(1) $\displaystyle\int \frac{1}{2+4x}\ dx$

Solution. Let $u = 2+4x$, so that $du = 4dx$. Therefore $dx = \frac{1}{4}du$. Hence we have

$$\int \frac{1}{2+4x}\ dx = \int \frac{1}{u}\ \left(\frac{1}{4}du\right) = \frac{1}{4}\int \frac{1}{u}\ du = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|2+4x| + C.$$

□

(2) $\displaystyle\int_0^{\pi/5} \sin(5\theta)\ d\theta$

Solution. Let $u = 5\theta$, so that $du = 5d\theta$, and hence $d\theta = \frac{1}{5}du$. Since we have a definite integral whose boundary values are given in terms of u, when making the change of variable we should also express the boundary conditions in the language of v. For this, note that when $\theta = 0$ we have $u = 0$, and when $\theta = \frac{\pi}{5}$ we have $u = \pi$. Hence

$$\int_{\theta=0}^{\pi/5} \sin(5\theta)\ d\theta = \int_{u=0}^{\pi} \sin(u)\left(\frac{1}{5}du\right) = \frac{1}{5}\int_{u=0}^{\pi} \sin(u)\ du = \frac{1}{5}\left[-\cos(u)+C\right]_{u=0}^{\pi}$$
$$= \frac{1}{5}\left[-\cos(\pi) + C - (-\cos(0)+C)\right] = \frac{2}{5}.$$

Alternatively, one could choose not to change the boundary conditions into the v-variable, assuming that one computes the antiderivative via substitution, expresses the answer back in the u-variable, and then applies the Evaluation Theorem. My experience seeing lots of students working on these kinds of problems suggests that it is MUCH BETTER to simply change the boundary conditions into the language of the new variable!

[Note that the integration constant simply vanishes when applying the evaluation theorem. For that reason, we won't continue to write it when computing a definite integral via the evaluation theorem.]

□

(3) $\displaystyle\int_1^3 \frac{u}{1+u^2}\ du$

Solution. Let $v = 1+u^2$, so that $dv = 2u\ du$. Hence we have $\frac{1}{2}dv = u\ du$. Again we need to express the boundary conditions in the new variable; observe that when $u = 1$ we have $v = 2$, and when $u = 3$ we have $v = 10$. So we conclude

$$\int_{u=1}^{3} \frac{u}{1+u^2}\ du = \frac{1}{2}\int_{v=2}^{10} \frac{1}{v}\ dv = \frac{1}{2}\left[\ln|v|\right]_{v=2}^{10} = \frac{1}{2}\left[\ln(10) - \ln(2)\right] = \frac{1}{2}\ln(5).$$

□

(4) $\displaystyle\int \frac{\sin(t)}{\cos^2(t)}\ dt$

Solution. Note that the denominator has the cosine function inside the squaring function, so let's try the substitution $u = \cos(t)$; this gives $du = -\sin(t)\ dt$, or $-du = \sin(t)\ dt$. This is good news, since we have a factor of $\sin(t)\ dt$ in our integrand. We have

$$\int \frac{\sin(t)}{\cos^2(t)}\ dt = \int \frac{1}{u^2}\ (-du) = -\int \frac{1}{u^2}\ du = -(-\frac{1}{u}) = \frac{1}{u} + C = \frac{1}{\cos(t)} + C = \sec(t) + C.$$

Though not necessary, it's good to occasionally check one's answer when computing antiderivatives. Fortunately the check is quite simple: one just differentiates the supposed antiderivative and makes sure that the given integrand is returned. In this case, we use familiar differentiation rules and some algebra:

$$\frac{d}{dt}[\sec(t)] = \sec(t)\tan(t) = \frac{1}{\cos(t)}\frac{\sin(t)}{\cos(t)} = \frac{\sin(t)}{\cos^2(t)}.$$

This is precisely the integrand we started with, so we must have correctly computed the antiderivative.

