# PROPERTIES OF LIMITS

## 1. Announcements

- Quiz 1 results posted on coursework
- Homework 2, homework 1 solutions, quiz 1 solutions posted on course webpage
- I've modified the syllabus slightly, so that Wednesday's class is on continuous functions

## 2. Recap

Last time we talked about limits and directional limits. In particular we

(1) gave an intuitive definition of what the limit of a function is;
(2) saw how to evaluate limits when given the graph of a function;
(3) said that $f(a)$ and $\lim_{x\to a} f(x)$ are not necessarily related.

## 3. Some loose ends

We have a few very important properties of limits to discuss, but before we get to them there are a few things I should have mentioned in class on Friday. The first is the following

**Fact.** $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal.

This fact comes in handy for evaluating limits of functions that look pretty tricky. We'll see an example of this later in the class.

The second is an example of a function which doesn't have a limit. In class on Friday we constructed some functions which didn't have a limit at $a$ because the directional limits didn't agree (though they did exist). The function $\sin(\frac{1}{x})$, however, doesn't even have a directional limit at 0!

You can see from the graph that as $x \to 0^+$, the function is not approaching a single value. Indeed, Steve pointed out in class that for any value $c$ in the interval $[-1, 1]$ that you like, there is a sequence of points approaching 0 whose outputs are approaching $c$. In this sense, then, the function fails to have a directional limit because it is approaching 'too many' values as $x \to 0^+$.

Before leaping into properties of limits, let's do a reality check on limits.

**Example.** Evaluate $\lim_{x\to a} x$ and $\lim_{x\to a} c$, where $c$ is an arbitrary constant.

*Solution.* The graphs of each of these functions is shown below. We can see that as $x \to a$, the function $f(x) = x$ is approaching the value $a$, whereas for the function $g(x) = c$ outputs are approaching $c$. Hence we have

$$\lim_{x\to a} x = a \text{ and } \lim_{x\to a} c = c.$$

□

## 4. Properties of Limits

If we are given functions $f(x)$ and $g(x)$ whose limits we understand, we can use properties of limits to evaluate limits of functions built out of $f$ and $g$. More specifically, we have the following:

**Theorem.** Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then

- $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- for a fixed real number $c$, $\lim_{x\to a}[cf(x)] = c\,(\lim_{x\to a} f(x))$
- $\lim_{x\to a}[(fg)(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$
- if $\lim_{x\to a} g(x)\neq 0$, then $\lim_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

Indeed, these same rules hold for directional limits, so that (for instance) if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^-} g(x)$ exist, then

$$\lim_{x\to a^-}[f(x)+g(x)] = \lim_{x\to a^-} f(x) + \lim_{x\to a^-} g(x).$$

**Corollary.** If $f(x)$ is a polynomial, then $\lim_{x\to a} f(x) = f(a)$.

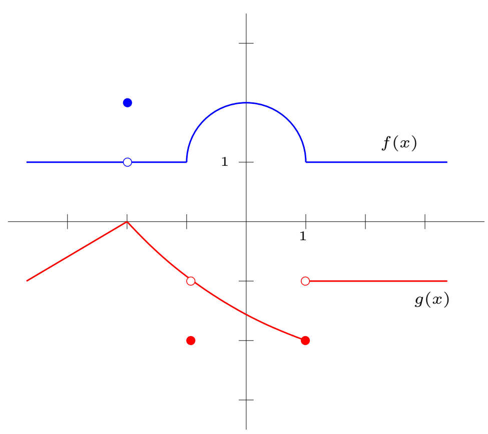
*Proof.* Since $f(x)$ is a polynomial, we know that $f(x) = a_n x^n + \cdots + a_1 x + a_0$. Then we have

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}[a_n x^n + \cdots + a_1 x + a_0] = \lim_{x\to a}[a_n x^n] + \cdots + \lim_{x\to a}[a_1 x] + \lim_{x\to a}[a_0]\\ &= a_n \lim_{x\to a}[x^n] + \cdots + a_1 \lim_{x\to a}[x] + a_0 = a_n\left(\lim_{x\to a} x\right)^n + \cdots + a_1\left(\lim_{x\to a} x\right) + a_0\\ &= a_n a^n + \cdots a_1 a + a_0 = f(a),\end{aligned}$$

where we may apply all the limit laws since $\lim_{x\to a} x$ and $\lim_{x\to a} a_0$ exist (and in fact equal $a$ and $a_0$, respectively, as per our previous example). $\square$

It isn't so crucial that you remember the proof, but instead just remember the handy fact that evaluating limits of polynomials is quite easy.

**Example.** Consider the graph of the functions $f(x)$ and $g(x)$ below.




What is $\lim_{x\to -2}[3f(x)+2g(x)]$? What is $\lim_{x\to 1}[3f(x)+2g(x)]$?

*Solution.* We can see from the graph that $\lim_{x\to -2} f(x) = 1$ and $\lim_{x\to -2} g(x) = 0$, and so to evaluate $\lim_{x\to -2}[3f(x) + 2g(x)]$ we can use the limit laws. We have

$$\lim_{x\to -2}[3f(x)+2g(x)] = \lim_{x\to -2}[3f(x)] + \lim_{x\to -2}[2g(x)] = 3\lim_{x\to -2} f(x) + 2\lim_{x\to -2} g(x) = 3\cdot 1 + 2\cdot 0 = 3.$$

To evaluate $\lim_{x\to 1}[3f(x) + 2g(x)]$ we would like to use the limit laws, but in this case $\lim_{x\to 1} g(x)$ does not exist. We cannot use the limit laws to conclude $\lim_{x\to 1}[3f(x) + 2g(x)]$ does not exist, so instead we will compare directional limits. From our earlier fact, we know that $\lim_{x\to 1}[3f(x) + 2g(x)]$ exists if and only if $\lim_{x\to 1^-}[3f(x) + 2g(x)]$ and $\lim_{x\to 1^+}[3f(x) + 2g(x)]$ exist and are equal.

