Day 12

Improper integrals of type II

WARM-UP PROBLEMS

"converges"

① Is $\int_1^\infty \frac{1}{1+x^2}\,dx$ finite, or does it fail to exist?

"diverges"

② Is $\int_{-\infty}^\infty e^{-x}\,dx$ finite? Or does it fail to exist?

③ Is $\int_2^\infty \frac{1}{x(\ln(x))^2}\,dx$ finite? Or does it fail to exist?

Lingo: if improper integral is finite, it "converges". Otherwise, it "diverges"

Remember last time we said that to use FToC II to compute an area problem $\int_a^b f(x)\,dx$, we need

1. we need $[a,b]$ to be finite
2. we need $f(x)$ to be continuous on $[a,b]$

If this fails, we have a "type I" improperness

If this fails, we have a "type II" improperness

To resolve type II improprieties, we

a) ensure the improperness appears at far end of interval

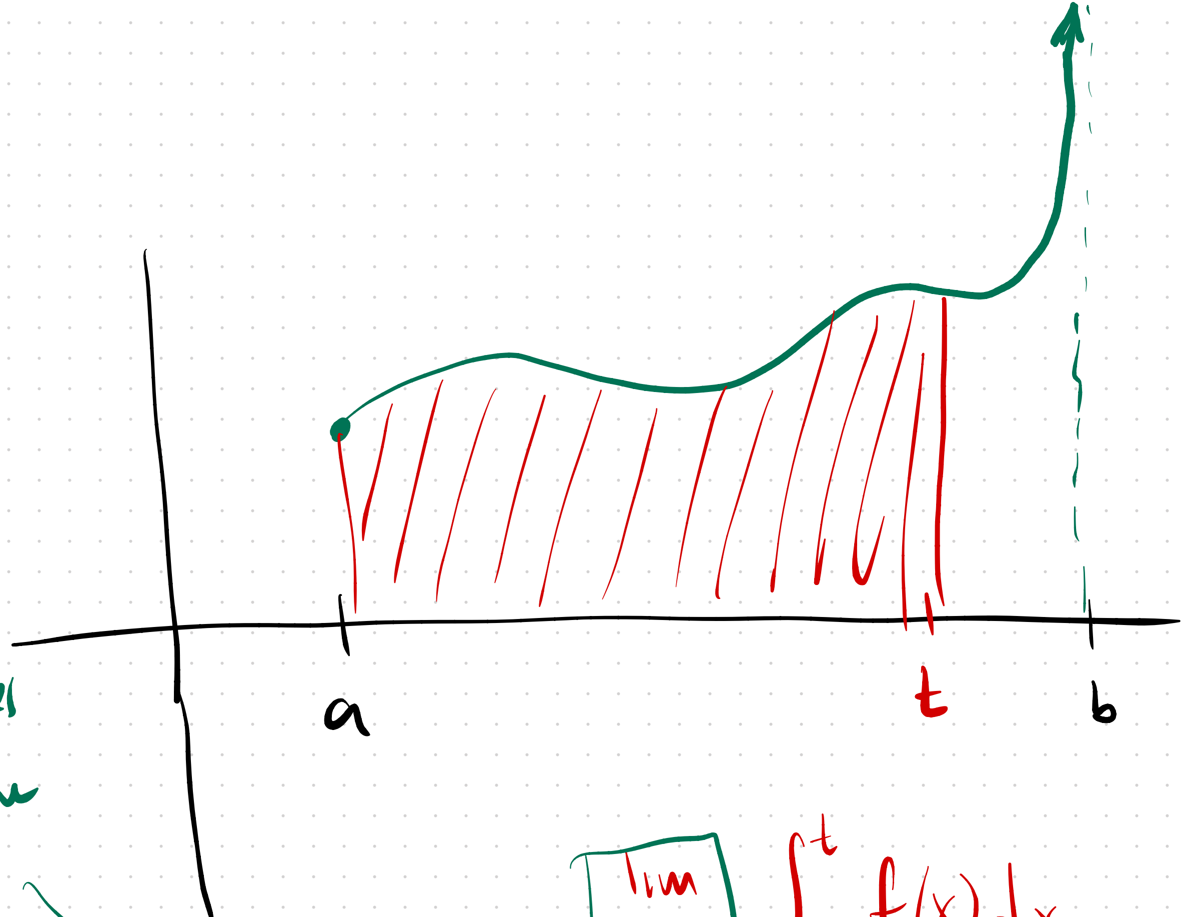
b) evaluate by using limits to approach source of improperness

