Day 6

product topology

Closed sets

We'll start off with one quick comment about metric topologies.

last time we saw two metric topologies on $\mathbb{R}^n$:

one induced by Euclidean metric $d_1(\vec{x},\vec{y}) = \sqrt{\sum (x_i - y_i)^2}$

another induced by $d_2(\vec{x},\vec{y}) = \max\{ |x_i - y_i| : 1 \le i \le n \}$.

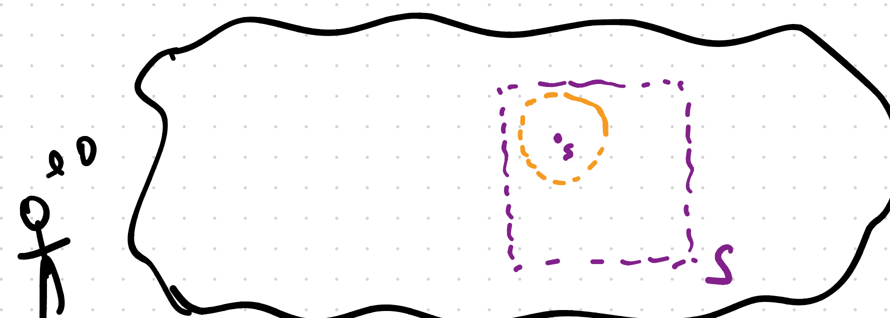
(Recall the open balls in d_2 look like open squares/open cubes, etc.

We had a challenge: show these two topologies are the same. Suggestion: fineness via bases.

For example, to say $\mathcal{T}_{d_1} \supseteq \mathcal{T}_{d_2}$, we need to argue that for any S a basis element in $\mathcal{T}_{d_2}$ and any $s \in S$, we can find some C a basis element in $\mathcal{T}_{d_1}$ with

$$s \in C \subseteq S$$

(Ditto for other direction)



Recall that we created metric topologies because we wanted to know if there were ways for "creating topologies".

For example: if we have $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$, can we create a "natural" topology on $X_1 \times X_2$?

ie, related to $\mathcal{T}_1$ and $\mathcal{T}_2$

Def'n / Theorem (Product topology)

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces. Then

$$\mathcal{B} = \{ U \times V : U \in \mathcal{T}_1 \text{ and } V \in \mathcal{T}_2 \}$$

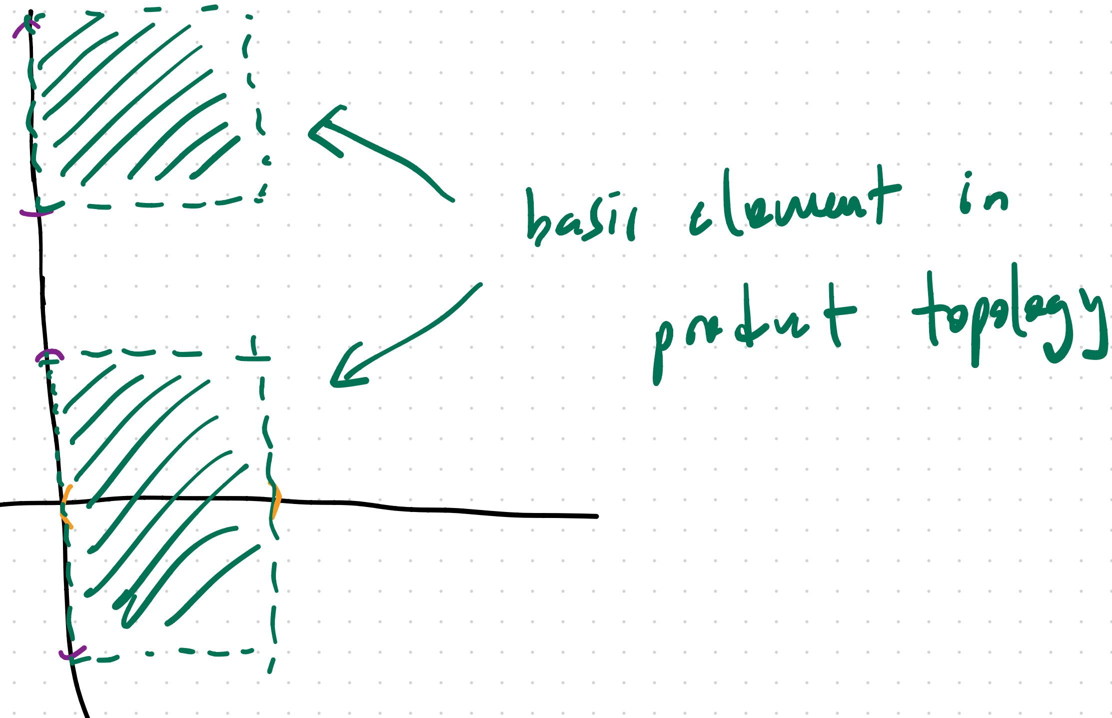
is a basis. The topology $\mathcal{T}_{\mathcal{B}}$ is called the product topology on $\underbrace{X_1 \times X_2}$.

