THE EXTREME VALUE THEOREM

1. Announcements

By and large, I was quite pleased with the results of the midterm. I thought people did well with what was a long and difficult test. I encourage you to look over the test to see your mistakes. If you'd like to talk to me about any of the grading, you're welcome to drop by and chat.

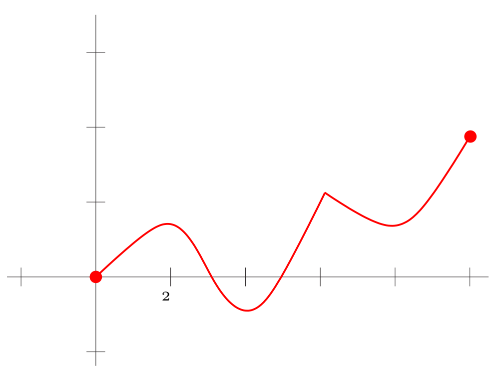
2. Absolute Maxima/Minima

Finding local maxima and minima of a function is important, but they only detect 'local' information. In particular, while we might have a local minimum at a point c, we do not know that c is the smallest value the function might attain. We will be interested in questions just like this in the next week. So, we adopt two new definitions.

Definition. A function $f(x)$ is said to have an absolute maximum at a point c if $f(c) \geq f(x)$ for all points x in the domain of f.

Again, there is a corresponding definition for an absolute minimum which you can guess. How are the definitions of absolute and relative extrema different? In the relative case, we only require that our point $f(c)$ beat values $f(x)$ where x is *close to* c, though in the relative case we insist that $f(c)$ beat all values $f(x)$ for x in the domain of f. Let's see the difference in action.

Example. Identify local maxima and minima and absolute maxima and minima for $f(x)$ defined by



Solution. We can see that the relative minima $x = 2, 4, 6$ and 8, with an absolute minimum at $x = 4$ and an absolute maximum at $x = 10$. □

Notice that one might wonder in this example if the endpoints are considered relative maxima or not. In many ways, this question is really more a matter of definitions, so I'm happy to have people consider these places either as relative maxima or not as relative maxima. There are benefits to either, and I won't take off if you label these as relative maxs or not as relative maxs. I am more than happy to talk to you about this if you like, so just let me know if this troubles you.

3. Extreme Value Theorem

We already have a technique for finding and identifying local maxima and minima involving the derivative. We do this by first finding where the first derivative is zero or undefined, the so-called critical points of $f(x)$. After we have identified critical points, we test each critical point with either the first or second derivative test to determine where it is a local max, a local min, or neither a local max nor a local min.

How might we find and identify absolute maxima and minima? In general, this is a harder problem to solve. For one, a function might not even have a relative maximum or minimum. For instance, consider the function $f(x)$ defined on $(0, 1)$ as pictured below. One can see that the function does not attain either a maximum value

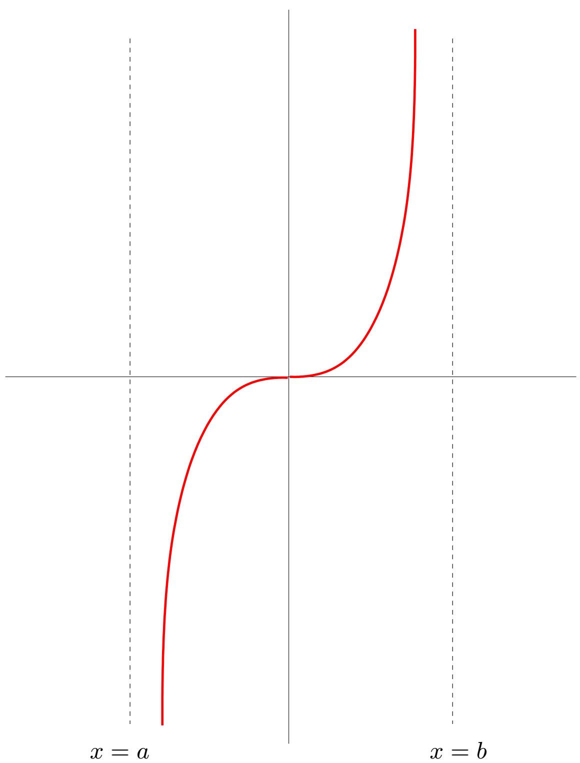


Figure 1. A function on an open interval which doesn't have absolute extrema

or a minimum value on the domain, since the function approaches both $+\infty$ and $-\infty$ on the domain.

So when will we know we can find extreme values (i.e., absolute maxima and minima)? In comes the

Theorem 3.1 (Extreme Value Theorem). *If $f(x)$ is a continuous function on $[a, b]$, then $f(x)$ has an absolute maximum c and an absolute minimum d.*

Just like the intermediate value theorem, if we ever want to use the extreme value theorem to conclude a function has an absolute maximum or minimum, it is very important that the hypotheses of the extreme value theorem are satisfied. The graph above gives an example of a continuous function on the *open* interval $(0, 1)$ which does not satisfy the conclusion of the intermediate value theorem. In fact, one can also construct a function $f(x)$ defined on a closed interval which doesn't have an absolute minimum nor an absolute maximum. Such a function is shown below. Notice that it fails to be continuous on the interval $[0, 1]$!

