The big ideas in calculus

- Functions are complicated
  - Easy to do +,-, ×, ÷, polynomials
  - Hard to do √, ∫, π, *e*, ln, ...

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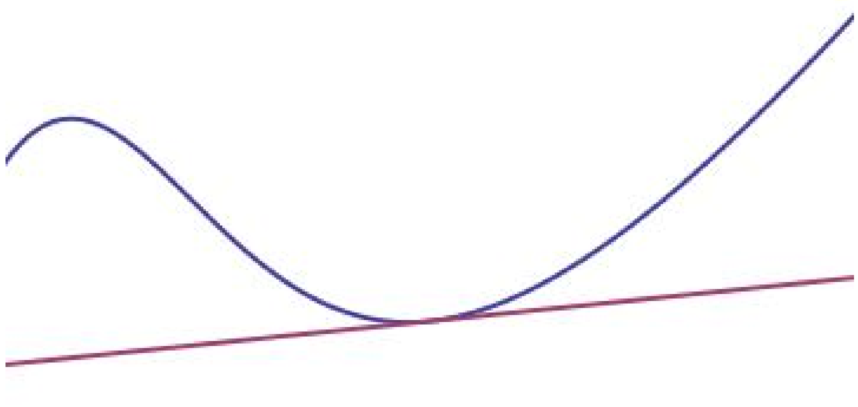
- Functions are complicated
  - Easy to do +,-, ×, ÷, polynomials
  - Hard to do $\sqrt{}$, $\int$, π, e, ln, ...
- Handy to be able to approximate a function
  - Punchline from Calc I: approximating functions with tangent lines
  - Punchline from Calc II: approximating functions with degree *n* polynomial

Taylor Approximation in action

- High degree Taylor approximations fit better

  Approximating with degree 1 poly isn't so awesome



Taylor Approximation in action

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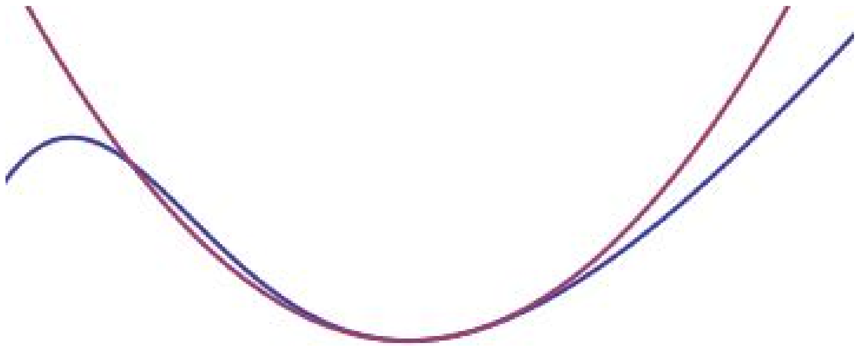
  Approximating with degree 2 poly is a little better



Taylor Approximation in action

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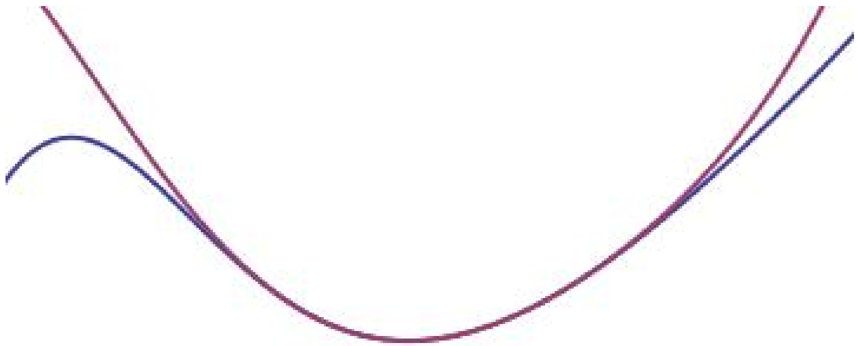
  Approximating with degree 5 poly is alright



Taylor Approximation in action

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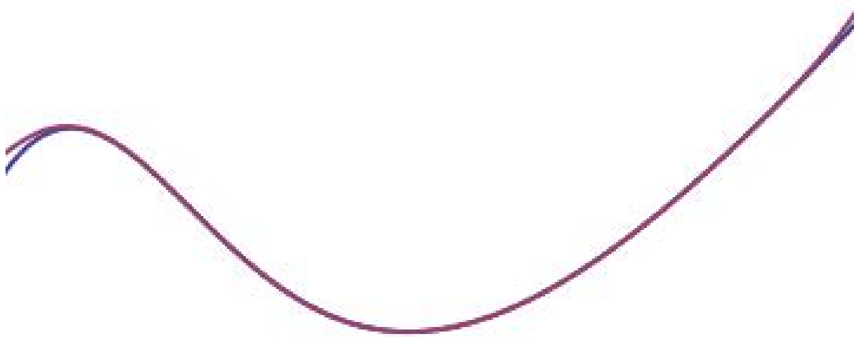
  Approximating with degree 15 poly is much better



What to approximate with?

