Linear algebra in your daily (digital) life

Andrew Schultz

December 10, 2025

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    - Google's PageRank algorithm

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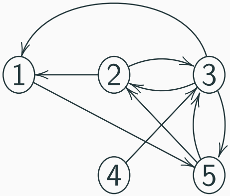
Voting in Directed Graphs

The general constraints

Suppose that G is a directed graph



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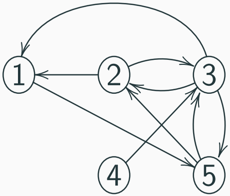

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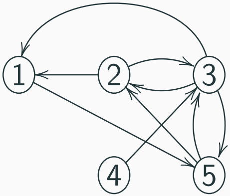

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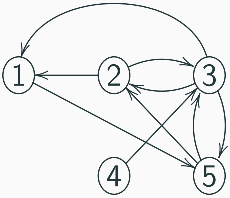

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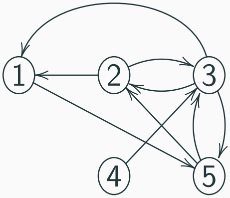

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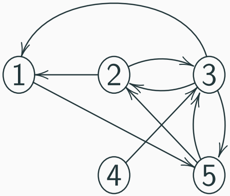

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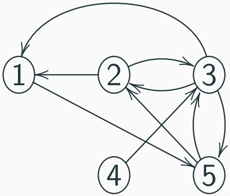

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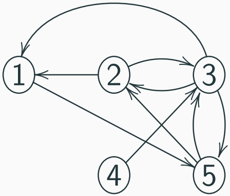

The first approach

$(j) \longrightarrow (i)$ is a vote for (i) from (j)

Form matrix L so that

$$L_{i,j} = \begin{cases} 1 & \text{, if } (j) \longrightarrow (i) \\ 0 & \text{, else} \end{cases}$$

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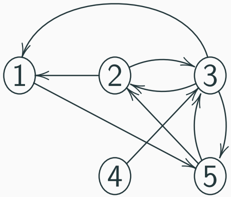

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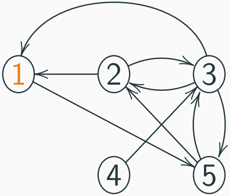

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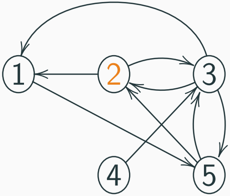

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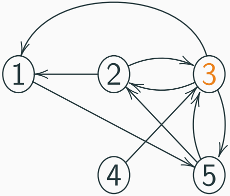

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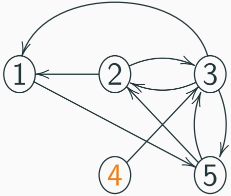

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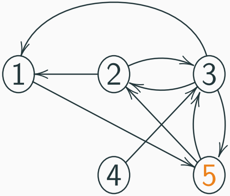

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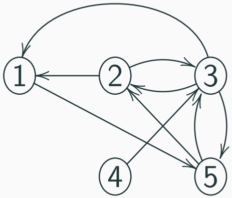

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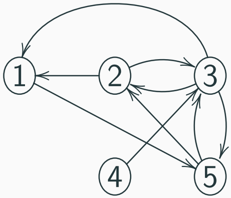
Page 3 wins (score 3), Pages 1,2,5 tie for second (score 2), Page 4 loses



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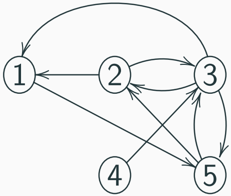

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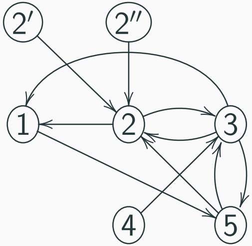

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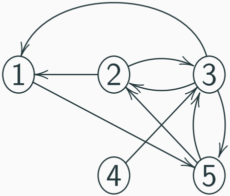
Each page gets a total of 1 vote:

$$\ell(j) = \sum_i L_{i,j}$$



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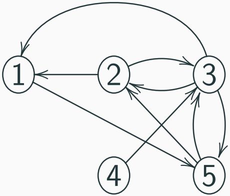

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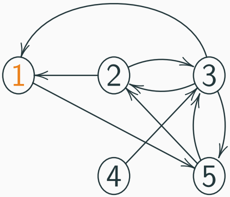

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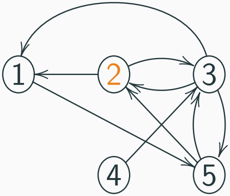

