COMPLEX EIGENVALUES AND TRANSITION MATRICES

1. Complex Eigenvalues, Redux

Yesterday we showed that

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \overbrace{\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}}^{R} \begin{pmatrix} a+bi & \\ & a-bi \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{-1}.$$

Today we will show

Theorem 1.1. *If A is a 2×2 matrix with complex eigenvalues $a \pm bi$, and if the corresponding eigenvectors are $\overrightarrow{v} \pm \overrightarrow{w} i$, then*

$$A = \left(\overrightarrow{w} \,\middle|\, \overrightarrow{v} \right) \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \left(\overrightarrow{w} \,\middle|\, \overrightarrow{v} \right)^{-1}.$$

The take away is that complex eigenvalues mean that our operator 'acts like' a rotation/scaling matrix.

Example. Consider the matrix

$$\begin{pmatrix} 0 & 2 \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

The characteristic polynomial is $\det \begin{pmatrix} -\lambda & 2 \\ -\frac{1}{2} & -\lambda \end{pmatrix} = \lambda^2 + 1$, and so has eigenvalues $\pm i$. One can compute an eigenvalue for $\lambda = i$ and find

$$E_i = \ker \begin{pmatrix} -i & 2 \\ -\frac{1}{2} & -i \end{pmatrix} \; i \; = \ker \begin{pmatrix} 1 & 2i \\ -\frac{1}{2} & -i \end{pmatrix} \; +\tfrac{1}{2} I \; = \ker \begin{pmatrix} 1 & 2i \\ 0 & 0 \end{pmatrix} = \operatorname{Span} \begin{pmatrix} -2i \\ 1 \end{pmatrix}.$$

Hence for this problem the vector $\overrightarrow{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the vector $\overrightarrow{w} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$. The theorem tells us that we should expect

$$\begin{pmatrix} 0 & 2 \\ -\frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}^{-1},$$

and indeed by carrying out this multiplication we see that the equality holds.

This example just confirms the theorem in a special case, but also illustrates how we go about producing the vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ in the theorem.

Proof. Proof of theorem We won't show this in class, but if $\overrightarrow{v} + i\overrightarrow{w}$ is an eigenvector for $a + ib$ then one can show $\overrightarrow{w} - i\overrightarrow{w}$ is an eigenvector for $a - ib$. Now with $\overrightarrow{v} \pm \overrightarrow{w}$ the eigenvectors corresponding to $a \pm ib$ we can change basis to diagonalize:

$$A = \left(\overrightarrow{w} v + i\overrightarrow{w} \,\middle|\, \right) \begin{pmatrix} a+ib & \\ & a-ib \end{pmatrix} \left(\overrightarrow{w} v + i\overrightarrow{w} \,\middle|\, \right)^{-1}.$$

But from the first equation on the first page we also know that

$$\begin{pmatrix} a+ib & \\ & a-ib \end{pmatrix} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix},$$

where in fact

$$\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}.$$

Plugging these into the previous equation gives

$$A = \left(\overrightarrow{w}v + i\overrightarrow{w} \,\middle|\, \overrightarrow{w}v - i\overrightarrow{w} \right) \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \left(\overrightarrow{w}v + i\overrightarrow{w} \,\middle|\, \overrightarrow{w}v - i\overrightarrow{w} \right)^{-1}$$

$$= \underbrace{\left(\overrightarrow{w}v + i\overrightarrow{w} \,\middle|\, \overrightarrow{w}v - i\overrightarrow{w} \right) \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}}_{S} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \underbrace{\left(\left(\overrightarrow{w}v + i\overrightarrow{w} \,\middle|\, \overrightarrow{w}v - i\overrightarrow{w} \right) \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \right)^{-1}}_{S^{-1}}$$

Hence we only have left to compute

$$S = \left(\overrightarrow{w}v + i\overrightarrow{w} \,\middle|\, \overrightarrow{w}v - i\overrightarrow{w} \right) \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} = \frac{1}{2i} \left(\overrightarrow{w}v + i\overrightarrow{w} \,\middle|\, \overrightarrow{w}v - i\overrightarrow{w} \right) \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$

$$= \frac{1}{2i} \left(\overrightarrow{v} + i\overrightarrow{w} - \overrightarrow{v} + i\overrightarrow{w} \,\middle|\, i\overrightarrow{v} - \overrightarrow{w} + i\overrightarrow{v} + \overrightarrow{w} \right) = \left(\overrightarrow{w} \,\middle|\, \overrightarrow{v} \right).$$

□

2. Regular Transition Matrices

Definition 2.1. A matrix A is called a transition matrix if all the entries of A are non-negative and the sum of the entries in any given column is 1.

Theorem 2.1. *The sum of the coordinate entries of $\overrightarrow{v}$ is equal to the sum of the coordinate entries of $A\overrightarrow{v}$ for any $\overrightarrow{v}$.*

Proof. Since $A\overrightarrow{v} = v_1\overrightarrow{a_1} + \cdots + v_n$, we use the fact that the coordinates of $\overrightarrow{a_i}$ add to 1 to prove that each of the summands has column sum equal to v_i. Hence the sum of has a coordinates of $A\overrightarrow{v}$ is $\sum v_i$, as desired. □

This theorem means that regular transition matrices model redistribution systems: systems which model systems of migrating populations (where population doesn't change with time). We've actually seen several examples of this kind of dynamical system already.

Example. One of your old homework problems involved a small town that had a mom and pop grocery store until, one day, an evil multinational Super Shop-orama opened up in town. Each week 20% of the shoppers at the tiny grocery store left to shop at the the new store, and 10% of shoppers at the fancy new store would leave the to shop at the little store the next week. So if we count the number of shoppers as a vector whose first coordinate is the number of shoppers at the mom and pop store, and the second coordinate the number of shoppers at Super Shop-orama, then the matrix of this system is

$$\begin{pmatrix} .8 & .1 \\ .2 & .9 \end{pmatrix}.$$

We saw another example where some crazy monks would give all their gold to their other two monk friends. This had matrix

$$\begin{pmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{pmatrix}.$$

Definition 2.2. A matrix is called a regular transition matrix if it is a transition matrix and all its entries are positive

This just means that a regular transition matrix is a transition matrix without any 0's.

Theorem 2.2 (Big theorem for regular transition matrices)**.** *If A is a regular transition matrix then* 1 *is an eigenvalue for A with algebraic (and geometric) multiplicity* 1*. There exists an eigenvector $\overrightarrow{v}$ with eigenvalue* 1 *that has all positive entries and so that the sum of the coordinate entries of $\overrightarrow{v}$ is* 1*. Moreover, for any initial condition $\overrightarrow{v_0}$ for the system A,*

$$\lim_{t\to\infty} A\overrightarrow{v_t} = c\overrightarrow{v},$$

where c is the sum of the coordinates of $\overrightarrow{v_0}$.

We'll use this theorem next class to explain how Google ranks webpages.