- We've done *polynomial* approximation
  - $f(x) = \sum c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$
  - Taylor series="polynomial approximation"

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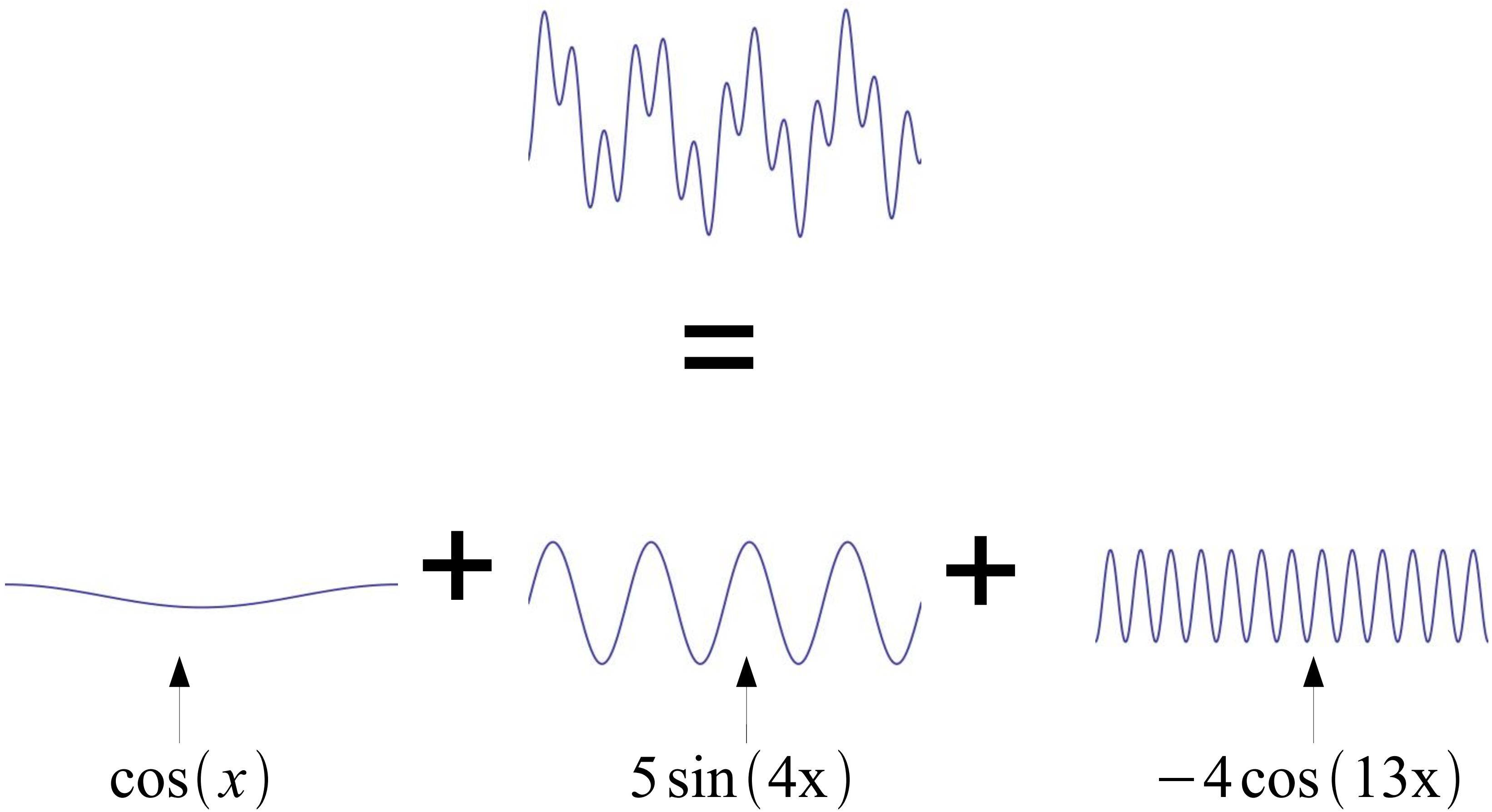
  - $f(x)=\sum c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$
  - Taylor series="polynomial approximation"
- We might also try trig approximation
  - $f(x)=\frac{a_0}{2} + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \ldots$
  - Fourier series = "approximate by trigs"

Fourier Series in action



Fourier series in action

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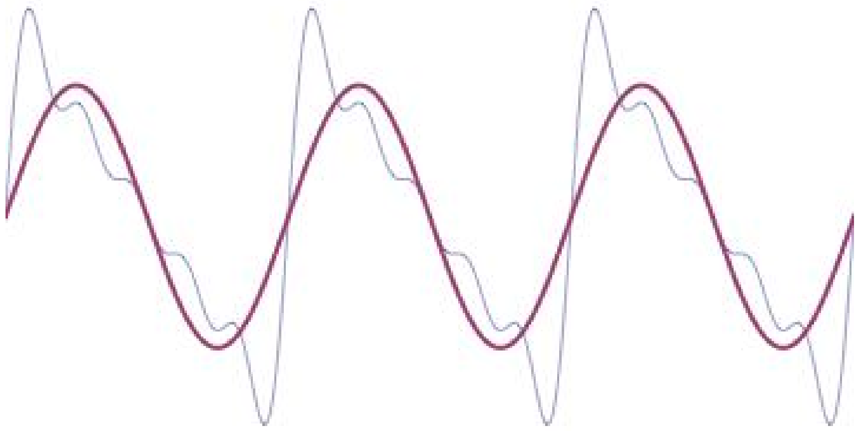

