COURSE NOTES - 01/12/05

1. Announcements

- My office hours for tuesday and wednesday will be changed starting next week (1/17). The new times will be Tuesday from 1:00 to 2:00 and Wednesday from 2:15-3:15. These changes have been made on the course webpage as well.
- Exercise 2.1.8, part (b), can be omitted in the homework.
- Some students have been curious about which sections they should read in the text. Most of the material in chapter 1 was presented in class differently from the corresponding sections from the book. Students who would like may read Chapter 1 in the book to gain a different perspective on this material, but this is not necessary. A reference for what we've discussed from Chapter 1 in class are the posted course notes, and we can discuss any questions about examples or course content in office hours or during class.

2. Recap

Last class we discussed some final function facts and were introduced to the tangent problem. Specific topics included:

- 3 properties of the logarithm function, including $\log(1) = 0$,
- the graph of the inverse of a function,
- even and odd functions,
- the beginnings of differential calculus: the tangent problem.

3. The tangent problem

The tangent problem asks the following question: given the graph of a function $f(x)$ and a point $P = (x_1, f(x_1))$ on the graph, what is the equation of the line tangent to the graph and passing through the point P?

When one tries to solve this problem, and important first step is to realize that half of the solution is already staring us in the face. Specifically, recall that the equation for a line requires knowing

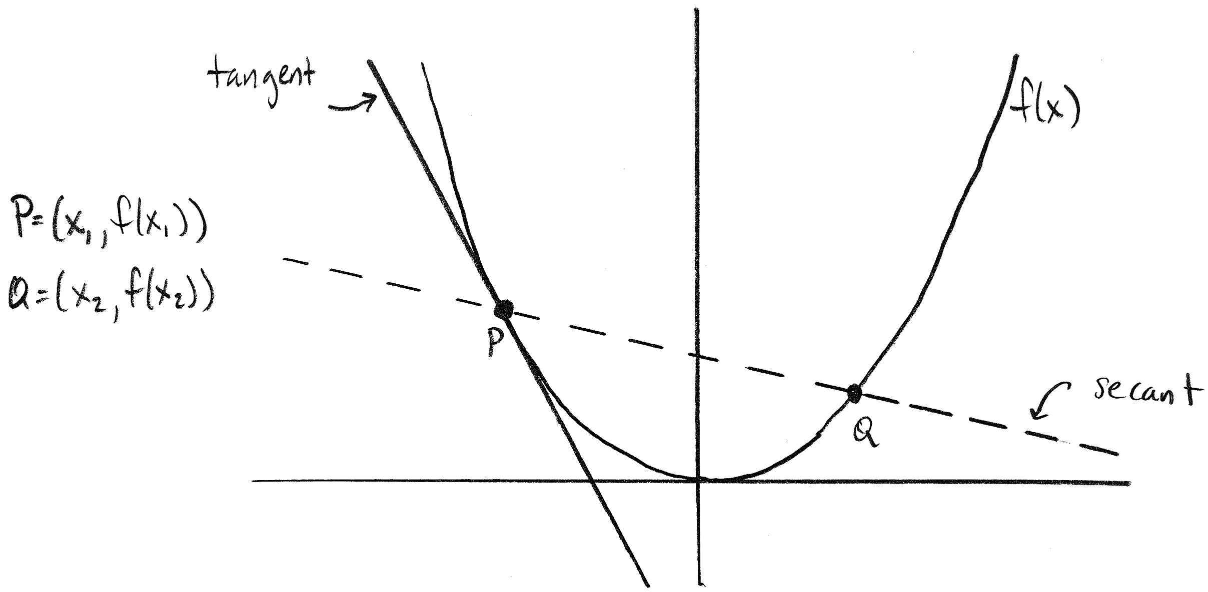
- the slope of the line, and
- a point on the line.

We are in the happy situation where we have already been given a point on the line (namely, the point P), so in fact to solve the tangent problem we need only know the slope of the tangent line.

How does one go about finding the slope of this tangent line? Our clever idea started by fix a random point $Q = (x_2, f(x_2))$ on the graph of f and compute the slope of the *secant line* passing through P and Q. We saw that this slope, which we'll call m_{PQ}, is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The clever part of the idea was to then let the point Q approach the point P; the secant line then approaches the tangent line we're interested in, and in particular if we keep track of what the slope m_{PQ} is doing as Q moves toward P, we can find the slope of the tangent line as desired!



