(1) (4 pts/part) Determine the Galois group $\text{Gal}(L/K)$ for each of these extensions $L: K$. Define the elements as precisely as possible. Here you should not assume that if $L: K$ is finite and normal, then the number of elements in $\text{Gal}(L/K)$ is $[L: K]$.
   (a) $\mathbb{Q}(\sqrt{7}) : \mathbb{Q}$
   (b) $\mathbb{Q}(\sqrt{2}) : \mathbb{Q}$
   (c) $\mathbb{Q}(\sqrt[4]{2}, \sqrt[5]{7}) : \mathbb{Q}$
   (d) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}$
   (e) $\mathbb{Z}_2(\zeta) : \mathbb{Z}_2$, where $\zeta$ is a root of $x^2 + x + 1 \in \mathbb{Z}_2[x]$  

Solution:
   (a) We have $\text{Gal}(L/K) \cong \mathbb{Z}_2 = \{\sigma_1, \sigma_2\}$, where $\sigma_1 : \sqrt{7} \mapsto -\sqrt{7}$ and $\sigma_2 : \sqrt{7} \mapsto \sqrt{7}$.
   (b) We have $\text{Gal}(L/K) \cong \{e\}$ consisting of just the identity map.
   (c) We have $\text{Gal}(L/K) \cong \mathbb{Z}_4 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$. Let $\omega = e^{2\pi i/5}$. Then the automorphisms are determined by $\sigma_1 : \omega \mapsto \omega$, $\sigma_3 : \omega \mapsto \omega^2$, $\sigma_4 : \omega \mapsto \omega^3$.
   (d) We have $\text{Gal}(L/K) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$, where the eight maps are given by the combinations $\sqrt{2} \mapsto \pm \sqrt{2}$, $\sqrt{3} \mapsto \pm \sqrt{3}$, $\sqrt{5} \mapsto \pm \sqrt{5}$.
   (e) We have $\text{Gal}(L/K) \cong \mathbb{Z}_2 = \{\sigma_1, \sigma_2\}$, where $\sigma_1 : \zeta \mapsto \zeta$ and $\sigma_2 : \zeta \mapsto \zeta + 1$.

(2) (3 pts/part) Let $\gamma = \sqrt{2 + \sqrt{2}}$. The purpose of this problem is to compute the Galois group of $\mathbb{Q}(\gamma) : \mathbb{Q}$.

(a) Compute the minimum polynomial $f \in \mathbb{Q}[x]$ of $\gamma$. Be sure to verify that $f$ is indeed irreducible. Compute all the roots of $f$.

(b) Let $\beta$ be the other positive root of $f$. By showing that $\beta = \frac{\sqrt{2}}{\gamma}$, prove that $\mathbb{Q}(\gamma)$ is a splitting field for $f$ over $\mathbb{Q}$.

(c) By considering the order of the $\mathbb{Q}$-automorphism $\alpha$ satisfying $\alpha(\gamma) = \beta$ (we know there is one by Theorem 7), prove that $\text{Gal}(\mathbb{Q}(\gamma)/\mathbb{Q}) \cong \mathbb{Z}_4$.

Solution:
   (a) We know that $\gamma^2 = 2 + \sqrt{2}$, so $(\gamma^2 - 2)^2 = 2$, or $\gamma^2 - 4\gamma + 2 = 0$. Therefore $f = x^2 - 4x + 2$ has $\gamma$ as a root. Since $f$ is irreducible by Eisenstein’s criterion, it is the minimum polynomial of $\gamma$. By the quadratic equation, one can easily see that the four roots of $f$ are given by $\pm\sqrt{2} \pm \sqrt{2}$.

(b) Let $\beta = \sqrt{2} - \sqrt{2}$. Note that $\gamma\beta = \sqrt{2 + \sqrt{2}} \sqrt{2 - \sqrt{2}} = \sqrt{2}$, so it follows that $\beta = \frac{\sqrt{2}}{\gamma}$. Clearly $\sqrt{2} \in \mathbb{Q}(\gamma)$, so $\beta \in \mathbb{Q}(\gamma)$ as well. Since the roots of $f$ are $\pm\gamma$ and $\pm\beta$, it follows that $\mathbb{Q}(\gamma)$ is the splitting field of $f$.