Here's a slightly less direct method that uses only the chain rule and basic trigonometric derivatives:

$$\frac{d}{dt}[\sec(t)] = \frac{d}{dt}\left[\frac{1}{\cos(t)}\right] = \frac{d}{dt}\left[(\cos(t))^{-1}\right] = -1\cdot(\cos(t))^{-2}\cdot(-\sin(t)) = \frac{\sin(t)}{\cos^2(t)}.$$

□

(5) $\displaystyle\int x^2 e^{-x^3}\, dx$

Solution. Since the function $-x^3$ is in the exponential function, we'll try $u = -x^3$. This gives $du = -3x^2\, dx$, or $-\frac{1}{3}\, du = x^2\, dx$. Since we have a factor of $x^2\, dx$ in our integrand, we're set to go:

$$\int x^2 e^{-x^3}\, dx = \int e^u\left(-\frac{1}{3}du\right) = -\frac{1}{3}\int e^u\, du = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{-x^3} + C.$$

□

(6) $\displaystyle\int_0^1 (u+1)(u^2+2u)^{10}\, du$

Solution. One could solve this by simply expanding the polynomial, but this would be computationally painful. Instead, let's use the substitution $v = u^2 + 2u$, so that $dv = (2u+2)du = 2(u+1)du$. Hence we have $\frac{1}{2}\, dv = (u+1)\, du$ (perfect for our integrand!). For the boundary conditions, when $u = 0$ we have $v = 0$, and when $u = 1$ we have $v = 3$. So we're left with

$$\int_{u=0}^1 (u+1)(u^2+2u)^{10}\, du = \int_{v=0}^3 v^{10}\left(\frac{1}{2}\, dv\right) = \frac{1}{2}\left[\frac{v^{11}}{11}\right]_{v=0}^3 = \frac{1}{22}\cdot 3^{11}.$$

□

(7) $\displaystyle\int \sec^2(x)\ \sqrt[3]{\tan(x)}\, dx$

Solution. Let $u = \tan(x)$, so that $du = \sec^2(x)\, dx$. After substituting we find

$$\int \sec^2(x)\ \sqrt[3]{\tan(x)}\, dx = \int \sqrt[3]{u}\, du = \int u^{1/3}\, du = \frac{3}{4}u^{4/3} + C = \frac{3}{4}(\tan(x))^{4/3} + C.$$

□

(8) $\displaystyle\int \frac{1}{x^2}\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)\, dx$

Solution. The obvious composition in this case has the function $\frac{1}{x}$ plugged into the function $t^2\sec(t)\tan(t)$, so let's go with the substitution $u = \frac{1}{x}$. This gives $du = -\frac{1}{x^2}\, dx$ which fits our integrand very nicely (since the factor of $\frac{1}{x^2}dx$ in our integrand will simply transform to $-du$). So we have

$$\int \frac{1}{x^2}\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)\, dx = -\int \sec(u)\tan(u)\, du = -\sec(u) + C = -\sec\left(\frac{1}{x}\right) + C.$$

□

(9) $\displaystyle\int \frac{x\sqrt{\arctan(x^2)}}{1+x^4}\ dx$

Solution. Let $u = \arctan(x^2)$, so that $du = \frac{1}{1+(x^2)^2} \cdot 2x\ dx$. This means that $\frac{1}{2}du = \frac{x}{1+x^4}dx$, which precisely matches a term in our integrand. So we have

$$\int \frac{x\sqrt{\arctan(x^2)}}{1+x^4}\ dx = \frac{1}{2}\int \sqrt{u}\ du = \frac{1}{2}\int u^{1/2}\ du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}u^{3/2} + C = \frac{1}{3}\left(\sqrt{\arctan(x^2)}\right)^3 + C.$$

$\square$

Solutions for "Problems requiring Integration by Parts".

