Day 34

Calculus of Power Series

Recall that a power series is a function of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

where the c_n are the "coefficients", and a is "center".

Big question: what values of x does f conv/div?

The key to answering this in practice: ratio test

Theorem (Interval of Convergence)

Suppose $f(x) = \sum c_n (x-a)^n$ is given. Then one of the following occurs

(i) The power series converges only for $x=a$ $(R=0)$

(ii) The power series converges for all x $(R=\infty)$

(iii) There exists some $R>0$ so that f converges for all x with $|x-a|<R$ and diverges for all x with $|x-a|>R$

The "R" is called the radius of convergence.

New topic for today: derivatives and antiderivatives of power series.

Thm (Calculus of power series) Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$.

Then (i) $\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$

(ii) $\int f(x)\,dx = \int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$

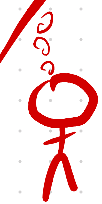
Moreover, both $\frac{d}{dx}[f(x)]$ and $\int f(x)\,dx$ have the same radius of convergence.

Ex We know that for $|x|<1$ we have

$$1-x+x^2-x^3+x^4-\cdots=\sum_{n=0}^{\infty}(-x)^n=\frac{1}{1-(-x)}=\frac{1}{1+x}$$

(Geometric series Theorem)

Original geometric series has radius of convergence $R=1$



Let's antidifferentiate both sides:

RHS: $\displaystyle\int \frac{1}{1+x}\,dx = \ln(1+x)+C$

LHS: $\displaystyle\int 1-x+x^2-x^3+x^4-\cdots\,dx = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}$$

So we get that for some value of C we have

$$\ln(1+x) + C = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

What is the value of C that makes this work?

We'll plug in $x=0$. The series returns 0, and $\ln(1+x)+C$ becomes $\ln(1)+C = 0+C = C$.

Hence we get $0+C=0$, so $C=0$.

Conclusion: we get

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum \frac{(-1)^n}{n+1} x^{n+1}$$

Good news: This still has radius of convergence $R=1$!

So: definitely converges for x in $(-1,1)$

definitely diverges for x in $(-\infty,-1)$ or $(1,\infty)$

Unknown (for now): does it converge at $x=-1$ or $x=+1$?

Does it converge at $x=+1$?

series is then $\underbrace{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots}_{\text{alternating harmonic series}}$, which converge

Conclusion:

$$\ln(2) = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$$

!!!!!!!!!!!!!!!