On conjectures of Mathai and Borel

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Abstract. Mathai [M] has conjectured that the Cheeger-Gromov invariant $\rho_{(2)} = \eta_{(2)} - \eta$ is a homotopy invariant of closed manifolds with torsion-free fundamental group. In this paper we prove this statement for closed manifolds M when the rational Borel conjecture is known for $\Gamma = \pi_1(M)$, i.e. the assembly map $\alpha\colon H_*(B\Gamma, \mathbb{Q}) \to L_*(\Gamma) \otimes \mathbb{Q}$ is an isomorphism. Our discussion evokes the theory of intersection homology and results related to the higher signature problem.

Let M be a closed, oriented Riemannian manifold of dimension $4k-1$, with $k \geq 2$. In [Ma] Mathai proves that the Cheeger-Gromov invariant $\rho_{(2)} \equiv \eta_{(2)} - \eta$ is a homotopy invariant of M if $\Gamma = \pi_1(M)$ is a Bieberbach group. In the same work, he conjectures that $\rho_{(2)}$ will be a homotopy invariant for all such manifolds M whose fundamental group Γ is torsion-free and discrete. This conjecture is verified by Keswani [K] when Γ is torsion-free and the Baum-Connes assembly map $\mu_{\max}\colon K_0(B\Gamma) \to K_0(C^*\Gamma)$ is an isomorphism. Yet it is now known that $\mu_{\max}$ fails to be an isomorphism for groups satisfying Kazhdan's property T. This paper improves on Keswani's result by showing that Mathai's conjecture holds for torsion-free groups satisfying the rational Borel conjecture, for which no counterexamples have been found.

As a consequence of a theorem of Hausmann [H], for every compact odd-dimensional oriented manifold M with fundamental group Γ, there is a manifold W with boundary such that Γ injects into $G = \pi_1(W)$ and $\partial W = rM$ for some multiple rM of M. Using this result, the author and Weinberger construct in [CW] a well-defined Hirzebruch-type invariant for M^{4k-1} given by

$$\tau_{(2)}(M) = \frac{1}{r}\left(\operatorname{sig}^G_{(2)}(\widetilde{W}) - \operatorname{sig}(W)\right),$$

where $\widetilde{W}$ is the universal cover of W. The map $\operatorname{sig}^G_{(2)}$ is a real-valued homomorphism on the L-theory group $L_{4k}(G)$ given by $\operatorname{sig}^G_{(2)}(V) = \dim_G(V^+) - \dim_G(V^-)$ for any quadratic form V, considered as an $\ell^2(G)$-module. This invariant $\tau_{(2)}$ is in general a diffeomorphism invariant, but is not a homotopy invariant when $\pi_1(M)$ is not torsion-free [CW]. It is now also known that $\tau_{(2)}$ coincides with $\rho_{(2)}$ by the work of Lück and Schick [LS].

The purpose of this paper is to show that the diffeomorphism invariant $\tau_{(2)}$, and subsequently $\rho_{(2)}$, is actually a homotopy invariant of M^{4k-1} if the fundamental group $\Gamma = \pi_1(M)$ satisfies the rational Borel conjecture, which states that the assembly map $\alpha\colon H_*(B\Gamma, \mathbb{Q}) \to L_*(\Gamma) \otimes \mathbb{Q}$ is an isomorphism if Γ is torsion-free. We include in our discussion some background in L-theory and intersection homology.

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Introduction

The classical work on cobordism by Thom implies that every compact odd-dimensional oriented manifold M has a multiple rM which is the boundary of an oriented manifold W. Hausmann [H] showed furthermore that, for every such M with fundamental group Γ, there is a manifold W such that $\partial W = rM$, for some multiple rM of M, with the additional property that the inclusion $M \hookrightarrow W$ induces an injection $\Gamma \hookrightarrow \pi_1(W)$. For such a manifold M^{4k-1} with fundamental group Γ, a higher "Hirzebruch type" real-valued function $\tau_{(2)}$ is defined in [CW] in the following manner:

$$\tau_{(2)}^{G}(M) = \frac{1}{r}\,(\,\mathrm{sig}\,_{(2)}^{G}(\widetilde{W}) - \mathrm{sig}\,(W)),$$

where $G = \pi_1(W)$ and $\widetilde{W}$ is the universal cover of W.