Hence, we are particularly interested in the quantity

$$\lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The definition of this limit is discussed below, but intuitively the idea is that the limiting value as Q approaches P of the slope of the secant line through P and Q should approach the slope of the tangent line, our ultimate goal.

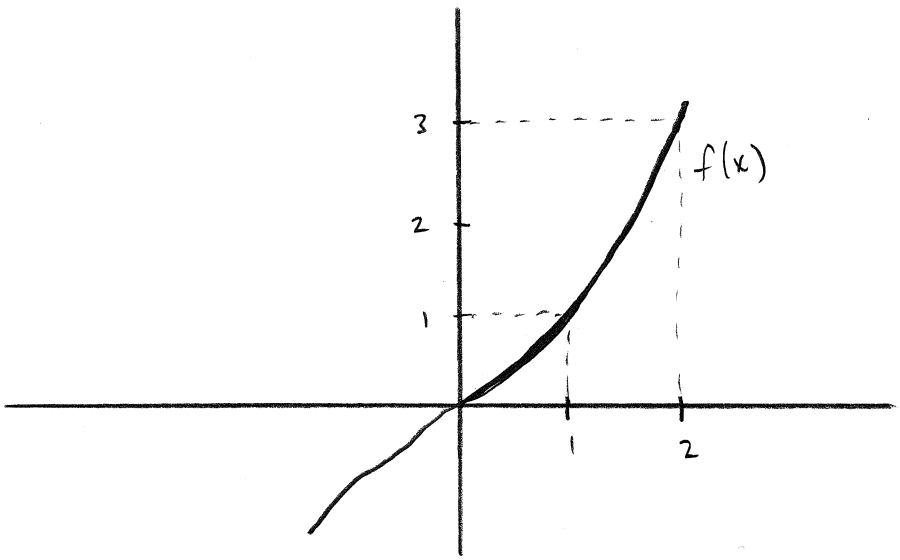
4. Limits

Thus, to answer the tangent problem we must first investigate limits of functions.

Intuitive Definition. The value $\lim_{x \to a} f(x)$, read 'the limit of $f(x)$ as x approaches a,' is the value outputs of f approach as inputs approach a. In other words, the limit captures of the information of where outputs are heading as inputs get closer and closer to a.

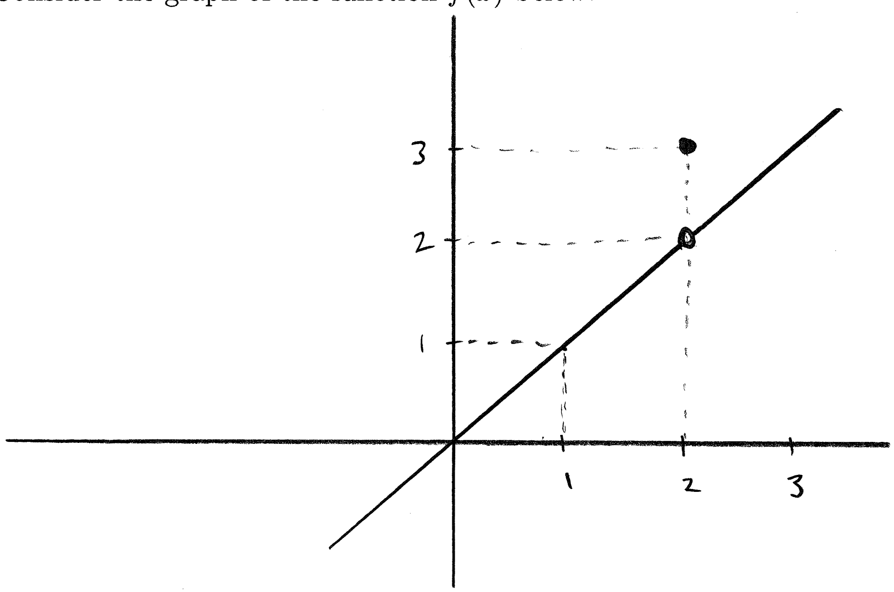
This definition is admittedly a bit vague, though perhaps it is best for us to work a few examples to get some idea for how to evaluate limits. For today we will be particularly interested in determining the limits of a function given the graph of that function. We will also discuss how one might guess at the limit of a function given a table describing the function, and we will also see how to draw the graph of a function satisfying prescribed limit information.