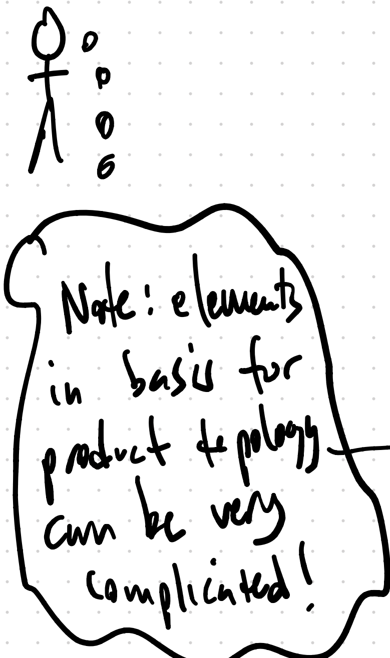
Cartesian product, aka $X_1 \times X_2 = \{ x_1 \times x_2 : x_1 \in X_1, x_2 \in X_2 \}$

usually written (x_1, x_2) ... just an ordered pair

Ex Note if $(X_1, T_1) = (X_2, T_2) = (\mathbb{R}, T_{3.2})$,

Then one open set is $(0,1) \times \left((-1,1) \cup (2,3)\right)$





Can we find a simpler basis for product topology?

Thm (Simpler basis for product topology)

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces with $\mathcal{B}_1$ a basis for $\mathcal{T}_1$ and $\mathcal{B}_2$ a basis for $\mathcal{T}_2$.

Then $\mathcal{B}_1 \times \mathcal{B}_2 = \{ B_1 \times B_2 : B_1 \in \mathcal{B}_1 \text{ and } B_2 \in \mathcal{B}_2 \}$

is a basis, and $\mathcal{T}_{\mathcal{B}_1 \times \mathcal{B}_2}$ is the product topology.

Pf We need to prove $\mathcal{B}_1 \times \mathcal{B}_2$ is a basis,

and we need to prove $\mathcal{T}_{\mathcal{B}_1 \times \mathcal{B}_2} = \underbrace{\mathcal{T}_{X_1 \times X_2}}_{\text{product topology.}}$

To see $\mathcal{B}_1 \times \mathcal{B}_2$ is a basis:

① Let $x_1 \times x_2 \in X_1 \times X_2$ be given. Since $x_1 \in X_1$ and $\mathcal{B}_1$ is a basis for X_1, we know there is some $B_1 \in \mathcal{B}_1$ with $x_1 \in B_1$. Likewise we have some $B_2 \in \mathcal{B}_2$ so that $x_2 \in B_2$. Then $x_1 \times x_2 \in B_1 \times B_2$.

② If $B_1 \times B_2$, $C_1 \times C_2 \in \mathcal{B}_1 \times \mathcal{B}_2$ and in

$x_1 \times x_2 \in (B_1 \times B_2) \cap (C_1 \times C_2)$, then we want some

$D_1 \times D_2 \in \mathcal{B}_1 \times \mathcal{B}_2$ so that $x_1 \times x_2 \in D_1 \times D_2 \subseteq (B_1 \times B_2) \cap (C_1 \times C_2)$

Since $x_1 \times x_2 \in (B_1 \times B_2) \cap (C_1 \times C_2)$, we have $x_1 \in B_1 \cap C_1$.

Since $\mathcal{B}_1$ is a basis, we have some $D_1 \in \mathcal{B}_1$ with

$x_1 \in D_1 \subseteq B_1 \cap C_1$. Likewise, we have $D_2 \in \mathcal{B}_2$ with

$x_2 \in D_2 \subseteq B_2 \cap C_2$. So $x_1 \times x_2 \in D_1 \times D_2 \subseteq (B_1 \times B_2) \cap (C_1 \times C_2)$

by definition of ordered pair.

