For each problem, do you think you should do a u-substitution or integration by parts? Whichever choice you make, examine the new antiderivative problem and ask yourself "Is this new antiderivative not worse than the one I started?" Then ask yourself what you think you'd do to tackle the new antiderivative problem.

(1) $\int x^2 \ln(x)\ dx$

Solution. There's no obvious composition in play, but there is a logarithm in the integrand. Hence integration by parts is a reasonable technique to employ. If we follow this route, LIATE says we should choose $u = \ln(x)$ and $dv = x^2\ dx$, which would yield $du = \frac{1}{x}\ dx$ and $v = \frac{1}{3}x^3$. Hence we'd have

$$\int x^2 \ln(x)\ dx = \frac{1}{3}x^3 \ln(x) - \int \frac{1}{x}\frac{1}{3}x^3\ dx = \frac{1}{3}x^3 \ln(x) - \frac{1}{3}\int x^2\ dx.$$

This latter integral could then be tackled using the anti-power rule. □

(2) $\int \frac{t+2}{\sqrt{t^2+4t}}\ dt$

Solution. Note that we have a composite function in the denominator, and that (a scaled version of) the derivative of the interior function appears as the numerator. Hence u-substitution is a natural first guess. If we let $u = t^2 + 4t$ (since it's the "interior" of the composite function), then we have $du = 2t + 4\ dt$. But this means that $t + 2\ dt = \frac{1}{2}\ du$, and so

$$\int \frac{t+2}{\sqrt{t^2+4t}}\ dt = \int \frac{\frac{1}{2}\ du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\ du.$$

This latter integral can be solved using the anti-power rule. □

(3) $\int x \sin(x)\ dx$

Solution. There's no composition in sight, nor is this integrand the derivative of some familiar function from differential calculus. Hence we should try integration by parts. LIATE says we should choose $u = x$ and $dv = \sin(x)\ dx$, which then gives $du = dx$ and $v = -\cos(x)$. So we have

$$\int x \sin(x)\ dx = -x\cos(x) - \int (-\cos(x))\ dx = -x\cos(x) + \int \cos(x)\ dx.$$

This final integral is one that we can compute "by observation," since $\cos(x)$ is a familiar derivative from differential calculus. □

(4) $\int x^{-2} e^{\frac{1}{x}}\ dx$

Solution. We see there's a composition (the function $\frac{1}{x}$ "plugged into" the exponential function), and so u-substitution is a natural bet. If we let $u = \frac{1}{x}$ (since it's the "interior" of the composition) then $du = -x^{-2}\ dx$ by the power rule. Fortunately we have precisely $x^{-2}\ dx$ in our integral, which we can see equals $-du$. So we get

$$\int x^{-2} e^{\frac{1}{x}}\ dx = \int e^u(-du) = -\int e^u\ du.$$

This last integral is a familiar one from differential calculus, so we can finish this computation without any further fancy techniques. □

(5) $\displaystyle\int \frac{\sin(\theta)}{1+\cos^2(\theta)}\ d\theta$

Solution. There are two natural ways to view a composition in our integrand, and so u-substitution could be a good bet. But what should we choose for u? One choice would be to think of the $\cos^2(\theta)$ term as $(\cos(\theta))^2$, so that $\cos(\theta)$ is the "interior" function. If we use this to give us $u = \cos(\theta)$, note that $du = -\sin(\theta)\ d\theta$. In out integrand we have precisely $\sin(\theta)\ d\theta$ in the numerator, which is then equal to $-du$ from our previous equation. So we get

$$\int \frac{\sin(\theta)}{1+\cos^2(\theta)}\ d\theta = \int \frac{-du}{1+u^2} = -\int \frac{1}{1+u^2}\ du.$$

This latter integral can then be tackled "by observation" since it's comes from one of our "familiar" derivative rules.

Note that there's another very natural way to see a composition in the integrand, and it leads to a different substitution. What is this alternate approach, and what happens if we take it instead? If we think of the integrand as $\sin(\theta)\left(1+\cos^2(\theta)\right)^{-1}$, then we might think of the "inner" function as $1+\cos^2(\theta)$. Note that if we do this then the chain rule gives us $du = 2\cos(\theta)(-\sin(\theta))\ d\theta$. The issue here is that our integrand has a factor of $\sin(\theta)\ d\theta$, but not a factor that looks like $\cos(\theta)\sin(\theta)\ d\theta$. Hence if we tried this approach, we wouldn't be able to complete the substitution, and the process would dead end. (When this sort of thing happens, it's a good sign that you simply need to think of another way to approach the integral!) □

(6) $\displaystyle\int \arctan(x)\ dx$

Solution. There's no composition in this integrand, and this isn't the derivative of some familiar function from differential calculus. Hence integration by parts is all we have left. LIATE says we should choose $u = \arctan(x)$ (since it's an **I**nverse trig function), which only leaves $dv = dx$. We then we get $du = \frac{1}{1+x^2}\ dx$ and $v = x$, and integration by parts says

$$\int \arctan(x)\ dx = x\arctan(x) - \int x\frac{1}{1+x^2}\ dx = x\arctan(x) - \int \frac{x}{1+x^2}\ dx.$$

Now for this latter integral, note that the numerator is a scalar multiple of the derivative of the denominator. This is a sign that a u-substitution — with $u = 1+x^2$ — will simplify the integral, and is the next step to pursue. □