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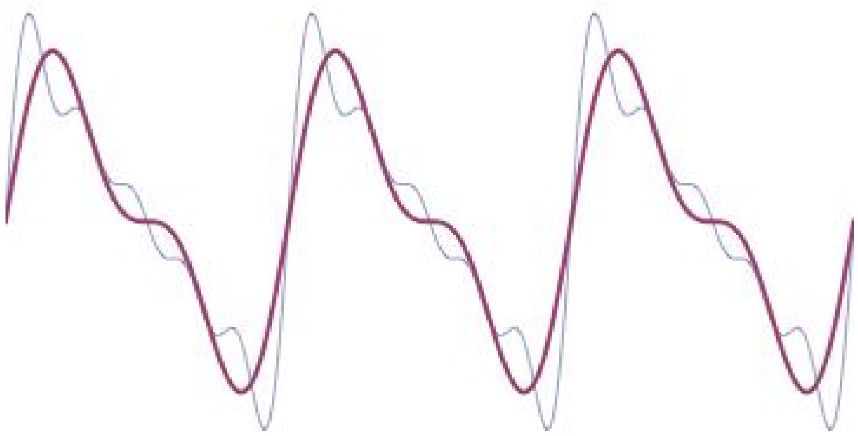

Fourier series in action

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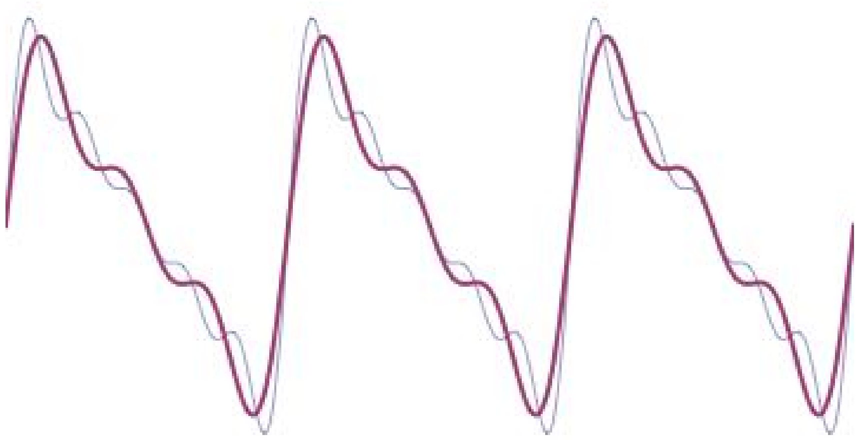
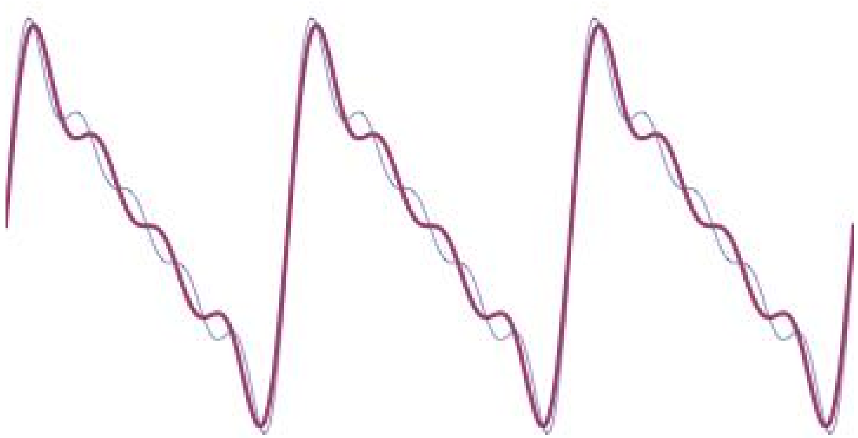


Fourier series in action

- "Order 1" approx is not so hot
- "Order 2" is better
- "Order 3" is betterer
- "Order 4" is pretty darn good



Fourier series “always” work

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Fourier series "always" work

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- That is, for a "nice" function f(x), you can find constants $a_0, a_1, b_1, a_2, b_2, \cdots$ so that

$$f(x)=\frac{a_0}{2}+a_1\cos(x)+b_1\sin(x)+a_2\cos(2x)+b_2\sin(2x)+\cdots$$

for every x in (-π,π).

How do we find coefficients?

