LOCAL EXTREMA AND CALCULUS

1. Announcements

Because of the test on Friday, things are a little different than usual. Important things to note:

- Homework 7 is due Wednesday.
- We'll have a review session Thursday, starting around 6:30. Like last time, I'll bring pizza. You bring questions.
- I'll have extra office hours this week: Tuesday from 3 to 4.

2. Recap

Last class period we introduced the so-called calculus/geometry dictionary. This is a powerful application of the ideas of calculus to studying the geometry of the graph of a function. More concretely, the dictionary gives a correspondence between information about a function's derivatives and information about the graph of the function. The main entries in the dictionary are

$f(x)$ is flat at a	$\leftrightarrow$	$f'(a) = 0$
$f(x)$ is increasing on (a, b)	$\leftrightarrow$	$f'(x) \geq 0$ on (a, b)
$f(x)$ is decreasing on (a, b)	$\leftrightarrow$	$f'(x) \leq 0$ on (a, b)
$f(x)$ is concave up on (a, b)	$\leftrightarrow$	$f''(x) > 0$ on (a, b)
$f(x)$ is concave down on (a, b)	$\leftrightarrow$	$f''(x) < 0$ on (a, b)
$f'(x)$ is increasing on (a, b)	$\leftrightarrow$	$f''(x) > 0$ on (a, b)
$f'(x)$ is decreasing on (a, b)	$\leftrightarrow$	$f''(x) < 0$ on (a, b)
Slopes of tangent lines of f are increasing on (a, b)	$\leftrightarrow$	$f''(x) \geq 0$ on (a, b)
Slopes of tangent lines of f are decreasing on (a, b)	$\leftrightarrow$	$f''(x) \leq 0$ on (a, b)

We'll normally be using the dictionary to translate derivative information into graphical information. Let's do an example.

Example. For the function $f(x) = x^3 + \frac{3}{2}x^2 - 18x + 1$, find all intervals on which the function is increasing, decreasing, concave up, and concave down.

Solution. To find intervals on which the function is increasing, the dictionary tells us we need to know the intervals on which $f'(x) \geq 0$. Similarly, intervals on which the function is decreasing correspond to intervals on which $f'(x) \leq 0$; intervals on which the function is concave up correspond to intervals on which $f''(x) \geq 0$; and intervals on which the function is concave down correspond to intervals on which $f''(x) \leq 0$. Hence, let's compute $f'(x)$ and $f''(x)$ and determine when these functions are positive and negative.

The power rule tells us that $f'(x) = 3x^2+3x-18 = 3(x^2+x-6) = 3(x+3)(x-2)$, and that $f''(x) = 6x+3 = 3(2x+1)$. Since we have factored $f'(x)$ it is easy to see that $f'(x) = 0$ at $x = -3, 2$. On the interval $(-\infty, -3)$ we see that $f'(x) > 0$ (you can see this by choosing a point from the interval, say -4, and computing the sign of $f'(x)$ there); on $(-3, 2)$ we have $f'(x) < 0$; and on $(2, \infty)$ we have $f'(x) > 0$. By the calculus/geometry dictionary, this means $f(x)$ is increasing on the intervals $(-\infty, -3)$ and $(2, \infty)$, and similarly $f(x)$ is decreasing on the interval $(-3, 2)$.

Now $f''(x) = 0$ at $x = -\frac{1}{2}$, and we can see that $f''(x) < 0$ on the interval $(-\infty, -\frac{1}{2})$ and $f''(x) > 0$ on the interval $(-\frac{1}{2}, \infty)$. The dictionary tells us that this implies $f(x)$ is concave down on $(-\infty, -\frac{1}{2})$ and concave up on $(-\frac{1}{2}, \infty)$. □

3. Local extrema and calculus

One of the terms I defined on Friday is the notion of local extrema of a function. Given the graph of a function $f(x)$, it is easy for us to spot local extrema. Without a graph, how can we determine extrema of a function? It turns out that calculus gives us a handy way to find extrema of a function.

Try to graph a function with a local max or min (your choice) at a point $x = a$. If you study your drawing, you will see that the derivative of your function at a will be either 0 or undefined. Hence, to find places where a function *might* have a local max or min, we need to find places where $f'(x) = 0$ or $f'(x)$ is undefined. In class on Friday we called such points *critical points* of $f(x)$. Notice that even though all local extrema are critical points, not all critical points are extrema.

Example. For the function $f(x) = x^3$, the point $x = 0$ is a critical point but not an extreme value.