To be more precise

<u>Def'n</u> (Type II improper integral) Suppose $f(x)$ is continuous on $[a,b)$, then we define

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

idea



t approaches b from the left

$$\boxed{\lim_{t \to b^-}} \int_a^t f(x)\,dx$$

Ex $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

Note: $\frac{1}{x^2}$ is discontinuous @ 0, but continuous elsewhere

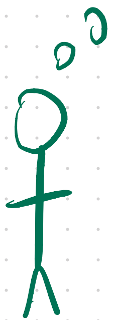
To get started, since the integrand is discontinuous on the left hand endpoint of $[0,1]$, we get

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx$$

function is discontinuous

Now $\lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-2}\,dx$

$$= \lim_{t\to 0^+} \left[\frac{1}{-1} x^{-1} \right]_t^1$$



t is tiny and positive

$\frac{1}{t}$ is then $\frac{1}{\text{tiny}}$

is huge (and positive)

$$= \lim_{t\to 0^+} \left[-\frac{1}{x} \right]_t^1$$

$$= \lim_{t\to 0^+} \left[-\frac{1}{1} - \left(-\frac{1}{t} \right) \right] = +\infty$$

So: $\int_0^1 \frac{1}{x^2}\,dx$ "diverges" to $+\infty$

Ex $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$

Like last time, integrand is discontinuous at $x=0$.

Like last time, this means

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^4 \frac{1}{\sqrt{x}}\,dx$$

Now $\displaystyle\lim_{t\to 0^+}\int_t^4 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^4 x^{-1/2}\,dx$

$$= \lim_{t\to 0^+}\left[\frac{1}{1/2}x^{1/2}\right]_t^4$$

$$= \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^4$$

$$= \lim_{t\to 0^+}\left[2\sqrt{4} - 2\sqrt{t}\right]$$

$$= 2\sqrt{4} - 0 = 4.$$

When t is tiny and positive, we get $\sqrt{t}$ is tiny (and positive)

So $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$ converges to 4

What happens if an integral is <u>really</u> improper?
(Ie, has discontinuities on the inside of the interval? or both endpoints of the interval? or is improper of types I and II at the same time?)

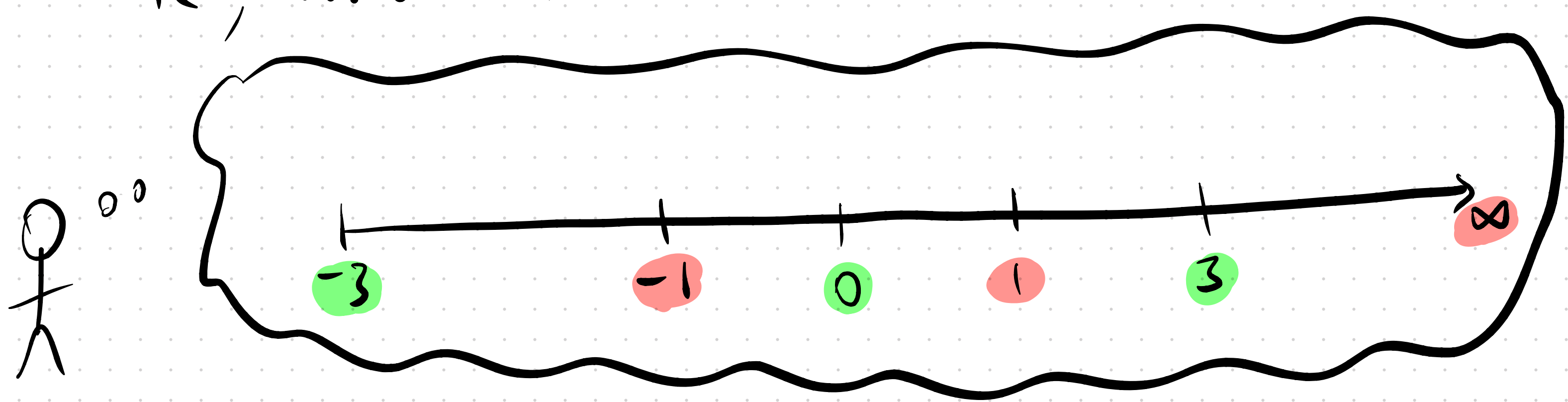
Key idea: break your integral into subintegrals, where each subintegral has only a single "improperness," and that improperness occurs at an endpoint.

Ex Does $\displaystyle\int_{-3}^{\infty} \frac{2x}{x^2-1}\,dx$ converge or diverge?