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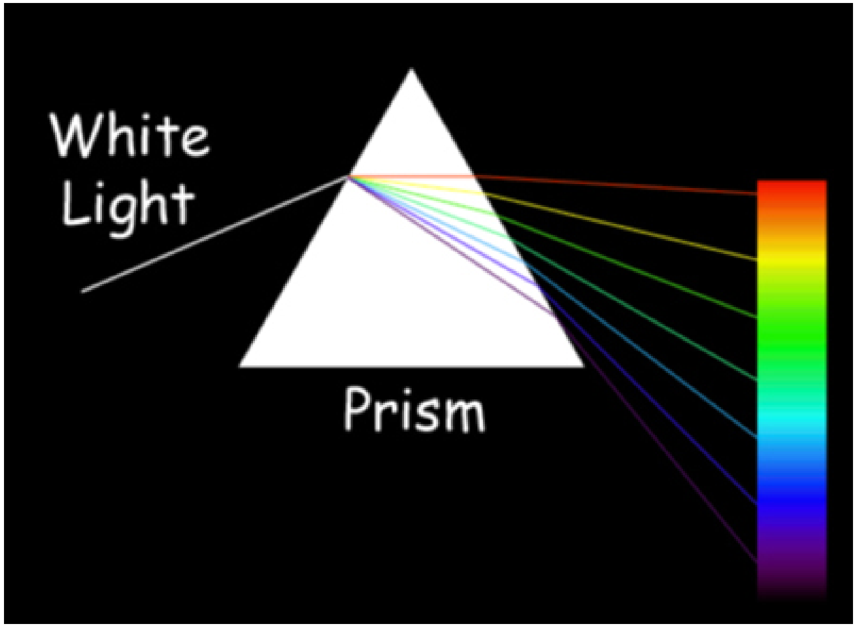
- For Fourier series, we get coefficients as

$$a_n = \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad b_n = \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Why do we care?

- Fourier series allow us to analyze anything that is “wavy”
  - Light
  - Sound
- Fourier series allow us to analyze anything on a closed interval
  - Digital Images

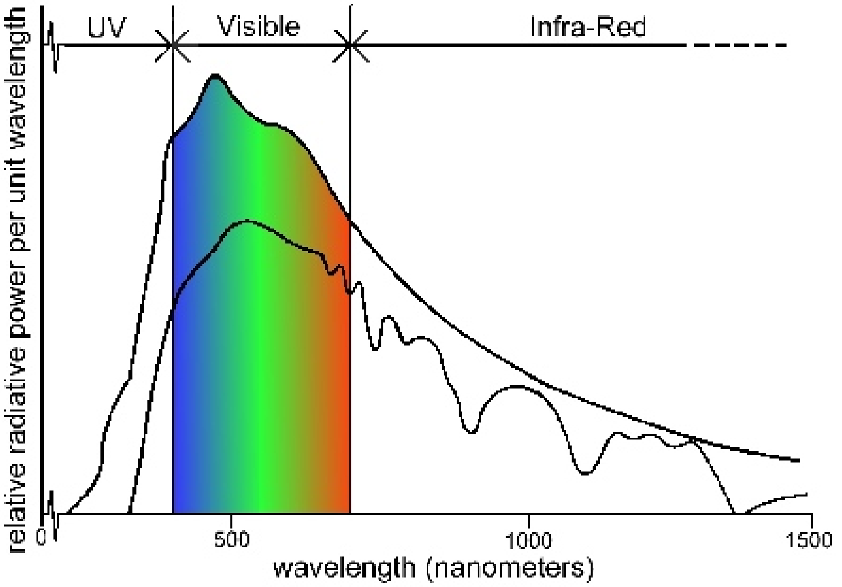
Analyzing light



- Sunlight is a sum of electromagnetic waves of various wavelengths
- Sunlight has a particular “amount” of each wavelength
- To determine “how much” of each wavelength is used, we compute the Fourier series of sunlight

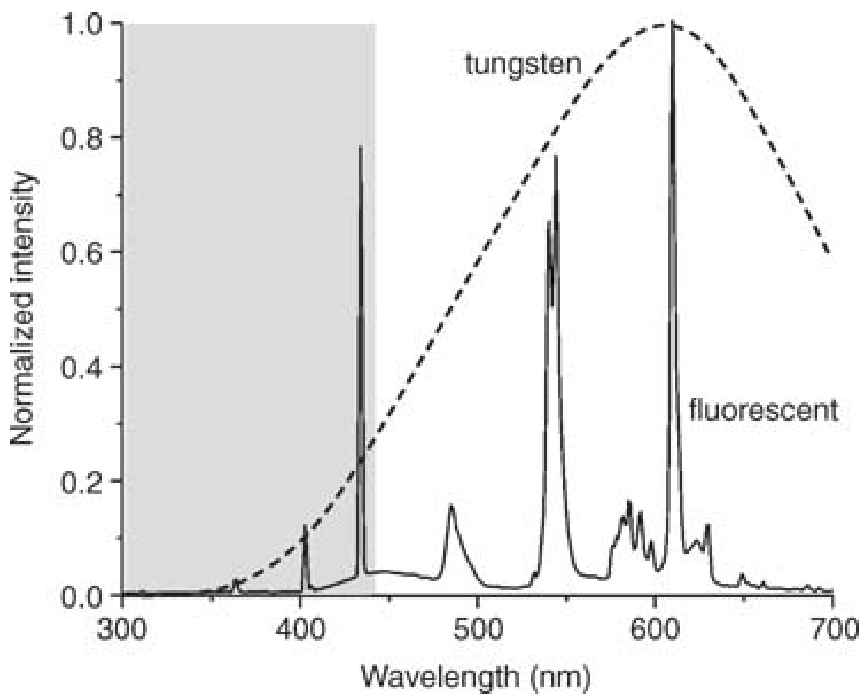
Analyzing light

- Sunlight has more blue than red (i.e. blue light is emitted with more power than red)
- Sunlight on the earth's surface is different from sunlight in space