Now

$$\lim_{x\to 1^-} f(x) = 1,\ \lim_{x\to 1^+} f(x) = 1,\ \lim_{x\to 1^-} g(x) = -2,\ \text{and}\ \lim_{x\to 1^+} g(x) = -1.$$

Using our limit laws for directional limits, we have

$$\lim_{x\to 1^-}[3f(x)+2g(x)] = \lim_{x\to 1^-}[3f(x)] + \lim_{x\to 1^-}[2g(x)] = 3\lim_{x\to 1^-} f(x) + 2\lim_{x\to 1^-} g(x) = 3\cdot 1 + 2\cdot -1 = 1$$

and

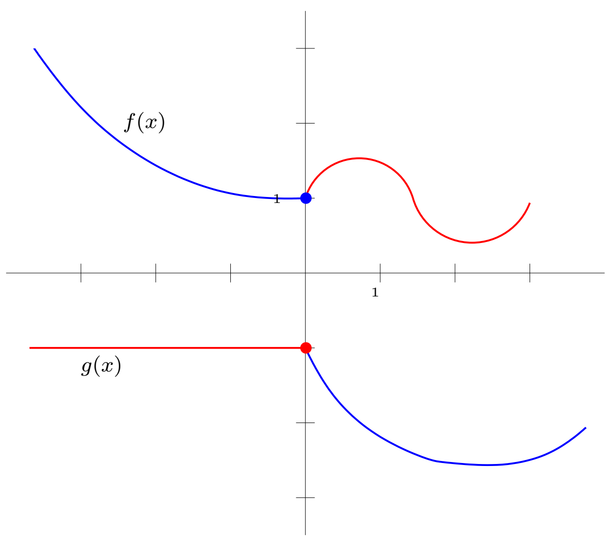
$$\lim_{x\to 1^+}[3f(x)+2g(x)] = \lim_{x\to 1^+}[3f(x)] + \lim_{x\to 1^+}[2g(x)] = 3\lim_{x\to 1^+} f(x) + 2\lim_{x\to 1^+} g(x) = 3\cdot 1 + 2\cdot -2 = -1.$$

Since these values don't agree, we conclude

$$\lim_{x\to 1}[3f(x) + 2g(x)] \text{ does not exist.}$$

□

**Example.** Consider the functions $f(x)$ and $g(x)$ depicted below. What is $\lim_{x\to 0}[f(x) + g(x)]$?




*Solution.* We cannot use the limit laws directly to evaluate $\lim_{x\to 0}[f(x) + g(x)]$, since neither $\lim_{x\to 0} f(x)$ not $\lim_{x\to 0} g(x)$ exist. Instead, we have to attempt to evaluate this limit by evaluating the directional limits and comparing.

We see that

$$\lim_{x\to 0^-} f(x) = 1,\ \lim_{x\to 0^+} f(x) = -1,\ \lim_{x\to 0^-} g(x) = -1,\ \text{and}\ \lim_{x\to 0^+} g(x) = 1.$$

Since the directional limits exist, we can compute the directional limit of the sum using the properties of directional limits:

$$\lim_{x\to 0^-}[f(x)+g(x)] = \lim_{x\to 0^-} f(x) + \lim_{x\to 0^-} g(x) = 1-1 = 0,$$

and

$$\lim_{x\to 0^+}[f(x)+g(x)] = \lim_{x\to 0^+} f(x) + \lim_{x\to 0^+} g(x) = -1+1 = 0.$$

Since the two directional limits exist and agree, we have

$$\lim_{x\to 0}[f(x)+g(x)] = 0.$$

□

This is a great example to keep in mind because it reinforces appropriate use of the limit laws: we can only apply them when the constituent limits exist, and when these constituent limits don't exist *we can't conclude anything* without further tinkering.

**Example.** Using the same functions as the previous example, what can one say about $\lim_{x\to 0^-} \dfrac{f(x)}{1+g(x)}$? What can one say about $\lim_{x\to 0^-} \dfrac{1-f(x)}{1+g(x)}$?

*Solution.* We would like to use the directional limit properties to say that the limit of these quotients is the quotient of the limits. Indeed, all of $f(x), 1+g(x)$ and $1-f(x)$ have limits as $x\to 0^-$. However, the limit rule on quotients can only be applied when the denonimator has a nonzero limit, and in both these example $\lim_{x\to 0^-}[1+g(x)] = 0$. Hence, we cannot conclude anything about either of these limits.

NB: In class on Wednesday we will see we can say something about the first limit, but that without futher information we can't say anything about the second. □

## 5. A preview of Continuity

In class Wednesday we're going to talk about continuous functions and some of the nice properties they enjoy.

**Definition.** A function $f(x)$ is continuous at $a$ if

$$\lim_{x\to a} f(x) = f(a).$$

We've already seen that polynomials satisfy this property, so polynomials are continuous functions. We'll see that lots of other functions are continuous, and we'll use this fact to make evaluating limits of functions lots easier (since you see from the definition that evaluating the limit of a continuous function is easy—you just evaluate the function at the point of interest!).

We'll also talk about the **Intermediate Value Theorem** in class on Wednesday, which is a very nice property enjoyed by continuous functions.