To see that this quantity can be made independent of (W, G), we consider any injection $G \hookrightarrow G'$. Let $W_{G'}$ be the G'-space induced from the G-action on $\widetilde{W}$ to G'. Now define

$$\tau_{(2)}^{G'}(M) = \frac{1}{r}\,(\,\mathrm{sig}\,_{(2)}^{G'}(W_{G'}) - \mathrm{sig}\,(W_{G'}/G')).$$

Since $\widetilde{W}$ is simply the G-cover W_G of W and the quotient $W_{G'}/G'$ is clearly diffeomorphic to W, we note that this definition of $\tau_{(2)}^{G'}$ is consistent with the above definition of $\tau_{(2)}^{G}$. However, by the Γ-induction property of Cheeger-Gromov [CG, page 8, equation (2.3)], we have

$$\begin{aligned}
\tau_{(2)}^{G'}(M) &= \tfrac{1}{r}\,(\,\mathrm{sig}\,_{(2)}^{G'}(W_{G'}) - \mathrm{sig}\,(W_{G'}/G')) \\
&= \tfrac{1}{r}\,(\,\mathrm{sig}\,_{(2)}^{G}(W_{G}) - \mathrm{sig}\,(W)) \\
&= \tau_{(2)}^{G}(M).
\end{aligned}$$

So from G one can pass to any larger group G' without changing the value of this quantity. Given two manifolds W and W' with the required bounding properties, we can use the larger group $G' = \pi_1(W) *_\Gamma \pi_1(W')$, which contains both fundamental groups $\pi_1(W)$ and $\pi_1(W')$. The usual Novikov additivity argument[2] proves that $\tau_{(2)}^{G}(M)$ is independent of all choices [CW]. We will henceforth refer to it as $\tau_{(2)}$.

In [CW] we prove that $\tau_{(2)}$ is a differential invariant but not a homotopy invariant of M^{4k-1} when $\pi_1(M)$ is not torsion-free. We will use ideas given by Weinberger in [W] which under the same conditions proves the homotopy invariance of the twisted ρ-invariant $\rho_\alpha(M) = \eta_\alpha(M) - \eta(M)$, where α is any representation $\pi_1(M) \to U(n)$. We supply brief expository remarks about intersection homology and algebraic Poincaré complexes in Sections I and II before proving the theorem in Section III.

[2]Novikov additivity, proved cohomologically in [AS], posits that signature is additive in the following sense: if Y is an oriented manifold of dimension $2n$ with boundary X and Y' is another such manifold with boundary $-X$, then $\mathrm{sig}\,(Y \cup_X Y') = \mathrm{sig}\,(Y) + \mathrm{sig}\,(Y')$. The additivity for $\mathrm{sig}\,_{(2)}^{G}$ is easy to argue on the level of $\ell^2(G)$-modules V endowed with nonsingular bilinear form. To see that this additivity corresponds to appropriate manifold glueings, we refer the reader to [F1] and [F2] of Farber, who puts L^2 cohomology groups into a suitable framework in which the same arguments can be repeated.

I. Intersection homology

Following Goresky and MacPherson, we say that a compact space X is a *pseudomanifold* of dimension n if there is a compact subspace Σ with $\dim(\Sigma) \leq n-2$ such that $X - \Sigma$ is an n-dimensional oriented manifold which is dense in X. We assume that our pseudomanifolds come equipped with a fixed stratification by closed subspaces

$$X = X_n \supset X_{n-1} = X_{n-2} = \Sigma \supset X_{n-3} \supset \cdots \supset X_1 \supset X_0$$

satisfying various neighborhood conditions (see [GM]). A *perversity* is a sequence of integers $\overline{p} = (p_2, p_3, \cdots, p_n)$ such that $p_2 = 0$ and $p_{k+1} = p_k$ or $p_k + 1$. A subspace $Y \subset X$ is said to be $(\overline{p}, i)$*-allowable* if $\dim(Y) \leq i$ and $\dim(Y \cap X_{n-k}) \leq i - k + p_k$. We denote by $IC_i^{\overline{p}}$ the subgroup of i-chains $\xi \in C_i(X)$ for which $|\xi|$ is $(\overline{p}, i)$-allowable and $|\partial\xi|$ is $(\overline{p}, i-1)$-allowable. If X is a pseudomanifold of dimension n, then we define the *i-th intersection homology group* $IH_i^{\overline{p}}(X)$ to be the i-th homology group of the chain complex $IC_*^{\overline{p}}(X)$. For any perversity $\overline{p}$, we have $IH_0^{\overline{p}}(X) \cong H^n(X)$ and $IH_n^{\overline{p}}(X) \cong H_n(X)$.