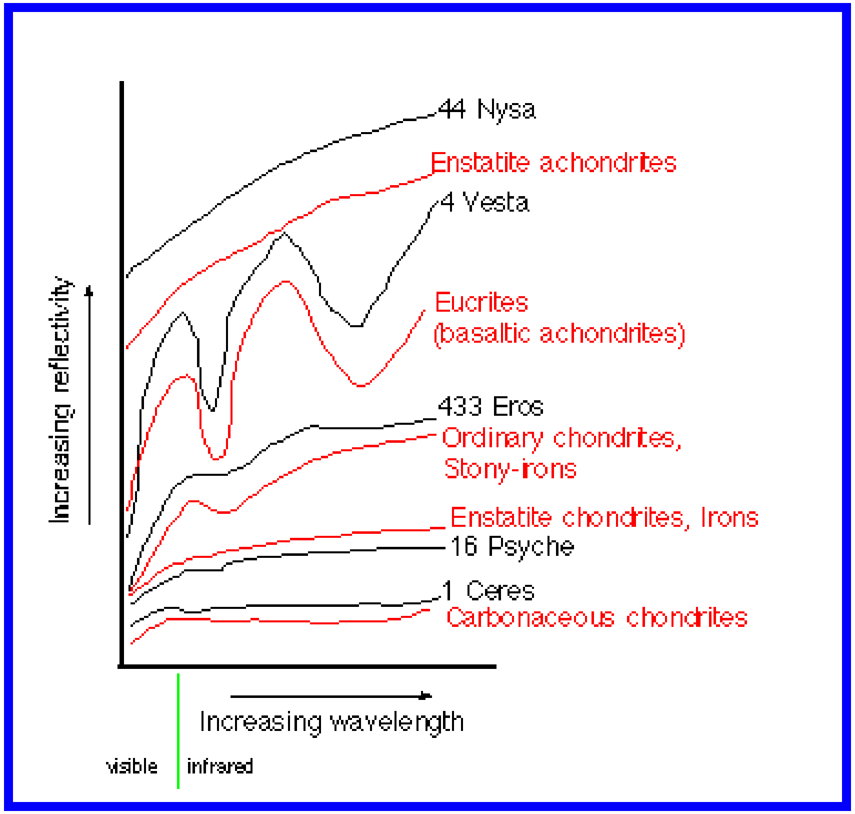
Analyzing light

- Different light sources have different electromagnetic spectra
- These spectra give a "signature" for the chemical composition of the light source
- Fourier analysis lets us compute a light source's "makeup"

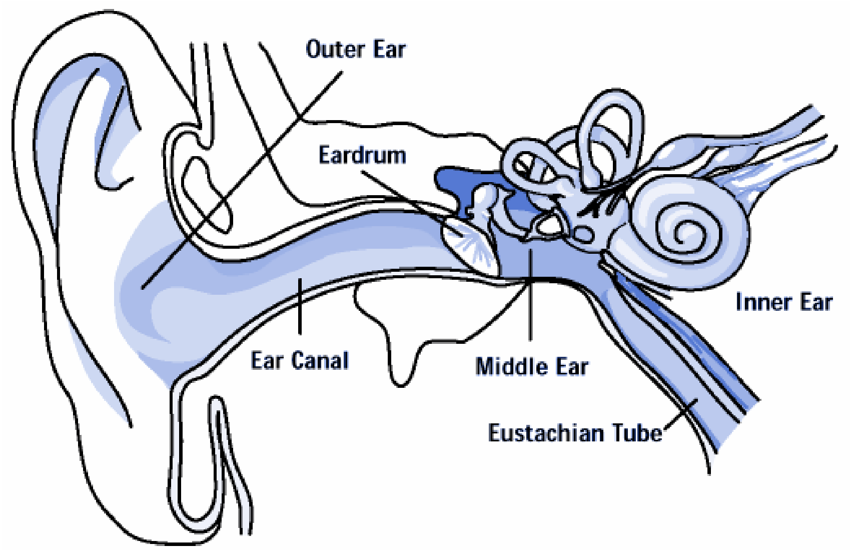


Analyzing light

- For example, it's possible to identify whether a given meteor comes from a given asteroid (!)