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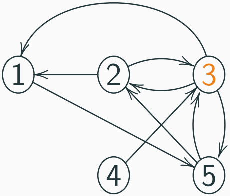

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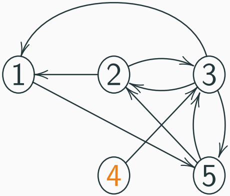

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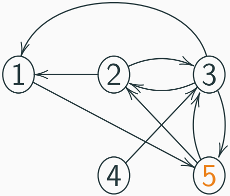

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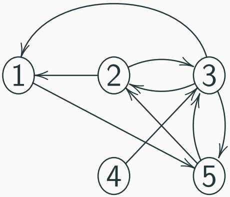

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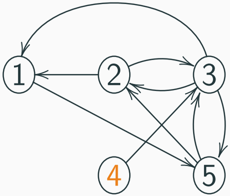

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$$\begin{pmatrix} 0.41 \\ 0.47 \\ 0.52 \\ 0 \\ 0.58 \end{pmatrix}$$ is 1-eigenvector

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Theorem

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Let $$PR = dW + (1-d)\begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

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- this row reduction would take about $30,000,000$ years.

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Fact: If an $n \times n$ matrix has positive entries and columns sum to 1, then 1 is the largest eigenvalue (in absolute value).

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Then

$$PR^k \overrightarrow{v} = c_1 \lambda_1^k \overrightarrow{v_1} + \cdots + c_{10^{10}} \lambda_{10^{10}}^k \overrightarrow{v_{10^{10}}}.$$

Fact: If an $n \times n$ matrix has positive entries and columns sum to 1, then 1 is the largest eigenvalue (in absolute value).

Hence $\lim_{k\to\infty} PR^k \overrightarrow{v}$ is an eigenvector with eigenvalue 1.

Who cares about matrix factorization?

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Singular Value Decomposition

The Singular Value Decomposition

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$$A = \left(\begin{array}{c|c|c} \overrightarrow{v}_1 & \cdots & \overrightarrow{v}_r \end{array} \right) \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \overrightarrow{w}_1 \\ \hline \vdots \\ \hline \overrightarrow{w}_c \end{pmatrix}.$$

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- simple expression for PsuedoInverse
- Quick test for numerical stability of matrix

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If $\sum_{i>s} \sigma_i$ is "insignificant," then we have $A \approx \sum_{i=1}^{s} A_i$

Who cares about matrix factorization?

Application to noise filtering

Noise Filtering

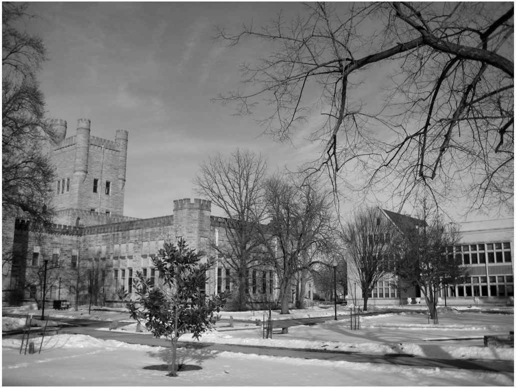
Who cares about matrix factorization?

Image compression through SVD

Matrix representations of images

You can think of an image as a matrix.

- Each pixel contains a gray value
- Gray values range from 0 to 255



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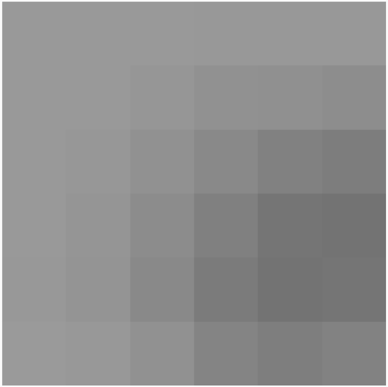

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153	153	153	152	152	152
153	153	150	145	144	141
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153	149	140	128	117	115
152	148	137	123	115	117
154	152	145	132	126	130