Example 1. Consider the graph of the function $f(x)$ below.



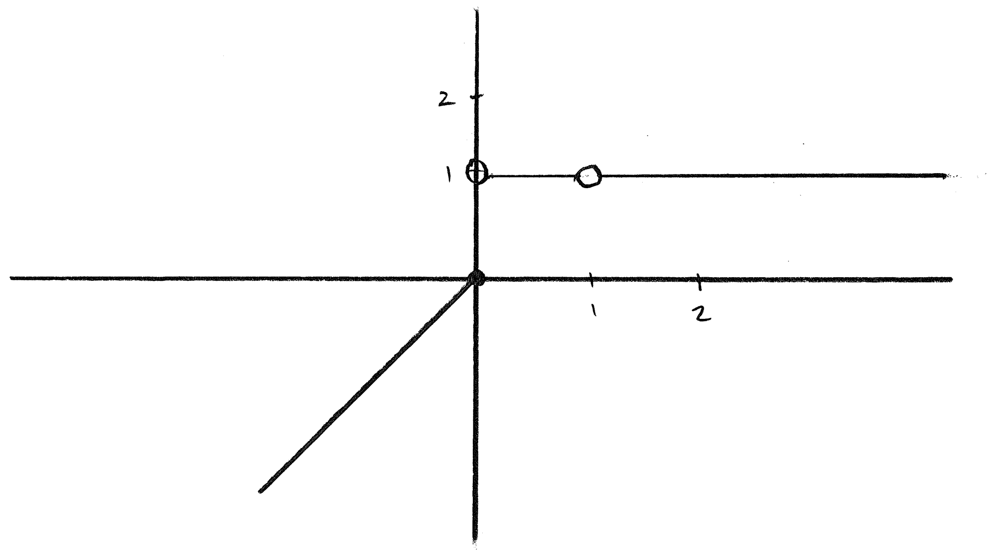
What are the values $\lim_{x \to 1} f(x)$ and $\lim_{x \to 2} f(x)$? For the first, as inputs approach the value 1, the graph shows that the function is approaching the value 2. Hence $\lim_{x \to 1} f(x) = 1$. In a similar way, as inputs approach 2 it seems the output of the function is approaching 3. Hence we have $\lim_{x \to 2} f(x) = 3$.

Example 2. Consider the graph of the function $f(x)$ below.



What are the values of $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$? As in the previous question, we see that the function approaches the value 1 as inputs approach 0, and also the function approaches the value 2 as inputs approach 1. Hence we have $\lim_{x \to 0} f(x) = 1$ and $\lim_{x \to 1} f(x) = 2$. Notice that in this case we have $f(1) \neq \lim_{x \to 1} f(x)$. This is an important lesson: the value of the limit of a function as $x \longrightarrow a$ is not necessarily related to the value of the function a.

Example 3. Consider the graph of the function $f(x)$ below.



What are the values of $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$? For the limit at $x = 2$, we see that the function takes the value 1 for all inputs around 2. Hence, as inputs approach 2, the function is constantly spitting out the value 1. So, $\lim_{x\to 2} f(x) = 1$. The limit at $x = 0$ is more problematic. As we approach 0 from the left, the function is tending toward the value 0. On the other hand, if we approach 0 from the right, the function is tending toward the value 1. Since there is no value that the function is approaching as inputs more toward 0 from an arbitrary direction, we say that $\lim_{x\to 0} f(x)$ does not exist.

This third example illustrates two important points. First, as in Example 2, it reminds us that $f(a)$ is not necessarily the same as the limit $\lim_{x\to a} f(x)$. In this case, $f(2)$ does not even exist! Second, it also gives us an example of a function and an input where the limit of the function does not exist. It also motivates the idea of a directional limit.