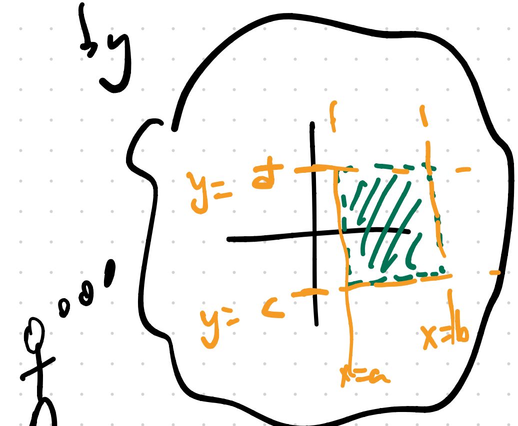
Now we'll show $\mathcal{T}_{\mathcal{B}_1 \times \mathcal{B}_2} = \mathcal{T}_{X_1 \times X_2}$. We'll prove by "fineness via bases".

So, note for $B_1 \times B_2 \in \mathcal{B}_1 \times \mathcal{B}_2$ we have $B_1 \in \mathcal{B}_1 \subseteq \mathcal{T}_1$ and $B_2 \in \mathcal{B}_2 \in \mathcal{T}_2$. So $B_1 \times B_2 \in \mathcal{T}_1 \times \mathcal{T}_2$. But $\mathcal{T}_1 \times \mathcal{T}_2$ is the basis for $\mathcal{T}_{X_1 \times X_2}$, so $\mathcal{T}_{\mathcal{B}_1 \times \mathcal{B}_2} \subseteq \mathcal{T}_{X_1 \times X_2}$.

For the other containment, let $U \times V$ be an element drawn from the basis $\mathcal{T}_1 \times \mathcal{T}_2$ that defines $\mathcal{T}_{X_1 \times X_2}$. We'll show: for any $u \times v \in U \times V$, we can find some $B_1 \times B_2 \in \mathcal{B}_1 \times \mathcal{B}_2$ so that $u \times v \in B_1 \times B_2 \subseteq U \times V$. Since $u \in U$ and since $\mathcal{B}_1$ is a basis on X_1, we have some $B_1 \in \mathcal{B}_1$ with $u \in B_1 \subseteq U$. Likewise we have some $B_2 \in \mathcal{B}_2$ with $v \in B_2 \subseteq V$. So by definition of Cartesian product we get $u \times v \in B_1 \times B_2 \subseteq U \times V$. $\blacksquare$

Ex let's revisit $\mathbb{R}^2$ under the product topology from $(\mathbb{R}, \mathcal{T}_{302})$. The result we just proved says we can use basis for $(\mathbb{R}, \mathcal{T}_{302})$ given by $\mathcal{C} = \{(a,b) : a,b \in \mathbb{R},\ a<b\}$ to create a basis on $\mathbb{R}^2$ given by

$$\mathcal{C}\times\mathcal{C} = \{(a,b)\times(c,d) : a,b,c,d \in \mathbb{R},\ a<b \text{ and } c<d\}$$



Based on the example we revisited at the start of this lecture, one can see that the topology we get from viewing $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$ is the same as our favorite topology on $\mathbb{R}^2$ (namely $\mathcal{T}_{302}$)

(In fact, this is true for $\mathbb{R}^n$ with $n \geq 2$).

We'll take a break from our quest for generating examples of new topologies to dig deeper into studying properties of a given topological space.

Central to this is notion of closed.

<u>Def'n</u> A set A is closed in a topological space $(X, \mathcal{T})$ iff $X - A$ is open (ie, $X - A \in \mathcal{T}$).

Ex X is closed because $X - X = \emptyset \in \mathcal{T}$.

Ex $\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$ is closed in $\mathbb{R}^2$. So we need to show

(Challenge: write each term as basis element in product topology)

$\mathbb{R}^2 - \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$ is open:

$$\mathbb{R}^2 - \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}x\\y\end{pmatrix} : y>0\right\} \cup \left\{\begin{pmatrix}x\\y\end{pmatrix} : y<0\right\}$$
$$\cup \left\{\begin{pmatrix}x\\y\end{pmatrix} : x>0\right\} \cup \left\{\begin{pmatrix}x\\y\end{pmatrix} x<0\right\}$$

Each of these terms is open, so the union is open.