Whenever $\overline{p} + \overline{q} \leq \overline{r}$, the intersection homology groups can be equipped with a unique product $\cap\colon IH_i^{\overline{p}}(X) \times IH_j^{\overline{q}}(X) \to IH_{i+j-n}^{\overline{r}}(X)$ that respects the intersection homology classes of dimensionally transverse pairs (C, D) of cycles. Let $\overline{m} = (0, 0, 1, 1, 2, 2, \ldots, 2k-2, 2k-2, 2k-1)$ be the middle perversity. If X is stratified with only strata of even codimension, and if $\dim(X) = 4k$, then the intersection pairing

$$\cap\colon IH_{2k}^{\overline{m}}(X) \times IH_{2k}^{\overline{m}}(X) \to \mathbb{Z}$$

is a symmetric and nonsingular form when tensored with $\mathbb{Q}$. We can define the signature $\operatorname{sig}(X)$ of X to be the signature of this quadratic form. If X is a manifold, this definition coincides with the usual notion of signature.

If X is a pseudomanifold, we say that X is a *Witt space* if $IH_k^{\overline{m}}(L, \mathbb{Q}) = 0$ whenever L^{2k} is the link of an odd-codimensional stratum of X. If X^q is a Witt space, there is a nondegenerate rational pairing

$$IH_i^{\overline{m}}(X, \mathbb{Q}) \times IH_j^{\overline{m}}(X, \mathbb{Q}) \to \mathbb{Q}$$

whenever $i + j = q$. If $q = 4k > 0$, then $IH_{2k}^{\overline{m}}(X, \mathbb{Q})$ is a symmetric inner product space. In this case, we define the Witt class $w(X)$ of X to be the equivalence class of $IH_{2k}^{\overline{m}}(X, \mathbb{Q})$ in the Witt ring $W(\mathbb{Q})$ of classes of symmetric rational inner product spaces. If $q = 0$, set $w(X)$ to be $\operatorname{rank}(H_0(X, \mathbb{Q})) \cdot \langle 1 \rangle$, and set it to zero if $q \not\equiv 0 \bmod 4$. If $(X, \partial X)$ is a Witt space with boundary, set $w(X) = w(\widehat{X})$, where $\widehat{X} = X \cup \operatorname{cone}(\partial X)$. Let $\operatorname{sig}_{\mathbb{Q}}\colon W(\mathbb{Q}) \to \mathbb{Z}$ be the signature homomorphism developed by Milnor and Husemoller [MH]. We define the signature $\operatorname{sig}(X)$ to the integer $\operatorname{sig}_{\mathbb{Q}}(w(X))$. See [S].

Denote by Ω_*^{Witt} the bordism theory based on Witt spaces; i.e. if Y is a Witt space, define $\Omega_n^{\text{Witt}}(Y)$ to be the classes $[X, f]$, where X is an n-dimensional Witt space and $f\colon X \to Y$ a continuous map, such that $[X_1, f_1] \sim [X_2, f_2]$ iff there is an $(n+1)$-dimensional Witt space W with $\partial W = X_1 \coprod X_2$ and a map $W \to Y$ that restricts to f_1 and f_2 on the boundary. Witt bordism enjoys the important properties that (1) there is a signature invariant defined on the cycle level which is a cobordism invariant, and (2) the signature can be extended to relative cycles $(X, \partial X)$ so that it is additive.

Lemma 1: Let M_1 and M_2 be homotopy equivalent manifolds of the same dimension. Suppose that their fundamental group Γ is torsion-free and satisfies the rational Borel conjecture. Then there is rationally a Witt cobordism between M and M' over $B\Gamma$.

Proof: It is well-known that, if Γ is torsion-free and satisfies the Borel conjecture, then it satisfies the Novikov conjecture, which asserts that, if $f\colon M \to B\Gamma$ is a map, then the generalized Pontrjagin number (or "higher signature")

$$f_*\left(L(M) \cap [M]\right) \in H_*(B\Gamma, \mathbb{Q})$$

is an oriented homotopy invariant. Here $L(M)$ is the Hirzebruch L-polynomial in terms of the Pontrjagin classes, and $[M]$ is the fundamental class of M. Let $f_i\colon M_i \to B\Gamma$ be the map classifying the universal cover of M_i. We would like to show that $[M_1, f_1]$ and $[M_2, f_2]$ rationally define the same class in $\Omega_*^{\mathrm{Witt}}(B\Gamma)$.

