Day 15

Uniform Continuity

In Math 302, you likely you discussed convergence of a sequence of functions:

<u>Def'n</u> (Pointwise convergence)

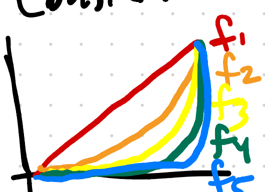
Suppose X, Y are topological spaces, and Y is a metric space. Suppose $f_n : X \to Y$ for each $n \in \mathbb{N}$. We say a function $f : X \to Y$ is a pointwise limit of $(f_1, f_2, f_3, \ldots)$ if for all $x \in X$ we have $(f_1(x), f_2(x), f_3(x), \ldots)$ have limit $f(x)$. This means:

for any $x \in X$ and any $\varepsilon > 0$, we can find $N \in \mathbb{N}$ so that for all $n \geq N$ we have $d(f_n(x), f(x)) < \varepsilon$.

Good news about pointwise limit: satisfy what you'd expect based on thinking of functions by their inputs.

Bad news: pointwise limit doesn't necessarily preserve continuity

<u>Ex</u> Consider $f_n : [0,1] \to [0,1]$ by $f_n(x) = x^n$. Each f_n is continuous, but pointwise limit is $f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$



Can we think of a new kind of convergence so that limits of continuous functions are continuous?

<u>Def'n</u> (Uniform Continuity)

Let X, Y be topological spaces where Y is a metric space, and (f_n) be a sequence of functions from X to Y. A function $f: X \to Y$ is called the uniform limit of (f_n) if: for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $n \geq N$ and all $x \in X$ we get $d(f_n(x), f(x)) < \varepsilon$.

Non-ex The functions $f_n : [0,1] \to [0,1]$ given by $f_n(x) = x^n$ have pointwise limit $f(x) = \begin{cases} 0 & \text{if } x<1 \\ 1 & \text{if } x=1, \end{cases}$

but they fail to converge uniformly. Challenge: prove it!

Fact: If (f_n) converge uniformly to f, then (f_n) converge pointwise to f.

Thm (Continuous functions are closed under uniform limits)

If X is a topological space and Y is a metric space, then if (f_n) converge uniformly to f and each f_n is continuous, then f is continuous too.

Pf (to be posted when slides go up)

Let V be open in Y, and we want to show that $f^{-1}(V)$ is open in X. For this, let $x \in f^{-1}(V)$ be given, and we aim to find an open $U \subseteq X$ with $x \in U \subseteq f^{-1}(V)$.

Now since $x \in f^{-1}(V)$, we know $f(x) \in V$. Since V is open and open balls form a basis for the topology on Y, we know there exists some $z \in Y$ and $\rho > 0$ so that $f(x) \in B(z, \rho)$. In fact, if we select $\varepsilon = \rho - d(z, f(x))$, we even have

$f(x) \in B(f(x), \varepsilon) \subseteq B(z, \rho) \subseteq V$ [using an argument we've seen a few times before]. Now, we know that since (f_n) converges uniformly to f, there exists some N so that for all $n \geq N$ and all $w \in X$ we have $d(f(w), f_n(w)) < \varepsilon/3$. Further, since f_N is continuous, we know that $U = f_N^{-1}(B(f_N(x), \varepsilon/3))$ is open. We will aim to show that $x \in U \subseteq f^{-1}(V)$.

First, to see $x \in U = f_N^{-1}(B(f_N(x), \varepsilon/3))$, note that $d(f_N(x), f_N(x)) = 0 < \varepsilon/3$, and so $x \in f_N^{-1}(B(f_N(x), \varepsilon/3)) = U$.

Now we check that $U \subseteq f^{-1}(V)$. Since $B(f(x), \varepsilon) \subseteq V$, we do this by showing any $y \in U$ has $f(y) \in B(f(x), \varepsilon)$. Now $y \in U$ means $f_N(y) \in B(f_N(x), \varepsilon/3)$, and so:

$$d(f(x), f(y)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) \quad \left[\begin{array}{c}\text{triangle}\\ \text{ineq.}\end{array}\right]$$

$$< \underset{\substack{\text{definition of } N \\ \text{in uniform continuity}}}{\varepsilon/3} + \underset{y \in U}{\varepsilon/3} + \underset{\substack{\text{definition of } N \\ \text{in uniform continuity}}}{\varepsilon/3}.$$

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Fun remark: The topics we discussed today deal with sequences of functions converging. So: can we think about functions as "points" inside some topological space?

Let $\text{Fun}(X,Y) = \{ f : X \to Y \}$. If d is a metric on Y we can define a metric on $\text{Fun}(X,Y)$:

$$\bar{d}(f,g) = \sup_{x \in X} \{ \max\{ d(f(x), g(x)), 1 \} \}$$

"uniform bounded metric on $\text{Fun}(X,Y)$"

Fact 1: uniform convergence of (f_n) to f is just convergence within metric space $\mathrm{Fun}(X,Y)$

Fact 2: Since uniform limits of continuous functions are continuous, This means

$$\mathcal{C} = \{ f : X \to Y \mid f \text{ is continuous} \} \subseteq \mathrm{Fun}(X,Y)$$

is closed under limits, and hence is a closed subspace of $\mathrm{Fun}(X,Y)$.