Ex Let $X = (0,1) \cup \{2\}$ endowed this with the subspace topology from $\mathbb{R}$. (see Homework 1, #1 and Homework 2, #3)

(ie: open in X means set is of the form $U \cap X$ for U open in $\mathbb{R}$)

Then $(0,1)$ is closed in X. Why? $X - (0,1) = \{2\}$ where $\{2\}$ is open in X since $\{2\} = (3/2, 5/2) \cap X$.

Closed sets satisfy some nice properties. The most foundational such property is

__Theorem__ (Topological properties of closed sets)
Let $(X, \mathcal{T})$ be any topological space. Then

1. X are both closed
2. If I is an index set and for all $i \in I$ we have a closed set K_i, then $\bigcap_{i \in I} K_i$ is closed.
3. If $n \in \mathbb{N}$ and for all $1 \leq i \leq n$ we have closed set K_i, then $K_1 \cup \cdots \cup K_n$ is closed.

In the video, I said I'd give the formal argument that the usual topology for $\mathbb{R}^n$ (the one generated by open balls under the Euclidean metric) matches the topology induced by the square metric.

So... here's the proof.

We'll use "fineness via bases." Let $\mathcal{B}_e$ denote the basis of open balls in the Euclidean metric d_e, and let $\mathcal{B}_s$ denote the basis of open balls in the square metric d_s:

$$\mathcal{B}_e = \{ B_{d_e}(\vec{x}, \rho) : \vec{x} \in \mathbb{R}^n, \rho \in \mathbb{R}_{>0} \}$$

$$\mathcal{B}_s = \{ B_{d_s}(\vec{x}, \rho) : \vec{x} \in \mathbb{R}^n, \rho \in \mathbb{R}_{>0} \}.$$

Start by letting $B_{d_e}(\vec{x}, \rho) \in \mathcal{B}_e$ be given, and let $\vec{y} \in B_{d_e}(\vec{x}, \rho)$ be given. Observe that we have shown already (on Day 4) that for $\rho' = \rho - d_e(\vec{x}, \vec{y})$ we have $B_{d_e}(\vec{y}, \rho') \subseteq B_{d_e}(\vec{x}, \rho)$

We now claim that for $M = \sqrt{\rho'/n}$ we have $B_{d_s}(\vec{y}, M) \subseteq B_{d_e}(\vec{y}, \rho')$; observe that this will imply $\mathcal{T}_{d_s} \supseteq \mathcal{T}_{d_e}$. So let $\vec{z} \in B_{d_s}(\vec{y}, M)$ be given, so that $\max\{|y_i - z_i| : 1 \leq i \leq n\} < M$. But then

we have $|y_i - z_i|^2 = (y_i - z_i)^2 < M^2$ for all $1 \leq i \leq n$,

so $d_e(\vec{y}, \vec{z}) = \sum_{i=1}^{n} (y_i - z_i^2) < nM^2 = n\left(\sqrt{\rho'/n}\right)^2 = \rho'$.

This gives $z \in B_{d_e}(\vec{y}, \rho')$, as desired.

Now for the other direction, let $B_{d_s}(\vec{x}, \rho)$ be given, and let $\vec{y} \in B_{d_s}(\vec{x}, \rho)$ be given.

Just as before, we can choose $\rho' = \rho - d_s(\vec{x},\vec{y})$ and we'll have $B_{d_s}(\vec{y},\rho') \subseteq B_{d_s}(\vec{x},\rho)$, and we'll aim to find some $\varepsilon > 0$ so that $B_{d_e}(\vec{y},\varepsilon) \subseteq B_{d_s}(\vec{y},\rho')$, at which point we'll get $\mathcal{T}_{d_s} \subseteq \mathcal{T}_{d_e}$, and we'll be done.

We claim that $B_{d_e}(\vec{y},\rho') \subseteq B_{d_s}(\vec{y},\rho')$. So let $z \in B_{d_e}(\vec{y},\rho')$, so that $d_e(\vec{y},\vec{z}) = \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2} < \rho'$.

If we had $d_s(\vec{y},\vec{z}) = \max\{|y_i - z_i| : 1 \le i \le n\} \ge \rho'$, this would mean that for some $1 \leq m \leq n$ we had $|y_m - z_m| \geq \rho'$.

But then we'd get

$$d_e(\tilde{y}, \tilde{z}) = \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2} \overset{(*)}{\geq} \sqrt{(y_m - z_m)^2} = |y_m - z_m| \geq \rho',$$

contrary to assumption. [Note: the inequality marked (*) holds since all summands are non-negative.]

$\blacksquare$