For a Witt space X, Siegel [S] constructs an L-class $L(X) \in H_*(X, \mathbb{Q})$ that generalizes the L-class of Goresky and MacPherson [GM], the latter of which is defined only on Whitney stratified pseudomanifolds with even-codimensional strata. If $f\colon X \to B\Gamma$ is the universal cover of X, then $f_*(L(X)) \in H_*(B\Gamma, \mathbb{Q})$ coincides with the higher signature given above. In addition, it is a bordism invariant in $\Omega_*^{\mathrm{Witt}}(B\Gamma)$. Rationally the Atiyah-Hirzebruch spectral sequence for Witt cobordism collapses, so we obtain an isomorphism

$$i\colon \Omega_n^{\mathrm{Witt}}(B\Gamma) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k \geq 0} H_{n-4k}(B\Gamma, \mathbb{Q})$$

such that $i([M, f])$ is the higher signature $f_*(L(M) \cap [M])$. But these signatures are homotopy invariants by assumption. ■

II. Algebraic Poincaré complexes

An *n-dimensional Poincaré complex* over a ring A with involution is an A-module chain complex C with a collection of A-module morphisms $\phi_s\colon C^{n-r+s} \to C_r$ such that the chain map ϕ_0 is a chain equivalence inducing abstract Poincaré duality isomorphisms $\phi_0\colon H^{n-r}(C) \to H_r(C)$. As defined by Wall [Wa], an *n-dimensional geometric Poincaré complex* X with fundamental group Γ is a finitely dominated CW-complex together with a fundamental class $[X] \in H_n(\widetilde{X}, \mathbb{Z})$ such that cap product with $[X]$ gives a family of $\mathbb{Z}\Gamma$-module isomorphisms

$$\cap\, [X]\colon H^r(\widetilde{X}) \longrightarrow H_{n-r}(\widetilde{X})$$

for $0 \leq r \leq n$ (there is additional business about an orientation group morphism $w\colon \Gamma_1(X) \to \mathbb{Z}_2$ for which one should consult [Ran2]). An n-dimensional geometric Poincaré complex X with fundamental group Γ naturally determines an n-dimensional *symmetric* Poincaré complex $(C(\widetilde{X}), \phi_{\widetilde{X}})$ over $\mathbb{Z}\Gamma$. Such symmetric complexes (as opposed to *quadratic* complexes, which define lower L-theory) over a ring A can be assembled into an abelian group $L^n(A)$ under a cobordism relation defined by abstract Poincaré-Lefschetz duality [M,Ran1], with addition given by

$$(C, \phi) + (C', \phi') = (C \oplus C', \phi \oplus \phi') \in L^n(A).$$

For a more detailed discussion on algebraic Poincaré complexes and in particular the manner in which algebraic Poincaré complexes are glued together along a common boundary, see [Ran3].

Lemma 2: Let M and M' be homotopy equivalent manifolds of the same dimension $4k-1$ with torsion-free fundamental group Γ. Suppose that Y is a rational Witt cobordism of M and M' over $B\Gamma$. If the Borel conjecture holds for Γ, then $\operatorname{sig}^{\Gamma}_{(2)}(Y_\Gamma) = \operatorname{sig}(Y)$, where Y_Γ is the induced Γ-cover of Y.

Proof: By the preceding lemma, there is a Witt cobordism between n copies of M and n copies of M'. Let C^h be the homotopy cylinder given by the homotopy equivalence $h : M \to M'$. Attach n copies of C^h to the rational Witt cobordism Y to form a space X. This space is usually not a pseudomanifold because the singular space is of codimension one, but it is an algebraic Poincaré space. Notice that all of these spaces come equipped with a map to $B\Gamma$. Since upper and lower L-theory coincide rationally, we may then consider X as an element of $L_*(B\Gamma) \otimes \mathbb{Q}$. Consider then the composition of maps

$$\Omega^{\mathrm{SO}}_*(B\Gamma) \otimes \mathbb{Q} \longrightarrow H_*(B\Gamma, \mathbb{Q}) \longrightarrow L_*(\Gamma) \otimes \mathbb{Q}\,.$$