(c) There are four elements of the Galois group $G = \text{Gal}(\mathbb{Q}(\gamma)/\mathbb{Q})$. Consider $\alpha \in G$ determined by $\gamma \mapsto \beta$. Then $\alpha(\sqrt{2}) = -\sqrt{2}$. Therefore $\alpha^2(\gamma) = \alpha(\beta) = \alpha(\sqrt{2}/\gamma) = -\sqrt{2}/\beta = -\gamma$. Therefore the order of $\alpha$ is 4, so $G \cong \mathbb{Z}_4$.

(3) Given $f \in K[x]$, we say the that Galois group of $f$ is the Galois group of the extension $L : K$ where $L$ is the splitting field of $f$ over $K$. Consider $f = (x^2 - 2)(x^2 - 3)(x^2 - 5) \in \mathbb{Q}[x]$.

(a) (3 pts) Determine the Galois group $G$ of $f$, listing all its elements using the $\mapsto$ notation.

(b) (4 pts) For each subgroup $H$ of $G$, compute $H^1$.

Solution:
(a) The roots of \( f \) are given by \( \pm \sqrt{2}, \pm \sqrt{3} \) and \( \pm \sqrt{5} \). Every automorphism in the Galois group of \( f \) is determined by its behavior on \( \sqrt{2}, \sqrt{3} \) and \( \sqrt{5} \). The eight automorphisms of \( f \) are given by

\[
\begin{align*}
\sigma_1 &: \; \sqrt{2} \mapsto \sqrt{2}, \quad \sqrt{3} \mapsto \sqrt{3}, \quad \sqrt{5} \mapsto \sqrt{5}, \quad (0, 0, 0); \\
\sigma_2 &: \; \sqrt{2} \mapsto -\sqrt{2}, \quad \sqrt{3} \mapsto \sqrt{3}, \quad \sqrt{5} \mapsto \sqrt{5}, \quad (1, 0, 0); \\
\sigma_3 &: \; \sqrt{2} \mapsto \sqrt{2}, \quad \sqrt{3} \mapsto -\sqrt{3}, \quad \sqrt{5} \mapsto \sqrt{5}, \quad (0, 1, 0); \\
\sigma_4 &: \; \sqrt{2} \mapsto \sqrt{2}, \quad \sqrt{3} \mapsto \sqrt{3}, \quad \sqrt{5} \mapsto -\sqrt{5}, \quad (0, 0, 1); \\
\sigma_5 &: \; \sqrt{2} \mapsto -\sqrt{2}, \quad \sqrt{3} \mapsto -\sqrt{3}, \quad \sqrt{5} \mapsto \sqrt{5}, \quad (1, 1, 0); \\
\sigma_6 &: \; \sqrt{2} \mapsto \sqrt{2}, \quad \sqrt{3} \mapsto -\sqrt{3}, \quad \sqrt{5} \mapsto -\sqrt{5}, \quad (0, 1, 1); \\
\sigma_7 &: \; \sqrt{2} \mapsto -\sqrt{2}, \quad \sqrt{3} \mapsto \sqrt{3}, \quad \sqrt{5} \mapsto -\sqrt{5}, \quad (1, 0, 1); \\
\sigma_8 &: \; \sqrt{2} \mapsto -\sqrt{2}, \quad \sqrt{3} \mapsto -\sqrt{3}, \quad \sqrt{5} \mapsto -\sqrt{5}, \quad (1, 1, 1).
\end{align*}
\]

The last column gives the corresponding element of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \).

(b) There are a total of 16 subgroups of the Galois group.