(1) $\int_0^{\pi} x\cos(2x)\,dx$. Let $f(x) = x$ and $g'(x)dx = \cos(2x)\,dx$. Then $f'(x) = 1$,
and $g(x) = \int \cos(2x)\,dx = \frac{1}{2}\sin(2x)$ (use a u-sub w/ $u = 2x$).

So $$\int_0^{\pi} x\cos(2x)dx = \frac{1}{2}x\sin(2x)\Big]_0^{\pi} - \frac{1}{2}\int_0^{\pi}\sin(2x)dx$$

$$= \left[\frac{1}{2}x\sin(2x) - \frac{1}{2}\left(-\frac{1}{2}\cos(2x)\right)\right]_0^{\pi} = \left[\frac{\pi}{2}\sin(2\pi) + \frac{1}{4}\cos(2\pi) - 0 - \frac{1}{4}\cos(0)\right]$$

$$= \frac{1}{4} - \frac{1}{4} = 0$$

(2) $\int xe^{3x}\,dx$. Let $f(x) = x$ and $g'(x)dx = e^{3x}\,dx$. Then $f'(x) = 1$, and
$g(x) = \frac{1}{3}e^{3x}$ (using a u-sub w/ $u = 3x$).

So $$\int xe^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C.$$

(3) $\int_1^e \ln(x^2)\,dx$. Let $f(x) = \ln(x^2)$ and $g'(x)dx = 1\cdot dx$, so $f'(x)dx = \frac{1}{x^2}\cdot 2x\,dx = \frac{2dx}{x}$
and $g(x) = x$.

So $$\int_1^e \ln(x^2)dx = x\ln(x^2)\Big]_1^e - \int_1^e x\cdot\frac{2}{x}\,dx = \Big[x\ln(x^2) - 2x\Big]_1^e$$

$$= \left[e\ln(e^2) - 2(e) - (\ln(1) - 2)\right]$$

$$= 2e - 2e + 2 = 2.$$

(4) $\int \arctan(x)\,dx$. Let $f(x) = \arctan(x)$ and $g'(x)dx = 1\cdot dx$, so $f'(x)dx = \frac{1}{1+x^2}\,dx$
and $g(x) = x$.

Then $\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx$. Using $u = 1+x^2$, we get

$$\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C.$$

Note: $\ln|1+x^2| = \ln(1+x^2)$ since $1+x^2 \geq 0$ for all x.

(5) $\int r^2 \ln(r)\, dr$. let $f(r) = \ln(r)$ and $g'(r)dr = r^2 dr$, so $f'(r) = \frac{1}{r}$ and $g(r) = \frac{1}{3}r^3$.

So $\int r^2 \ln(r) dr = \frac{1}{3} r^3 \ln(r) - \frac{1}{3}\int r^3 \cdot \frac{1}{r}\, dr = \frac{1}{3}\left(r^3 \ln(r) - \frac{1}{3} r^3 \right) + C$

(6) $\int \sin^2(x)\, dx$. let $f(x) = \sin(x)$ and $g'(x)\,dx = \sin(x)\,dx$, so $f'(x) = \cos(x)$ and $g(x) = -\cos(x)$.

Then $\int \sin^2(x)\,dx = -\cos(x)\sin(x) - \left(-\int \cos^2(x)\,dx\right) = -\cos(x)\sin(x) + \int \cos^2(x)\,dx$

Now $\cos^2(x) = 1 - \sin^2(x)$, so $\int \cos^2(x)\,dx = \int (1-\sin^2(x))dx = \int dx - \int \sin^2(x)\,dx$

$= x - \int \sin^2(x)\,dx$,

so we get

$$\int \sin^2(x)\,dx = -\cos(x)\sin(x) + \int \cos^2(x)\,dx = -\cos(x)\sin(x) + x - \int \sin^2(x)\,dx.$$

Adding $\int \sin^2(x)\,dx$ to both sides gives $2\int \sin^2(x)dx = -\cos(x)\sin(x) + x$,

and so $\int \sin^2(x)\,dx = \frac{1}{2}\left(-\cos(x)\sin(x) + x\right) + C$.