We can quickly see a type I improperness "at ∞"

We have type II impropernesses whenever $x^2-1=0$

ie, when $x=-1$ or $x=1$



We chop up $[-3,\infty)$ into subintervals

$$[-3,-1],\ [-1,0],\ [0,1],\ [1,3],\ [3,\infty).$$

Key fact: each subinterval has only one improperness, and it happens at an endpoint.

If any of these 5 subintegrals diverges so does the original

Evaluate $\displaystyle\int_{-3}^{\infty} \frac{2x}{x^2-1}\,dx$ as

$$\int_{-3}^{-1} \frac{2x}{x^2-1}\,dx + \int_{-1}^{0} \frac{2x}{x^2-1}\,dx + \int_{0}^{1} \frac{2x}{x^2-1}\,dx + \int_{1}^{3} \frac{2x}{x^2-1}\,dx + \int_{3}^{\infty} \frac{2x}{x^2-1}\,dx$$

Solutions to Warm Up Problems

① Is $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ finite, or does it fail to exist?

This is improper only at ∞, so we get

$$\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{1+x^2}\,dx = \lim_{R\to\infty}\Big[\arctan(x)\Big]_1^R$$

$$= \lim_{R\to\infty}\Big[\arctan(R) - \arctan(1)\Big]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

arctan(R) means: what angle θ gives $\tan(\theta) = R$. When R is big, we know $\theta \approx \frac{\pi}{2}$ has $\sin(\theta) \approx 1$ and $\cos(\theta) \approx 0$, so $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{1}{\text{tiny}} = \text{BIG!}$ So $\arctan(\text{big}) \approx \pi/2$.

② Is $\int_{-\infty}^{\infty} e^{-x}\,dx$ finite? Or does it fail to exist?

Since e^{-x} is continuous on $(-\infty, \infty)$, we evaluate as

$$\int_{-\infty}^{\infty} e^{-x}\,dx = \int_{-\infty}^{0} e^{-x}\,dx + \int_{0}^{\infty} e^{-x}\,dx.$$

Remember that for the left side to converge, both integrals on the right have to converge.

We'll examine $\int_{-\infty}^{0} e^{-x}\,dx$ first.

Now $\displaystyle\int_{-\infty}^{0} e^{-x}\,dx = \lim_{R\to-\infty}\int_{R}^{0} e^{-x}\,dx = \lim_{R\to-\infty}\left[-e^{-x}\right]_R^0$

$$= \lim_{R\to-\infty}\left[-e^{-0}-\left(-e^{-R}\right)\right]$$

$$= \lim_{R\to-\infty} -1 + \frac{1}{e^{R}}$$

$$= +\infty$$

When R is a big negative number, e^R is super close to 0. So e^{-R} is 1/tiny, which is huge!

Since $\displaystyle\int_{-\infty}^{0} e^{-x}\,dx$ diverges, we don't even need to compute $\displaystyle\int_{0}^{\infty} e^{-x}\,dx$. We already know $\displaystyle\int_{-\infty}^{\infty} e^{-x}\,dx$ diverges.

③ Is $\int_2^\infty \frac{1}{x(\ln(x))^2}\,dx$ finite? Or does it fail to exist?

The integrand is only problematic for negative values of x (since then we can't compute $\ln(x)$) or when $x=0$ (since then the denominator is $0 \cdot \ln(0)$... which is really bad!)

So we do a type I improperness:

$$\int_2^\infty \frac{1}{x(\ln(x))^2}\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln(x))^2}\,dx.$$

Since the antiderivative here is harder, lets first do it on its own, then worry about the related (limit of an) area problem

$$\int \frac{1}{x\,(\ln(x))^2}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{1}{-1}u^{-1} + C$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{\ln(x)} + C.$$

ln(x) is "inside" another function, and its derivative is in the integrand! u-sub is a good idea

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

Now that we know the antiderivative, we compute

$$\lim_{R\to\infty} \int_2^R \frac{1}{x\,(\ln(x))^2}\,dx = \lim_{R\to\infty} \left[-\frac{1}{\ln(x)} \right]_2^R$$

$$= \lim_{R\to\infty} \left[-\frac{1}{\ln(R)} - \left(-\frac{1}{\ln(2)} \right) \right]$$

$$= 0 + 1/\ln(2) = 1/\ln(2).$$

Remember $\ln(R) = z$ means $e^z = R$. So if R is big, we want to know: what do we raise e to in order to get a big number? Answer: a big number!

But if $\ln(R)$ is big when R is big, then $\frac{1}{\ln(R)}$ is super close to 0