Lecture 5 Bonus

u-substitution problems

Procedure for using u-substitution

0. Determine if u-substitution is a reasonable thing to try.

→ Is there a composition?

→ Is the derivative of the "interior" function appear in integrand?

1. Let u be the interior function
2. Compute du/dx and write du in terms of dx & x stuff
3. Rewrite the integral in terms of the substituted variable
4. Is the new integral "better" than before?
5. Evaluate the new integral

Ex $\displaystyle\int (\underbrace{\ln(x)}_{u})^2 \underbrace{\frac{1}{x}\,dx}_{du}$

Let $u = \ln(x)$, Then

$\dfrac{du}{dx} = \dfrac{1}{x}$. So

$du = \dfrac{1}{x}\,dx$

$= \displaystyle\int u^2\,du$

$= \dfrac{1}{3}u^3 + C$

$= \dfrac{1}{3}(\ln(x))^3 + C$

Ex $$\int \frac{\sin(x)}{\cos(x)+1}\,dx = \int \sin(x)\cdot(\underbrace{\cos(x)+1}_{u})^{-1}\,dx$$

$-du$

$$= \int u^{-1}(-du)$$

$$= \int u^{-1}\cdot -1\cdot du$$

$$= -1\int u^{-1}\,du$$

$$= -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\cos(x)+1| + C$$

let $u = \cos(x)+1$, so $\frac{du}{dx} = -\sin(x)$, hence

$du = -\sin(x)\,dx$. Negate both sides to get

$$-du = \sin(x)\,dx$$

Ex $\displaystyle\int \frac{\sin(x)}{(\cos(x))^2+1}\,dx$

Problem! If we try to substitute in the u-variable, the denominator becomes u and numerator is just $\sin(x)\,dx$. It has no "$\cos(x)$" term to give du.

What would happen if we chose $u=(\cos(x))^2+1$ in the same manner as the last problem? Then

$$\frac{du}{dx} = 2(\cos(x))^1 \cdot (-\sin(x))$$

hence

$$du = -2\cos(x)\cdot\sin(x)\,dx$$

Ex $\displaystyle\int \frac{\sin(x)}{(\cos(x))^2+1}\,dx$

$$= \int \frac{-du}{u^2+1}$$

$$= -\int \frac{1}{u^2+1}\,du$$

$$= -\arctan(u) + C$$

$$= -\arctan(\cos(x)) + C$$

We need to try something else. Let's choose u to be $\cos(x)$ instead!

Since $u=\cos(x)$, we get $\frac{du}{dx} = -\sin(x)$, hence

$du = -\sin(x)\,dx$. So

$$-du = \sin(x)\,dx$$

Ex $$\int \frac{(\arctan(u))'}{1+u^2}\,du$$

$$= \int (\arctan(u))' \cdot \frac{1}{1+u^2}\,du$$

$$= \int t'\,dt$$

$$= \frac{1}{2}t^2 + C$$

$$= \frac{1}{2}(\arctan(u))^2 + C$$

Note: u is already the original variable's name! So we'll do a t-substitution.

So, let's choose $t = \arctan(u)$, so that

$$\frac{dt}{du} = \frac{1}{u^2+1}, \text{ so}$$

$$dt = \frac{1}{u^2+1}\,du.$$

Ex Let's consider $\int x^5 e^{x^3}\,dx$ and rewrite using a u-substitution. (Note: we won't solve it "all the way" for now, but just substitute.)

Since x^3 is "plugged into" the exponentiation function, we'll try $u = x^3$. Hence $\frac{du}{dx} = 3x^2$, so $du = 3x^2 dx$.

Now $\int x^5 e^{x^3}\,dx = \int x^3 e^{x^3} x^2\,dx = \int u e^u \left(\frac{1}{3} du\right)$

$= \frac{1}{3}\int u e^u\,du$