(7) $\int e^\theta \sin(\theta)\,d\theta$. let $u = \sin(\theta)$ and $dv = e^\theta d\theta$, so $du = \cos(\theta)\,d\theta$ and $v = e^\theta$.

Then $\int e^\theta \sin(\theta)\,d\theta = e^\theta \sin(\theta) - \int e^\theta \cos\theta\, d\theta$.

To evaluate $\int e^\theta \cos\theta\, d\theta$, let $r = \cos\theta$ and $ds = e^\theta d\theta$, so $dr = -\sin\theta\, d\theta$ and $s = e^\theta$.

Then $\int e^\theta \cos\theta\, d\theta = e^\theta \cos\theta + \int e^\theta \sin\theta\, d\theta$. Plugging this back in gives $\int e^\theta \sin(\theta)\,d\theta = e^\theta \sin(\theta) - e^\theta \cos\theta - \int e^\theta \sin\theta\, d\theta$. Solving this equation for $\int e^\theta \sin\theta\, d\theta$ gives $\int e^\theta \sin(\theta)\,d\theta = \frac{1}{2}\left[e^\theta \sin(\theta) - e^\theta \cos(\theta)\right) + C$.

(1) $\int x^5 e^{x^3}\,dx$. Start off with the substitution $u = x^3$, so

$$\int x^5 e^{x^3}dx = \int x^3 \cdot x^2 e^{x^3}dx = \tfrac{1}{3}\int u e^u du.$$ Now evaluate

this integral using integration by parts, with $f(u) = u$

and $g'(u)du = e^u du$. Then $f'(u) = 1$ and $g(u) = e^u$, so

$$\tfrac{1}{3}\int ue^u du = \tfrac{1}{3}\left(ue^u - \int e^u du\right) = \tfrac{1}{3}ue^u - \tfrac{1}{3}e^u + C = \tfrac{1}{3}x^3 e^{x^3} - \tfrac{1}{3}e^{x^3} + C.$$

(2) $\int \sin(\sqrt{2x})\,dx$. Start off with the substitution $r = \sqrt{2x}$, so

$dr = \sqrt{2}\,\frac{1}{2\sqrt{x}}\,dx$, so that $dx = \sqrt{2}\sqrt{x}\,dr = r\,dr$. Then

$\int \sin(\sqrt{2x})dx = \int r\sin(r)dr$. Now use integration by parts: $u = r$ $du = dr$ $dv = \sin(r)$ $v = -\cos(r)$

So $\int r\sin(r)dr = -r\cos(r) + \int \cos(r)dr = -r\cos(r) + \sin(r) + C$

$= -\sqrt{2x}\cos(\sqrt{2x}) + \sin(\sqrt{2x}) + C.$

(3) $\int t^3\ln(1+t^2)\,dt$. Start off with the substitution $w = 1+t^2$, so $dw = 2t\,dt$.

Then $t^2 = w-1$, so $\int t^3\ln(1+t^2)dt = \int t^2\ln(1+t^2)\,t\,dt = \tfrac{1}{2}\int (w-1)\ln(w)dw.$

$= \tfrac{1}{2}\int w\ln(w)dw - \tfrac{1}{2}\int \ln(w)dw$

Both these integrals require integration by parts, and result in

$\tfrac{1}{2}\int w\ln(w)dw - \tfrac{1}{2}\int \ln(w)dw = \tfrac{1}{2}w^2\ln(w) - \tfrac{1}{4}w^2 - \tfrac{1}{2}w\ln(w) - \tfrac{1}{2}w + C$

$= \tfrac{1}{2}(1+t^2)^2\ln(1+t^2) - \tfrac{1}{4}(1+t^2)^2 - \tfrac{1}{2}(1+t^2)\ln(1+t^2) - \tfrac{1}{2}(1+t^2) + C$