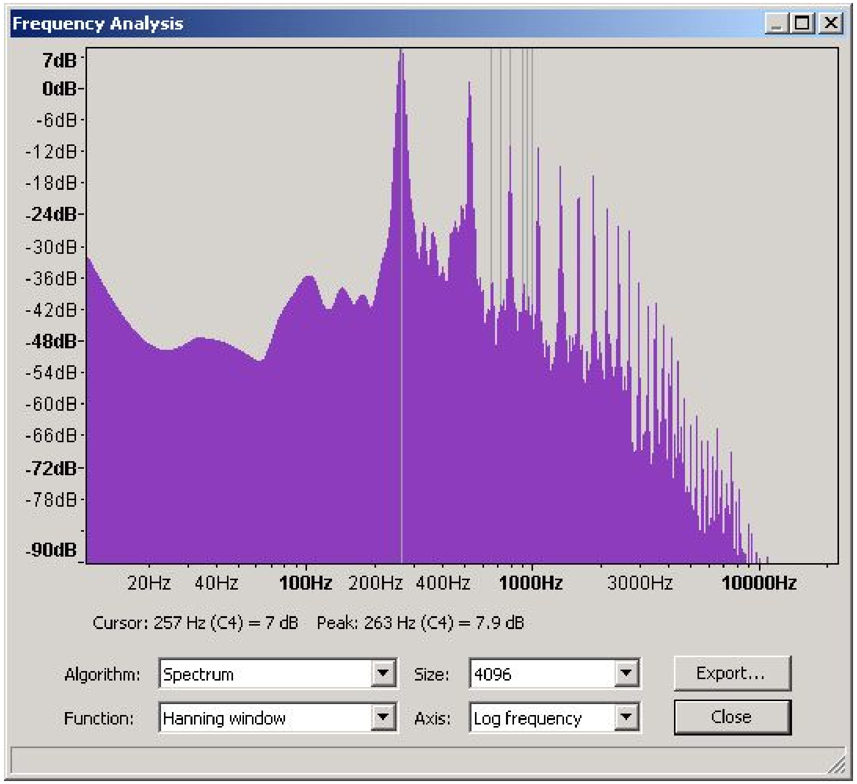


Analyzing sound



- The sound we hear is composed of sound waves with different wavelengths
- Humans hear wavelengths from 20-20,000 Hz

Analyzing sound



- Sound sources are rarely “pure”
- Middle C is about 261 Hz
- A piano playing middle C has a different sound composition than a guitar playing middle C

Let there be synths

- By studying how an instrument “makes” its sound through Fourier analysis, we can get a computer to imitate the sound of the instrument

Autotune

"Autotune" is produced by

- Analyzing a sound source from a mic
- Seeing a sound wave with undesirable frequency
- "Moving" that sound wave to a higher frequency
- Playing the "moved" sound from the speaker

How mp3 compression works

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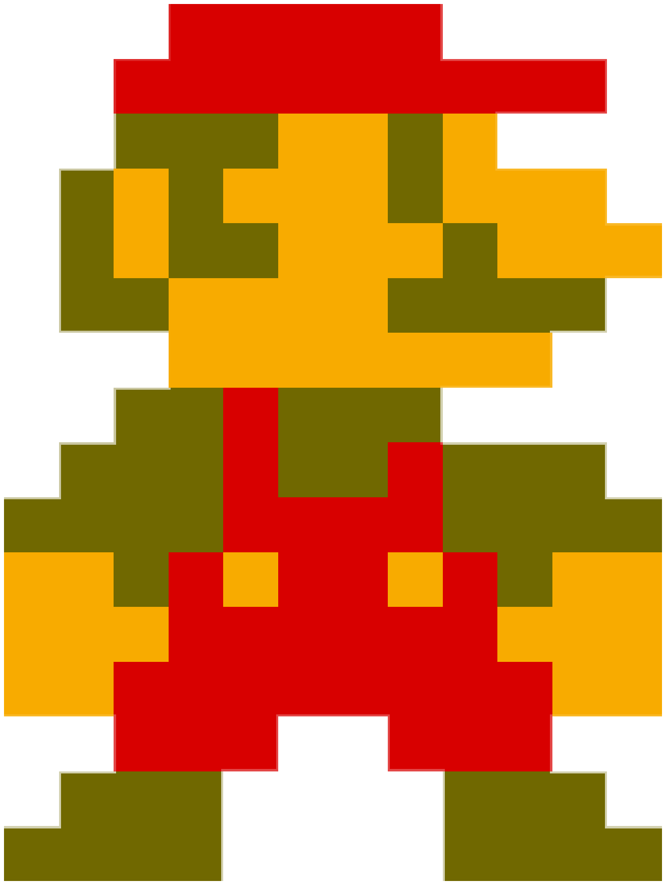
  - McGurk effect (what you hear may depend on what you see!)

How mp3 compression works

- Do a Fourier analysis of a sound
- Determine which constituent sounds aren't “heard” by your ear
- Throw those sound sounds away, keep the sounds you can hear
- Reconstitute the sound from the preserved constituent pieces