The first map is well known to be surjective, and the second is surjective by our assumption that the Borel conjecture holds for Γ. Therefore there is a closed manifold X' without boundary bordant to X as algebraic Poincaré complexes. Atiyah's Γ-index theorem [A] shows that $\operatorname{sig}^{\Gamma}_{(2)}(X'_\Gamma) = \operatorname{sig}(X')$, so that the cobordism invariance of signature implies that $\operatorname{sig}^{\Gamma}_{(2)}(X_\Gamma) = \operatorname{sig}(X)$. For a topological proof of the Γ-index theorem for signature, see the appendix of [CW].

By Novikov additivity, we may decompose the complex X to arrive at the equation

$$\operatorname{sig}^{\Gamma}_{(2)}(Y_\Gamma) + n \cdot \operatorname{sig}^{\Gamma}_{(2)}(C^h_\Gamma) = \operatorname{sig}(Y) + n \cdot \operatorname{sig}(C^h).$$

By Novikov additivity, the homotopy cylinder C^h enjoys the convenient property that $\operatorname{sig}^{\Gamma}_{(2)}(C^h_\Gamma) = \operatorname{sig}(C^h)$, so $\operatorname{sig}^{\Gamma}_{(2)}(Y_\Gamma) = \operatorname{sig}(Y)$. ∎

III. The main theorem

Theorem: Let M be an oriented compact manifold of dimension $4k-1$. If $\pi_1(M)$ is torsion-free, then the real-valued quantity $\tau_{(2)}(M)$ is a homotopy invariant.

Proof: Let M and M' be homotopy equivalent closed manifolds of dimension $n = 4k-1$. There are closed manifolds W and W' such that $\partial W = rM$ and $\partial W' = sM'$ with injecting fundamental groups. Construct s copies of W and r copies of W' and attach rs Witt cobordisms Y between the boundary components M and M'. Call this space X with $H = \pi_1(X)$. Note that both

$\pi_1(W)$ and $\pi_1(W')$ inject into H. Since the map $\Omega^{\mathrm{SO}}(BH) \otimes \mathbb{Q} \to \Omega^{\mathrm{Witt}}(BH) \otimes \mathbb{Q}$ is surjective (see [Cu]), it follows that X is Witt cobordant to a smooth manifold X'. Hence they share the same signatures. Note that X' has the property that $\mathrm{sig}\,^H_{(2)}(X'_H) = \mathrm{sig}\,(X')$, and hence $\mathrm{sig}\,^\Gamma_{(2)}(X'_\Gamma) = \mathrm{sig}\,(X')$ by Atiyah's Γ-index theorem. By the observation above, we can then conclude that $\mathrm{sig}\,^\Gamma_{(2)}(X_\Gamma) = \mathrm{sig}\,(X)$.

By Novikov additivity, we can partition X along the original attachments to obtain

$$s \cdot \mathrm{sig}\,^\Gamma_{(2)}(W_\Gamma) + rs \cdot \mathrm{sig}\,^\Gamma_{(2)}(Y_\Gamma) - r \cdot \mathrm{sig}\,^\Gamma_{(2)}(W_\Gamma) = s \cdot \mathrm{sig}\,(W) + rs \cdot \mathrm{sig}\,(Y) - r \cdot \mathrm{sig}\,(W').$$

By Lemma 2, the cylindrical terms cancel, leaving

$$\frac{1}{r}\left(\mathrm{sig}\,^\Gamma_{(2)}(W_\Gamma) - \mathrm{sig}\,(W)\right) = \frac{1}{s}\left(\mathrm{sig}\,^\Gamma_{(2)}(W'_\Gamma) - \mathrm{sig}\,(W')\right).$$

Equivalently $\tau_{(2)}(M) = \tau_{(2)}(M')$, so $\tau_{(2)}$ is a homotopy equivalence. ■

Remark: Given the identification of $\tau_{(2)}$ and $\rho_{(2)}$, one can conjecture that $\rho_{(2)}$ is homotopy invariant iff the fundamental group of M is torsion-free (see [CW]). However, recent work of [NW] makes use of secondary invariants for groups with unsolvable word problems, which, a fortiori, are not residually finite. Potentially, the results of this paper can have applications to the geometry of certain moduli spaces.

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