<table>
<thead>
<tr>
<th>( H )</th>
<th>( H^\dagger )</th>
<th>( H )</th>
<th>( H^\dagger )</th>
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<tbody>
<tr>
<td>( \langle \sigma_1 \rangle )</td>
<td>( \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) )</td>
<td>( \langle \sigma_2, \sigma_3 \rangle )</td>
<td>( \mathbb{Q}(\sqrt{5}) )</td>
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<td>( \mathbb{Q}(\sqrt{2}, \sqrt{3}) )</td>
<td>( \langle \sigma_3, \sigma_4 \rangle )</td>
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<tr>
<td>( \langle \sigma_4 \rangle )</td>
<td>( \mathbb{Q}(\sqrt{5}, \sqrt{6}) )</td>
<td>( \langle \sigma_2, \sigma_6 \rangle )</td>
<td>( \mathbb{Q}(\sqrt{15}) )</td>
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<tr>
<td>( \langle \sigma_5 \rangle )</td>
<td>( \mathbb{Q}(\sqrt{2}, \sqrt{15}) )</td>
<td>( \langle \sigma_3, \sigma_7 \rangle )</td>
<td>( \mathbb{Q}(\sqrt{10}) )</td>
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<tr>
<td>( \langle \sigma_6 \rangle )</td>
<td>( \mathbb{Q}(\sqrt{3}, \sqrt{10}) )</td>
<td>( \langle \sigma_1, \sigma_5 \rangle )</td>
<td>( \mathbb{Q}(\sqrt{6}) )</td>
</tr>
<tr>
<td>( \langle \sigma_7 \rangle )</td>
<td>( \mathbb{Q}(\sqrt{3}, \sqrt{15}) )</td>
<td>( \langle \sigma_5, \sigma_6 \rangle )</td>
<td>( \mathbb{Q}(\sqrt{30}) )</td>
</tr>
<tr>
<td>( \langle \sigma_8 \rangle )</td>
<td>( \mathbb{Q}(\sqrt{6}, \sqrt{10}, \sqrt{15}) )</td>
<td>( \langle \sigma_1, \sigma_2, \sigma_3 \rangle )</td>
<td>( \mathbb{Q} )</td>
</tr>
</tbody>
</table>

(4) (4 pts/part) Find the Galois group of \( x^3 - 5 \) over the following fields. List all the subgroups \( H \) and the fixed field \( H^\dagger \).

(a) \( \mathbb{Q} \)

(b) \( \mathbb{Z}_3 \)

(c) \( \mathbb{Z}_7 \)

Solution:

(a) The Galois group is \( S_3 \), each automorphism determined by its behavior on \( \sqrt[3]{5} \) and \( \omega = e^{2\pi i/3} \). The six elements are as follows:

\[
\begin{align*}
\sigma_1 &: \; \sqrt[3]{5} \mapsto \sqrt[3]{5}, \quad \omega \mapsto \omega; \\
\sigma_2 &: \; \sqrt[3]{5} \mapsto \sqrt[3]{5} \omega, \quad \omega \mapsto \omega; \\
\sigma_3 &: \; \sqrt[3]{5} \mapsto \sqrt[3]{5} \omega^2, \quad \omega \mapsto \omega; \\
\sigma_4 &: \; \sqrt[3]{5} \mapsto \sqrt[3]{5}, \quad \omega \mapsto \omega^2; \\
\sigma_5 &: \; \sqrt[3]{5} \mapsto \sqrt[3]{5} \omega, \quad \omega \mapsto \omega^2; \\
\sigma_6 &: \; \sqrt[3]{5} \mapsto \sqrt[3]{5} \omega^2, \quad \omega \mapsto \omega^2.
\end{align*}
\]

There are six subgroups of \( S_3 \).

<table>
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<th>( H^\dagger )</th>
</tr>
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<tbody>
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<td>( \mathbb{Q}(\sqrt[3]{5}, \omega) )</td>
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<tr>
<td>( \langle \sigma_2 \rangle )</td>
<td>( \mathbb{Q}(\omega) )</td>
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<tr>
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<td>( \mathbb{Q}(\sqrt[3]{5}) )</td>
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<tr>
<td>( \langle \sigma_4 \rangle )</td>
<td>( \mathbb{Q}(\sqrt[3]{5} \omega^2) )</td>
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<tr>
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<td>( \langle \sigma_6 \rangle )</td>
<td>( \mathbb{Q}(\sqrt[3]{5} \omega^2) )</td>
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</tbody>
</table>

(b) Since \( f = (x + 1)^3 \), it follows that the Galois group is \( \langle e \rangle \) and \( \langle e \rangle^\dagger = \mathbb{Z}_3 \).
(c) Note first that $f$ is irreducible over $\mathbb{Z}_7$. Let $\gamma$ be a root. Then $f = (x - \gamma)(x - 2\gamma)(x - 4\gamma)$. Hence the Galois group is $\mathbb{Z}_3$. Hence $\langle e \rangle^1 = \mathbb{Z}_7(\gamma)$ and $\mathbb{Z}_3^1 = \mathbb{Z}_7$. 