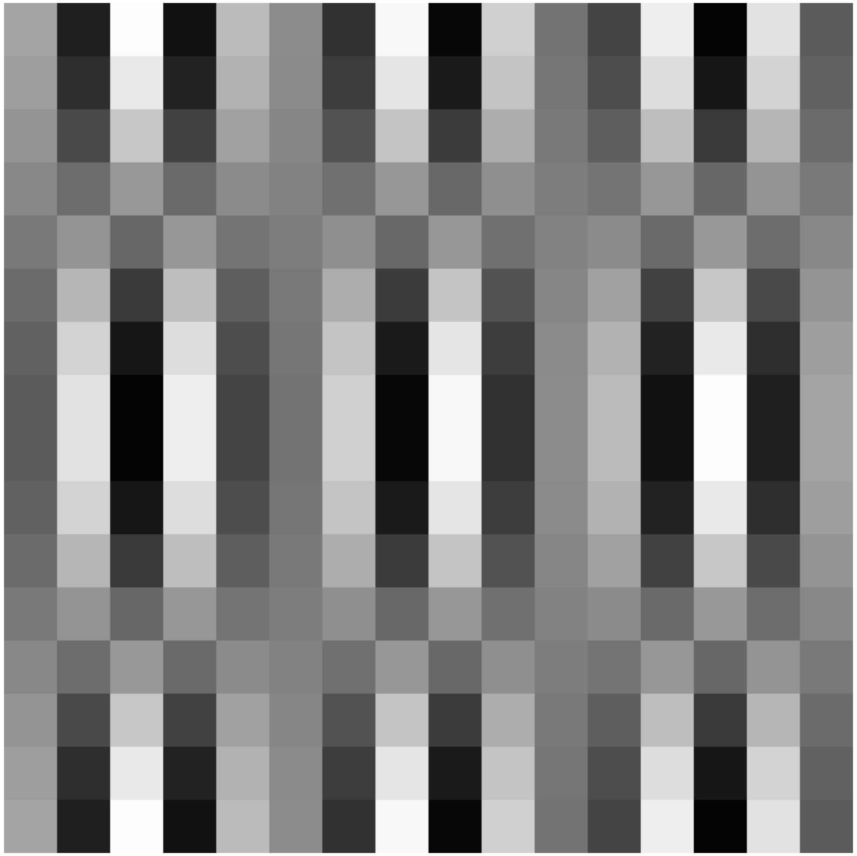
Images are wavy too

- An image that is 12 pixels by 16 pixels has 12*16 pieces of information
- Usually we think of that info “pixel by pixel”
- This is a “local” approach



Thinking globally instead of locally

- There are other ways to store image information
- Can express the image as a sum of "wavy" pictures
- Each piece has global impact
- Both horizontal and vertical frequencies



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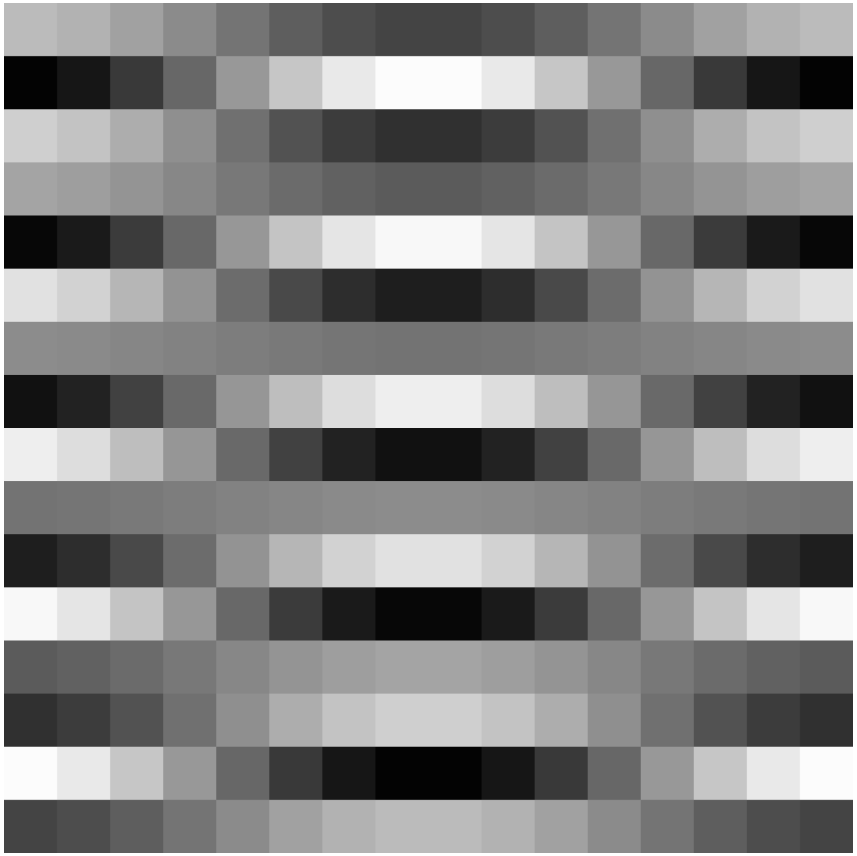