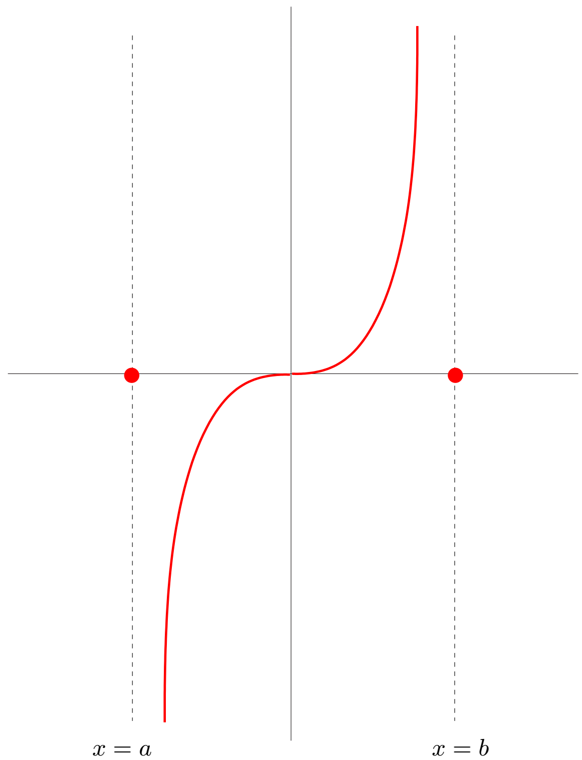


FIGURE 2. A function on a closed interval which doesn't have absolute extrema

4. Finding absolute extrema for continuous functions on closed intervals

So how can we use the extreme value theorem to find absolute maxima or minima? We'll adopt a 4 step technique:

FINDING ABSOLUTE MAXIMA AND MINIMA

- 1^{st} Verify that $f(x)$ is continuous on a closed interval of interest
- 2^{nd} Find critical points of $f(x)$ that lie in the closed interval of interest
- 3^{rd} Evaluate $f(x)$ at critical points and endpoints in a chart
- 4^{th} Pick out the absolute maximum by finding which critical point/endpoint gives the largest function value (similarly for absolute minimum)

It is *critical* if you're going to use this process that you verify the first step. If you don't first check that the function is continuous on a closed interval, this method can fail! Let's try this out.

Example. Find the absolute maxima and minima of the function $f(x) = x\sqrt{x - x^2}$ on the interval $[0, 1]$.

Solution. We want to use the procedure above, so first we need to verify that $f(x)$ is continuous on $[0, 1]$. But since x is continuous everywhere and $\sqrt{x - x^2}$ is defined on $[0, 1]$ (and hence continuous there, since it's an algebraic function), we see that $f(x)$ is continuous on $[0, 1]$ as desired.

For the second step, we need to find critical points of $f(x)$. This means we need to compute $f'(x)$. Now

$$\frac{d}{dx}\left[x\sqrt{x-x^2}\right] = \sqrt{x-x^2} + x\,\frac{1}{2\sqrt{x-x^2}}\,(1-2x) = \sqrt{x-x^2} + \frac{x-2x^2}{2\sqrt{x-x^2}}$$

$$= \frac{2(x-x^2)+x-2x^2}{2\sqrt{x-x^2}} = \frac{3x-4x^2}{2\sqrt{x-x^2}}.$$

Now critical points of f are where $f'(x) = 0$ or when $f'(x)$ is undefined. Zeroes of $f'(x)$ occur when the numerator of the derivative is zero. Now $x(3 - 4x) = 0$ only when $x = 0$ or $x = \frac{3}{4}$. $f'(x)$ is undefined when the denominator is 0, but this happens on when $x - x^2 = 0$, so that $x = 0, 1$. Hence, our only critical point is $x = 0, \frac{3}{4}, 1$.

Now we need to evaluate $f(x)$ at the critical points and endpoints of the interval.

x	$f(x)$
0	$f(0) = 0\sqrt{0-0} = 0$
$\frac{3}{4}$	$f(\frac{3}{4}) = \frac{3}{4}\sqrt{\frac{3}{4} - \frac{9}{16}} > 0$
1	$f(1) = 1\sqrt{1-1} = 0$

Since 0 is the smallest value we see for $f(x)$ in this chart, we have that 0 and 1 are both absolute minima of $f(x)$. Since $f(\frac{3}{4}) > 0$, this means $x = \frac{3}{4}$ is an absolute maximum of the function $f(x)$. □

Example. Find the absolute maxima and minima of the function $f(x) = \dfrac{e\log(x)}{x}$ on the interval $[1, e^2]$.