(7) $\displaystyle\int e^{\sec(t)}\frac{\sin(t)}{\cos^2(t)}\ dt$

Solution. Note that we have a composition, with $\sec(t)$ being "plugged into" the exponential. So if we choose $u = \sec(t)$ (since it's the "interior" of the composition), then we get $du = \sec(t)\tan(t)\ dt$. Note then that if we use the definition of trig functions, we can re-express our differential as $du = \sec(t)\tan(t)\ dt = \frac{1}{\cos(t)}\frac{\sin(t)}{\cos(t)}\ dt$, and we have precisely this factor in our integrand. Hence we get

$$\int e^{\sec(t)}\frac{\sin(t)}{\cos^2(t)}\ dt = \int e^u\ du.$$

This latter integral is one of our familiar antiderivatives, so is easy to knock down. □

(8) $\displaystyle\int \sin(x)e^x \; dx$

Solution. There's no composition in this integrand, nor is it a familiar antiderivative that comes from a derivative rule covered in differential calculus. So we're left with integration by parts. LIATE says we should choose $u = \sin(x)$ and $dv = e^x \; dx$, and if we do this then we get $du = \cos(x) \; dx$ and $v = e^x$. Hence we get

$$\int \sin(x)e^x \; dx = \sin(x)e^x - \int \cos(x)e^x \; dx.$$

Now this latter integral looks a lot like our original integral; it's certainly not a "better" integral, but it's also not obviously "worse." Given this, the next thing for us to try is to tackle this integral in the same way we did our last one: using integration by parts, this time with $u = \cos(x)$ and $dv = e^x \; dx$.

Though this problem doesn't ask us to take this second step, we'll carry through with the calculation here to see how things shake out. Our choices of u and v give us $du = -\sin(x) \; dx$ and $v = e^x$, and so

$$\int \cos(x)e^x \; dx = \cos(x)e^x - \int (-\sin(x))e^x \; dx = \cos(x)e^x + \int \sin(x)e^x \; dx.$$

If we plug this into our first integration by parts formula we get

$$\int \sin(x)e^x \; dx = \sin(x)e^x - \left(\cos(x)e^x + \int \sin(x)e^x \; dx\right) = \sin(x)e^x - \cos(x)e^x - \int \sin(x)e^x \; dx.$$

Observe that if we add $\int \sin(x)e^x \; dx$ to both sides we get

$$2\int sin(x)e^x \; dx = \sin(x)e^x - \cos(x)e^x.$$

If we divide both by sides by 2, we'll have computed our original integral. □

(9) $\displaystyle\int \ln(\sqrt[3]{t}) \; dt$

Solution. There's a composition in this integrand, so u-substitution is a natural approach to take. On the other hand, integration by parts LOVES logarithms (it's the first letter in LIATE after all), so that might be profitable as well. Let's see how each shakes out.

We'll do integration by parts first, since it ultimately falls out a bit faster. (If it's not your first instinct, that's ok! You'll see what happens with u-substitution below.) Using LIATE we choose $u = \ln(\sqrt[3]{t})$ and $dv = dt$, which then gives us $du = \frac{1}{\sqrt[3]{t}} \cdot \left(\frac{1}{3}t^{-2/3}\right) \; dt = \frac{1}{3t} \; dt$ (using the chain rule and some exponent arithmetic), while $v = t$. So integration by parts gives

$$\int \ln(\sqrt[3]{t}) \; dt = t\ln(\sqrt[3]{t}) - \int t\frac{1}{3t} \; dt = t\ln(\sqrt[3]{t}) - \frac{1}{3}\int dt.$$

This latter antiderivative is one we can calculate "by observation."

What if, instead, we thought to do this problem with a substitution? Here the natural choice is to make $u = \sqrt[3]{t} = t^{1/3}$ (since it's the interior of the composition), so then we must have $du = \frac{1}{3}t^{-2/3} \; dt$. Unfortunately there's no $t^{-2/3}$ term for us to substitute in the integrand. On the other hand, our du equation can be "solved for dt" to yield $dt = 3t^{2/3} \; du$, and since $u = \sqrt[3]{t} = t^{1/3}$ this means we get $dt = 3u^2 \; du$. Hence our substitution gives

$$\int \ln(\sqrt[3]{t}) \; dt = \int \ln(u)\left(3u^2 \; du\right) = 3\int u^2 \ln(u) \; du.$$

This latter integral is now one which suggests integration by parts a bit more strongly, since there's no composition but there is a product (including a factor that's a logarithm).

What's the takeaway message here? It isn't that one approach is "right" and the other is "wrong." In fact, if anything the message here is that we had two natural approaches we might choose, and both helped push us toward solving the problem. The key when you make a choice for how to proceed with an integral is to carefully follow where it leads.

[Note: One last approach that might be worth pointing out is that the rules of logarithms give us $\ln(\sqrt[3]{t}) = \ln(t^{1/3}) = \frac{1}{3}\ln(t)$, so that our problem is just $\frac{1}{3}\int \ln(t)\ dt$. This problem certainly looks less daunting, and is less ambiguous in terms of approaches one might take. On the other hand, it requires one to feel comfortable recognizing and using logarithm properties.] □

(10) $\displaystyle\int \frac{\sqrt{1+\ln(x)}}{x}\ dx$

Solution. Here we have a composition, with $1 + \ln(x)$ plugged into the square root function, so the natural first choice is u-substitution. If we let $u = 1 + \ln(x)$ (since this is the "interior" of the composition), then we get $du = \frac{1}{x}\ dx$. Fortunately we have precisely a factor of $\frac{1}{x}\ dx$ in our integral, so our substitution should carry through nicely. We get

$$\int \frac{\sqrt{1+\ln(x)}}{x}\ dx = \int \sqrt{1+\ln(x)}\left(\frac{1}{x}\ dx\right) = \int \sqrt{u}\ du = \int u^{1/2}\ du.$$

This latter integral is now ready to be tackled using the anti-power rule. □