Lecture 3 Bonus

Why FToC II works

What does FToC II say?

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$.

Recall from Calc I, That for any point x_0, we have

$$F'(x_0) = \lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{(x_0+h) - x_0}$$

$$= \lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h}$$

Hence, for "small h", $F'(x_0) \approx \dfrac{F(x_0+h) - F(x_0)}{h}$

Now consider $F(b) - F(a)$

By the last slide (with $x_0 = a$, $x_0 + h = b$, $h = b - a$,

we get

$$\frac{F(b) - F(a)}{b-a} \approx F'(a) = f(a)$$

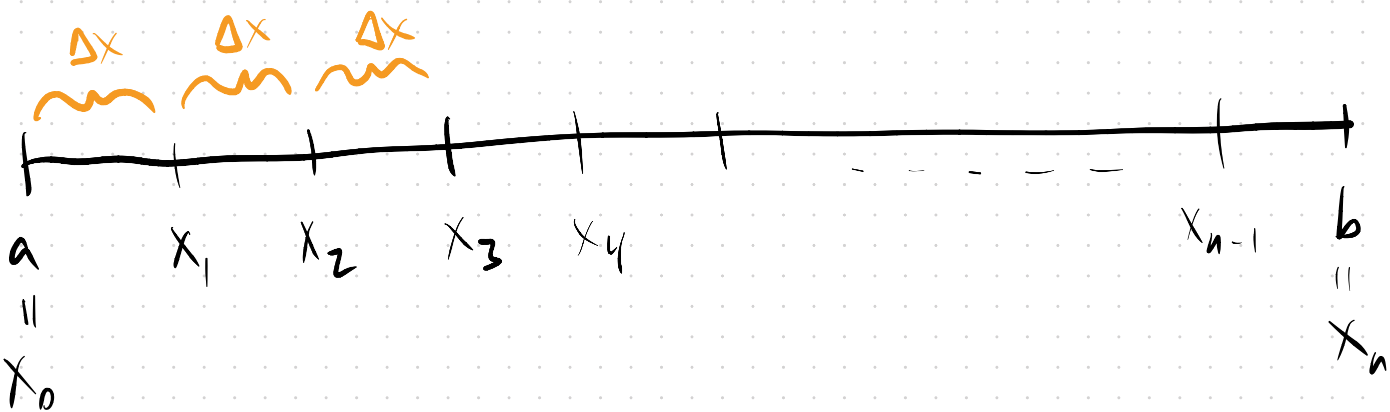
Hence $F(b) - F(a) \approx f(a) * (b-a)$ ... but only if $b-a$ is tiny.

How to get a better approximation on $F(b)-F(a)$?

Problem: $b-a$ is probably too big!

Solution: create intermediate points!

$$F(b)-F(a) = \big(F(b)-F(x_{n-1})\big) + \big(F(x_{n-1})-F(x_{n-2})\big) + \cdots$$
$$+ \big(F(x_3)-F(x_2)\big) + \big(F(x_2)-F(x_1)\big) + \big(F(x_1)-F(a)\big)$$



Idea: we can approximate "good approximation"

$$F(x_{i+1}) - F(x_i) \approx F'(x_i) * \Delta x = f(x_i) * \Delta x$$

Since we can make Δx really tiny, this is a good approximation! So, we get

$$F(b) - F(a) = \Big(F(b) - F(x_{n-1})\Big) + \cdots + \Big(F(x_2) - F(x_1)\Big) + \Big(F(x_1) - F(a)\Big)$$

$$\approx \underbrace{f(x_{n-1})\,\Delta x + \cdots + f(x_1)\,\Delta x + f(a)\,\Delta x}_{\text{left hand Riemann sum!}}$$

So if we take a limit on both sides

1. The Riemann sum become the definite integral
2. The approximation becomes an equality

Upshot:

$$F(b) - F(a) = \int_a^b f(x)\,dx$$