Solution. We want to use the procedure above, so first we need to verify that $f(x)$ is continuous on $[1, e^2]$. Now $\log(x)$ is a nice, continuous function whenever $x > 0$, so in particular it's continuous on the interval $[1, e^2]$ we're interested in. Similarly, the denominator would only cause a problem with continuity if it vanished on the interval we're concerned with. But since 0 is not in the interval $[1, e^2]$, having that x in the denominator doesn't mess up continuity. We conclude that $f(x)$ is a continuous function on $[1, e^2]$ as desired.

For the second step, we need to find critical points of $f(x)$. This means we need to compute $f'(x)$. Now

$$\frac{d}{dx}\left[\frac{e\log(x)}{x}\right] = e\left[\frac{x\frac{d}{dx}[\log(x)] - \log(x)\frac{d}{dx}[x]}{x^2}\right] = \frac{e(1-\log(x))}{x^2}.$$

Now critical points of f are where $f'(x) = 0$ or when $f'(x)$ is undefined. Zeroes of $f'(x)$ occur when the numerator of the derivative is zero. Now $e(1 - \log(x)) = 0$ only when $1 - \log(x) = 0$, which happens only if

$1 = \log(x)$, and we know that this means exactly $x = e$. $f'(x)$ is undefined only when the denominator is 0 or when the numerator is undefined, but these things only happen when $x \leq 0$. Since we're in the interval $[1, e^2]$, we don't have to worry about this. Hence, our only critical point is $x = e$.

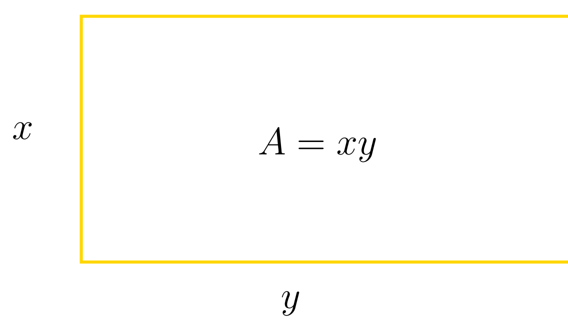
Now we need to evaluate $f(x)$ at the critical point e and the endpoints of the interval: 1 and e^2.

x	$f(x)$
1	$f(1) = \frac{e\log(1)}{1} = \frac{e \cdot 0}{1} = 0$
e	$f(e) = \frac{e\log(e)}{e} = \frac{e \cdot 1}{e} = 1$
e^2	$f(e^2) = \frac{e\log(e^2)}{e^2} = \frac{e \cdot 2}{e} = \frac{2}{e}$

Since 0 is the smallest value we see for $f(x)$ in this chart, we have that 1 is the input which gives the absolute minimum of this function on $[1, e^2]$. To find the maximu, we just need to determine which is larger: 1 or $\frac{2}{e}$. But since $e \approx 2.7$, we see that $\frac{2}{e} < 1$. Hence e is the input which gives the absolute maximum of this function on $[1, e^2]$. □

Example. A crazy billionaire gives you 10 meters of gold wire and asks you to construct a rectangle with maximum area. If you succeed, he'll give you and your math professor \$1,000,000. What rectangle will you construct? What will be its dimensions and area?

Solution. I'll operate on the assumption that you'll want to win the million bucks, and hence you'll try to maximize the area of the resultant rectangle. How will you do this? Let's call x the height of the rectangle and y its width.



One can see that the resultant rectangle has area $A = xy$. We would like to maximize this function, but we don't know how to in its current form: it's a function of two variables, and we don't know how to maximize such a function. So what do we do?

The fact that we have 10 meters of wire comes to the rescue, since it tells us the perimeter of our figure is 10. Specifically, we have $10 = 2x + 2y$. We can now solve for y in terms of x and rewrite our area function:

$10 = 2x + 2y \Leftrightarrow y = 5 - x$, and hence $A = xy = x(5 - x) = 5x - x^2$. This is the kind of function we can maximize, but first we need to know the domain of the function.

For this, notice that we could make a rectangle with no height (so that $x = 0$). This would be dumb, but we could do it. We could make another dumb rectangle: one without width, so that $x = 5$ (think about why this is 5 and not 10). The possible rectangles we could construct sit somewhere between these two stupid extremes, so we see that x lives in the interval $[0, 5]$. Now we're really in business since we have reduced our original problem to the following: maximize the function $A(x) = 5x - x^2$ on the interval $[0, 5]$.

Since $A(x)$ is continuous on $[0, 5]$ (it's a polynomial, so in fact it's continuous everywhere), we can find the extreme values of A by computing the value of of $A(x)$ at critical points and endpoints. We proceed to find critical points: $A'(x) = 5 - 2x$, and so the only critical point we have is $x = \frac{5}{2}$.

Now we need to evaluate $A(x)$ at the critical point and the endpoints of the interval.

x	$A(x)$
0	$A(0) = 0$
$\frac{5}{2}$	$A(\frac{5}{2}) = \frac{25}{4}$
5	$A(5) = 0$

So to maximize the area of our rectangle we will make $x = \frac{5}{2}$ (and so $y = \frac{5}{2}$ since $y = 5 - x$), and the resultant area will be $\frac{25}{4}$.

We can restate our result: of all rectangles with a fixed perimeter, a square maximizes area. $\square$