So how can we determine whether a function has, say, a local maximum at a point *without looking at its graph*? There are actually a few answers. **In all cases, we are assuming that it has already been shown that $f'(a) = 0$ or is undefined. We'll also assume that $f(x)$ is continuous at a.**

(1) Bare Hands The definition of a local max says that a is a local max of f if for any point b near a, $f(a) \geq f(b)$. Hence, we could find a point very close to a and to its left, say b_l, and determine if $f(a) \geq f(b_l)$. If we also found a point b_r very close to a and to its right, we could then test if $f(a) \geq f(b_r)$. If both of these inequalities hold, we're golden.
    The downside to this procedure is twofold. First, it isn't clear how close the points b_l and b_r need to be to a to count at 'near a.' Second, we would have to compute the actual value of $f(b_l)$ and $f(b_r)$, which can be pretty difficult without a calculator.

(2) Second derivative test We might notice that at a local max, the function in question is often concave down. Hence, if we can show that $f''(a) < 0$, then we can immediately conclude that f has a maximum at a. The upside to this technique is that it is very quick (provided you have the second derivative). The downside is that it might happen that $f''(a) = 0$. In this situation, one cannot conclude from the second derivative alone whether f has a max or a min at a. Indeed, you can't even say if the point is an extreme value!

(3) First derivative test We might also examine the slopes of tangent lines near a to get a feeling for what the graph of f looks like near a. In particular, if we find a point very close to a and to its left, call it b_l, with $f'(b_l) > 0$ and another point very close to a and to its right, call it b_r, with $f'(b_r) < 0$, then we can be assured that the function has a maximum at a. Why? Sketch a line segment with positive slope whose right endpoint is a, and a line segment with negative slope whose left endpoint is a. What does

it look like? Sort of like a $\bigwedge$, right? And the tip of this $\bigwedge$ is the point $(a, f(a))$. So you can see that a is a local max.

In a similar way, you can verify that

- if $f'(b_l) < 0$ and $f'(b_r) > 0$, then f has a minimum at a;
- if $f'(b_l) > 0$ and $f'(b_r) > 0$, then f has neither a max nor a min at a; and
- if $f'(b_l) < 0$ and $f'(b_l) < 0$, then f has neither a max nor a min at a.

What are good parts to this approach? First, when we have to find points b_l and b_r, we can determine what 'close by' actually means using the IVT. In particular, provided our derivative is a continuous function (which it almost always is), all we need to do to choose b_l is to make sure $f'(x)$ has no zeros on the interval (b_l, a), and similarly to choose b_r we only have to make sure $f'(x)$ has no zeros on the interval (a, b_r). The second reason it is a good approach is that we don't actually have to compute $f'(b_l)$ or $f'(b_r)$. Instead, we only have to compute whether these numbers are positive or negative. In general, this is far easier to do than computing the actual derivatives at these points. Finally, this technique is great because it works every time.

Example. Find all possible extreme values for the function $f(x) = x^3 + \frac{3}{2}x^2 - 18x + 1$. Use these to find the actual extreme values of $f(x)$.

Solution. We have already seen that $f'(x) = 3(x+3)(x-2)$, so that the critical points of $f(x)$ are precisely $x = -3$ and $x = 2$ (notice that in this case, $f'(x)$ is defined everywhere, and hence the critical points of $f(x)$ are exactly those x which satisfy $f'(x) = 0$).

Now are $x = -3$ and $x = 2$ local extrema? Let's focus on $x = -3$ first. To see if it is a local max or min, we'll use the second derivative test described above. We know that $f''(x) = 3(2x+1)$, and so $f''(-3) = 3(-5) < 0$. By the second derivative test, $f(x)$ has a local maximum at $x = -3$.

To determine if $x = 2$ is a local extreme we'll use the first derivative test (just to shake things up). To choose points to the left and right of $x = 2$ we just need to make sure our left point is larger than -3. I'll take $b_l = 0$ and $b_r = 3$. At these points I have $f'(0) = 3(3)(-2) < 0$ and $f'(3) = 3(6)(1) > 0$. The first derivative test tells me that $x = 2$ is a local minimum of $f(x)$. □

Example. Find all local extrema of the function $f(x) = xe^x$.

Solution. Possible extrema of $f(x)$ occur at critical points. Since $f'(x) = xe^x + e^x = (x+1)e^x$, the only critical point is at $x = -1$ (notice that $e^{\text{anything}} > 0$, and so $(x+1)e^x$ is 0 only when $x+1 = 0$). Is $x = -1$ a local max or min? I'll use the first derivative test to answer this question. I can pick my points b_l and b_r freely since there are no critical points aside from $x = -1$, so I'll choose $b_l = -2$ and $b_r = 0$. I then have $f'(-2) = (-1)e^{-2} < 0$ and $f'(0) = (1)e^0 > 0$ (notice I've used $e^{\text{anything}} > 0$), and the first derivative test tells me $f(x)$ has a local minimum at $x = -1$. □