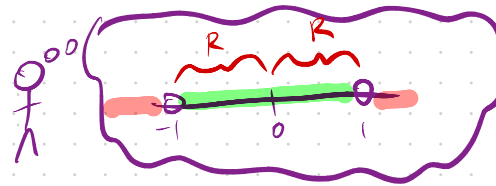
Day 35

Building new power series

At the end of our last class, we took the power series representation $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots$ for $\underbrace{|x|<1}$



$(-1, 1)$

and we integrated to find

$$\ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

for x in $(-1, 1]$

Cool corollary: $\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

Ex let's do another example of generating a "new" power series in terms of calculus on an "old" power series

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

$$= \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for $|-x^2| < 1$ ← same as saying $|x^2| < 1$, which is same as saying $|x| < \sqrt{1} = 1$

by the geometric series Theorem

Int. of conv. is $(-1, 1)$

Integrating both sides gives:

LHS: $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

RHS: $\int 1 - x^2 + x^4 - x^6 + \cdots\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+1}$$

So: $\arctan(x) + C = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

Questions: ① interval of convergence? ② What's the value of C?

We know it still has radius of convergence 1, so definitely converges on $(-1,1)$ and definitely diverges on $(-\infty,-1)$ and $(1,\infty)$.

Does it converge at $x=1$? Yes!

$$1-\frac{1^3}{3}+\frac{1^5}{5}-\frac{1^7}{7}+\cdots = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

Use Alternating Series Theorem, and find it converges.

Does it converge at $x=-1$?

$$(-1) - \frac{(-1)^3}{3} + \frac{(-1)^5}{5} - \frac{(-1)^7}{7} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \cdots$$

Use Alternating series Theorem to say: Converges

So: interval of convergence is $[-1,1]$.

$$\frac{1}{1-\frac{x}{2}} = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + \cdots$$

by GST for $\left|\frac{x}{2}\right| < 1 \rightsquigarrow |x| < 2$

$\rightsquigarrow$ int is $(-2, 2)$

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