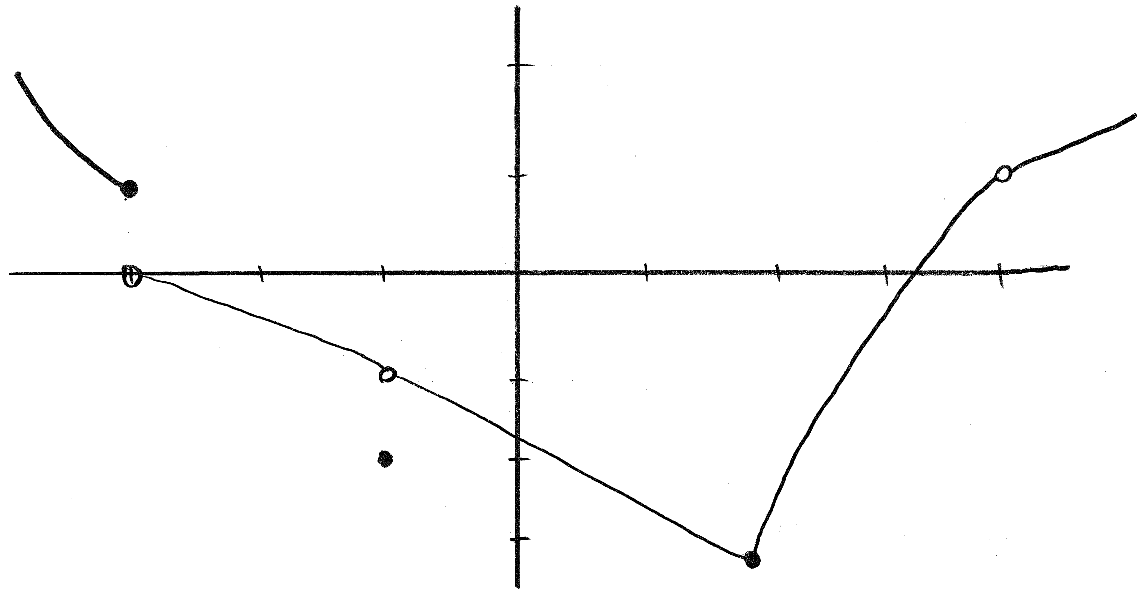
Intuitive Definition. The value $\lim_{x\to a^-} f(x)$, read 'the limit of $f(x)$ as x approaches a from the left,' is the value outputs of f approach as inputs *to the left of* a approach a. In other words, the limit captures of the information of where outputs are heading as inputs get closer and closer to a from the left. In a similar way, we define $\lim_{x\to a^+} f(x)$, read 'the limit of $f(x)$ as x approaches a from the right,' is the value outputs of f approach as inputs *to the right of* a approach a.

Example 3, again. Consider the graph of the function $f(x)$ from Example 3.

What are the values of $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$? As we noted above, as inputs approach 0 from the left, the function approaches the value 0; as inputs approach 0 from the right, the function approaches the value 1. Hence we have

$$\lim_{x\to 0^-} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to 0^+} f(x) = 1.$$

Example 4. Consider the graph of the function $f(x)$ below.



Then we have the following:

$\lim_{x\to -3} f(x)$ does not exist	$\lim_{x\to -3^-} f(x) = 1$	$\lim_{x\to -3^+} f(x) = 0$
$\lim_{x\to -1} f(x) = -1$	$\lim_{x\to 2} f(x) = -3$	$\lim_{x\to 4} f(x) = 1$

Example 5. Consider a function $f(x)$ as follows:

x	f(x)
5.5	3.7
5.25	3.2
5.1	3.1401
5.01	3.1400001
5.0001	3.140000000001

Guess the value of $\lim_{x\to 5^+} f(x)$? It seems from the chart provided that $\lim_{x\to 5^+} f(x) = 3.14$, since the chart shows a few inputs approaching 5 from the right whose outputs are approaching 3.14.

Given the information at hand, our guess is a very good one. We have to be careful, though, to remember that our answer is only a guess for what the limit should be. No chart can provide enough information for us to *know* any limit...they just give us a good idea.

Finally, we make a few comments concerning when limits exist. One thing we have seen in a few examples is that $\lim_{x\to a} f(x)$ does not exist when $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$. In fact, it's true that

$$\lim_{x\to a} f(x) \text{exists if and only if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$

Another remark worth remembering is that directional limits can also fail to exist. For instance, in the graph of the function below, as inputs approach the value 0 outputs are wildly running between -1 and 1 without ever settling down toward one fixed value. In this case, $\lim_{x\to 0} f(x)$ does not exist.