Keeping only "significant" terms

According to our theory, if there are s-many significant singular values, then

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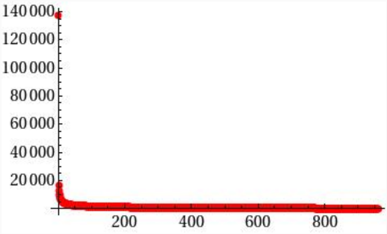
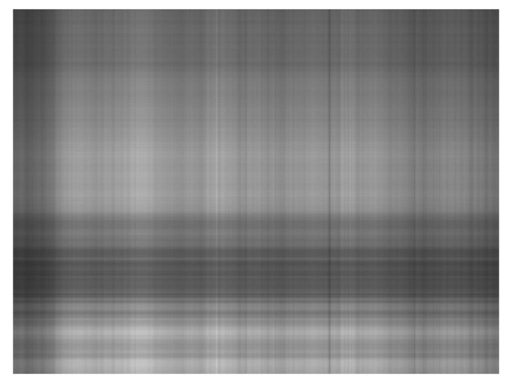


$$\sigma_1 \approx 138,000$$
$$\sigma_2 \approx 17,000$$
$$\sigma_{50} \approx 2,200$$
$$\sigma_{200} \approx 900$$

Some approximations

Let's see what our truncated matrix "looks like"

- s=1
- Compression: 0.18%



Some approximations

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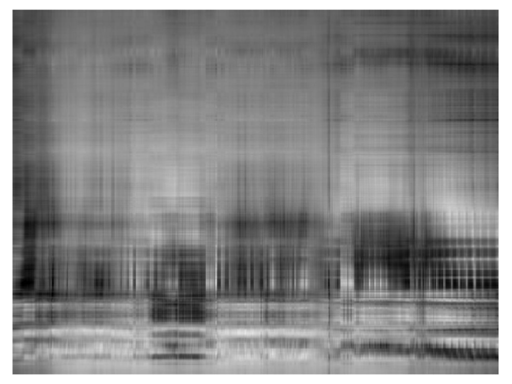
- s=5
- Compression: 0.9%



Some approximations

Let's see what our truncated matrix "looks like"

- s=10
- Compression: 1.8%

Some approximations

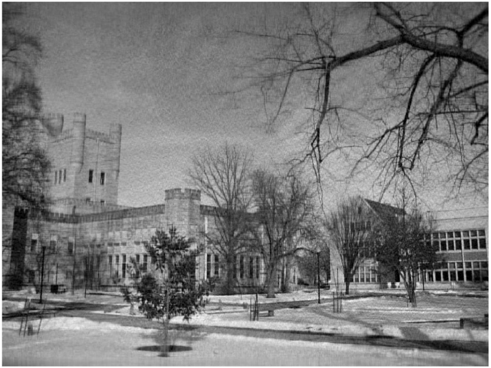
Let's see what our truncated matrix "looks like"

- s=25
- Compression: 4.5%

Some approximations

Let's see what our truncated matrix "looks like"

- s=100
- Compression: 18%



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- controlling quality of compressed image
- doesn't require us to keep track of a basis
- takes advantage of properties of images

Who cares about change of basis?

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New coordinate systems

The "usual" way of thinking about a matrix

Typically we think of a matrix in terms of its entries.

$$A = \sum_{i,j} a_{ij} E(i,j)$$

where $E(i,j)$ is the matrix with a 1 in the ith row, jth column.

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- Each $E(i,j)$ solely responsible for its local behavior.
- Deleting an a_{ij} completely wipes out pixel info.

Rewriting the matrix

What if we chose a different basis for $n \times n$ matrices?

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- deleting $c_{i,j}$ has gradual (though global) effect

A potential basis

We'll choose an basis of $\mathbb{R}^n$ from Fourier series

$$\overrightarrow{f}_i = \alpha_i \left\{ \cos\left[\frac{\pi}{n} \left(\frac{2j+1}{2} \right) i \right] \right\}_{j=0}^{n-1}$$

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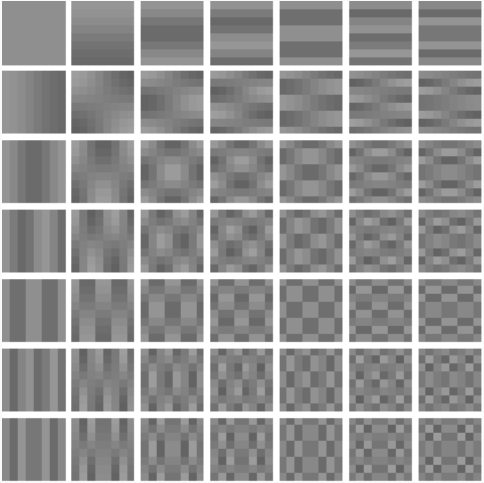
$$B(i,j) = \begin{pmatrix} \overrightarrow{f}_0 \\ \hline \vdots \\ \hline \overrightarrow{f}_{n-1} \end{pmatrix} E(i,j) \left(\begin{array}{c|c|c} \overrightarrow{f}_0 & \cdots & \overrightarrow{f}_{n-1} \end{array} \right)$$

Seeing the new basis ($n = 8$)



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Compute coefficients $c_{i,j}$ such that $A = \sum_{i,j} c_{i,j} B(i,j)$ by

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Who cares about change of basis?

