longrightarrow \cdots \longrightarrow & \lim\limits_{k\to\infty} H_r(N(C'_k), \mathbb{L}(e)) \\
\downarrow & & \downarrow & & \downarrow \\
L_r^{bdd}(N(C'_m)) & \xrightarrow[g^r_{m,m+1}]{} & L_r^{bdd}(N(C'_{m+1})) & \longrightarrow \cdots \longrightarrow & \lim\limits_{k\to\infty} L_r^{bdd}(N(C'_k))
\end{array}$$

It is sufficient for our purposes to prove that the map $A^\infty\colon \lim\limits_{k\to\infty} H_n(N(C'_k), \mathbb{L}(e)) \longrightarrow \lim\limits_{k\to\infty} L_n^{bdd}(N(C'_k))$ is an isomorphism since the maps ϕ_k constructed in Section II give rise to a commutative diagram

$$\begin{array}{ccc}
H_r(M, \mathbb{L}(e)) & \longrightarrow & L_r^{bdd}(M) \\
\downarrow & & \downarrow \\
\lim\limits_{k\to\infty} H_r(N(C'_k), \mathbb{L}(e)) & \longrightarrow & \lim\limits_{k\to\infty} L_r^{bdd}(N(C'_k))
\end{array}$$

for which the vertical maps are isomorphisms by the condtion that M is a uniformly contractible manifold with bounded geometry.

We will prove the surjectivity of A^∞. The same argument shows injectivity as well. Any element $x \in \lim\limits_{k\to\infty} L_r^{bdd}(N(C'_k))$ may be considered as an element in $L_r^t(N(C'_m))$ for some real $t > 0$ and sufficiently large integer m. By Lemma 1, for any positive integer j and real number $u > 0$ the function $g^r_{1,j}$ may therefore be considered a map with domain $L_r^u(N(C'_1))$ and range $L_r^{a_j(u)}(N(C'_j))$, where $a_j(u) = c(u)\sqrt{\frac{f(R_j)^3}{R_j}}$ as in Lemma 1. An element $x \in L_r^t(N(C'_m))$ may be pulled back to an element $y \in L_r^t(N(C'_1))$ via $g^r_{1,m}$, so that $g^r_{1,j}(y)$, which by abuse of notation we will still call x, lies in $L_r^{a_j(t)}(N(C'_j))$ for all positive integers $j > m$.

To prove the surjectivity of A^∞, consider the constants c_0 and $\varepsilon_0 > 0$ (which depend on r, the dimension of our original space M) guaranteed by Lemma 2. Recall that $\ell \equiv \dim N(C'_j) \leq f(j)$ and $R_{j+1} \geq 4R_j$ for all j. The assumption on the asymptotic dimension growth f of M states that $f(j) \leq K \log j$ for some constant K. Now

$$c(t)\, c_0^{\ell} \sqrt{\frac{f(R_j)^3}{R_j}} \leq c(t)\, c_0^{K \log j} \sqrt{\frac{f(R_j)^3}{R_j}} = c(t)\, j^{K \log c_0} \sqrt{\frac{f(R_j)^3}{R_j}}.$$

Given the growth rate of R_j and $f(j)$, we can find an integer J such that the above quantity is less than ε_0 for all $j \geq J$. Therefore $a_j(t) < \varepsilon_0 / c_0^{\ell}$ for all $j \geq J$, where $\ell = \dim N(C'_j)$. By Lemma 2, this condition then yields a bijective map

$$H_r(N(C'_j), \mathbb{L}(e)) \to L_r^{a_j(t)}(N(C'_j))$$

for all $j \geq J$. By the diagram

$$\begin{array}{ccc} & & H_r(N(C'_j), \mathbb{L}(e)) \\ & \swarrow & \downarrow \\ L_r^{a_j(t)}(N(C'_j)) & \longrightarrow & L_r^{bdd}(N(C'_j)) \end{array}$$

we see that $x \in L_r^{a_j(t)}(N(C'_j))$ has a preimage in $H_r(N(C'_j), \mathbb{L}(e))$ for all sufficiently large j, so the map A^∞ is a surjection. □

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