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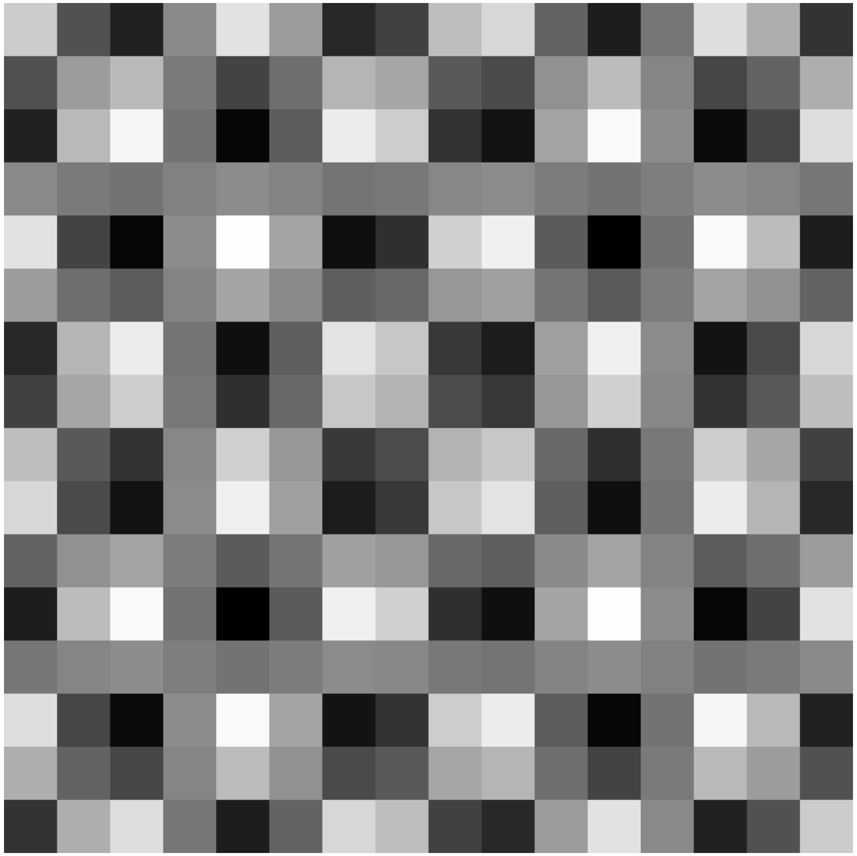


How JPEG compression works

- Your eye isn't good at detecting differences in brightness for “high frequency” components
- Throwing away most high frequency components doesn't affect image too much

JPEG at work

- The original image is 355 pixels wide, 355 pixels tall (126,000 pieces of information)

JPEG at work

- The original image is 355 pixels wide, 355 pixels tall (126,000 pieces of information)
- Here's the image with only “low frequency” compenents (20Hz or smaller)
  - Low frequency is bad at detail
  - .3% compression

JPEG at work

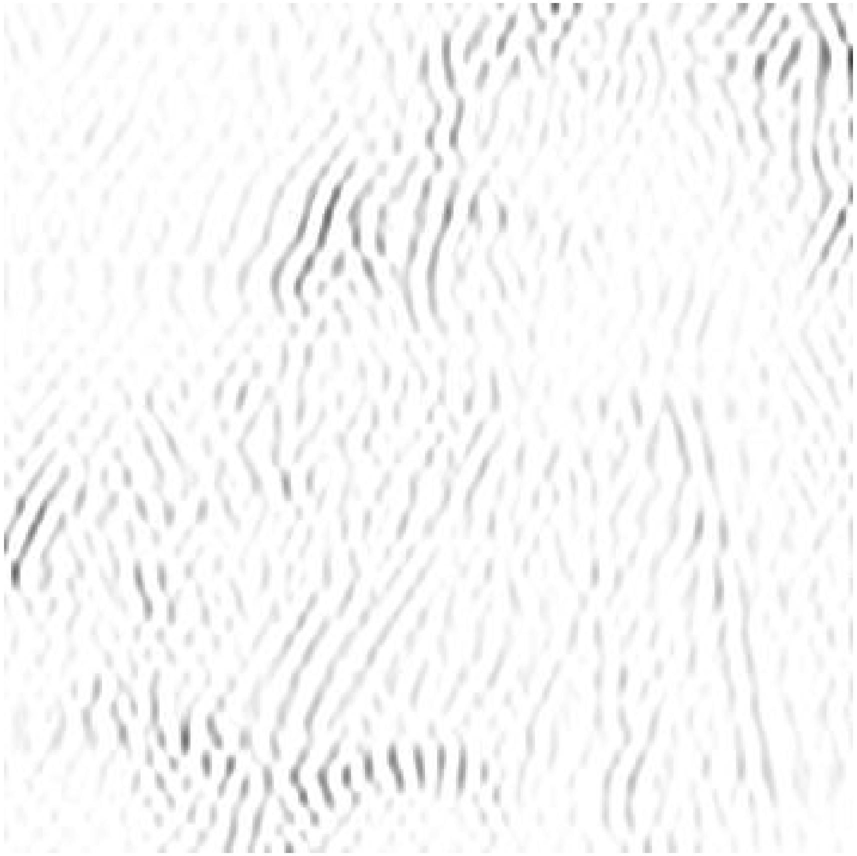
- The original image is 355 pixels wide, 355 pixels tall (126,000 pieces of information)
- Here's the image with only “low frequency” compenents (50Hz or smaller)
  - 2% compression

JPEG at work

- Here's the image with "medium" frequency in horizontal direction, "low" frequency in vertical
  - 2% compression
  - "horizontal detail"

JPEG at work

- Here's the image with “medium” frequency in vertical direction, “low” frequency in horizontal
  - 2% compression
  - “vertical detail”



JPEG at work

- Here's the image with only “mixed frequency” compenents (at most 50Hz in on direction, between 50Hz and 100Hz in another)
  - 4% compression
  - Good at seeing detail, bad at “big picture”

JPEG at work

- Here's the image with “mixed frequency” compenents and low frequency
  - 6% compression
  - A pretty good approximation!