The human eye and $\mathcal{B}$

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Then decompression is

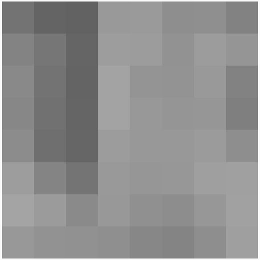
- De-quantize
- Change back to standard coordinates
- Reassemble the 8×8 blocks

Working through an example

Extract an 8×8 block

Working through an example

Extract an 8×8 block



$$\longrightarrow \begin{pmatrix} 115 & 100 & 98 & 153 & 154 & 142 & 143 & 130 \\ 131 & 118 & 101 & 157 & 156 & 146 & 156 & 149 \\ 137 & 115 & 100 & 163 & 148 & 147 & 153 & 130 \\ 135 & 113 & 101 & 163 & 152 & 149 & 150 & 127 \\ 140 & 111 & 102 & 156 & 152 & 152 & 155 & 142 \\ 157 & 132 & 116 & 153 & 150 & 151 & 159 & 160 \\ 164 & 155 & 138 & 152 & 144 & 141 & 151 & 161 \\ 152 & 146 & 145 & 143 & 135 & 132 & 142 & 159 \end{pmatrix}$$

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Working through an example

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$$\left[\frac{\begin{pmatrix} 1112 & -61 & -14 & 44 & 57 & 34 & -32 & -26 \\ -43 & -36 & -43 & 25 & 13 & 12 & -15 & -8 \\ 2 & 12 & 12 & -26 & -8 & -16 & 7 & 10 \\ 2 & -14 & 1 & 7 & 6 & -3 & 1 & 2 \\ -25 & -4 & -16 & 0 & -1 & 2 & 5 & 3 \\ -2 & 12 & -6 & 1 & -3 & 2 & -1 & -2 \\ -9 & -1 & -2 & 3 & 0 & 5 & 2 & 0 \\ -4 & 2 & -2 & 1 & -1 & 3 & 1 & -1 \end{pmatrix}}{\begin{pmatrix} 16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\ 12 & 12 & 14 & 19 & 26 & 58 & 60 & 55 \\ 14 & 13 & 16 & 24 & 40 & 57 & 69 & 56 \\ 14 & 17 & 22 & 29 & 51 & 87 & 80 & 62 \\ 18 & 22 & 37 & 56 & 68 & 109 & 103 & 77 \\ 24 & 35 & 55 & 64 & 81 & 104 & 113 & 92 \\ 49 & 64 & 78 & 87 & 103 & 121 & 120 & 101 \\ 72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \end{pmatrix}}\right] \longrightarrow$$

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We're down to 16 pieces of information!

Reconstituting our image

Here's the result of reversing this process:

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Compressed

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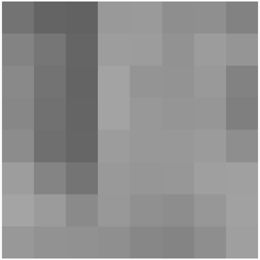
- Average difference $= 5.73$
- Std Dev $= 4.22$

Seeing is believing

Original



Compressed

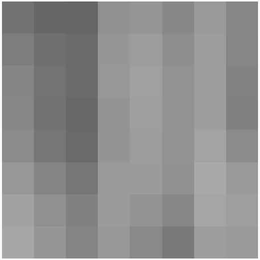


Image Processing

We can use these ideas to do some image processing as well

Image Processing

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Note: Here I won't split the image into 8×8 blocks – I want all the information about the image simultaneously

Image Processing

Smooth Part: $B(i, j)$ components for small j, small i

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Horizontal Edges: $B(i, j)$ components for small j, large i

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Vertical Edges: $B(i, j)$ components for small i, large j

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Image Processing

Scattered Edges: $B(i, j)$ components for large i, large j

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Who cares about change of basis?

The human ear and $\mathcal{B}$

How MP3 compression works

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- Filter out "unnecessary" data using psychoacoustics

Psychoacoustics

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- Temporal Masking - Some weak sounds aren't heard if played right after (or right before!) a louder sound
- Hass effect - If the same tone hits one ear just before another, then your brain perceives it as coming only from the first direction

Thanks!

Thank you!