A Brief History of Algebra

(a.k.a., What's algebraic about modern algebra?)

What we think algebra is

* High school algebra teaches rules for manipulating equations
  * e.g. ``multiplication distributes across addition''

* The theoretical culmination of high school algebra is the quadratic equation: solutions to $ax^2+bx+c=0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When algebra began

* Most of high school algebra is at least 2000 years old, though almost always phrased geometrically

  If a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments.

* Most notation we use is less than 500 years old
  * = and + weren't used until about 1550
  * Negative numbers weren't accepted until the late 1500s
  * Analytic geometry (coordinate systems for points) wasn't invented until around the 1650s

The Algebraic revolution

* Solving a general cubic equation was the main algebraic quest in the 1400 & 1500s
* In early 1500s, del Ferro solved $x^3+mx = n$

$$x = \sqrt[3]{\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}} + \sqrt[3]{\frac{n}{2}-\sqrt{\frac{n^2}{4}+\frac{p^3}{27}}}$$

* Ferro kept his work secret for fear of being challenged, but passed his work on to a student named Fior

More progress on cubics

* Niccolo Fontana Tartaglia solved a broader class of cubic equations
* Tartaglia also kept his results secret
* Fior challenged Tartaglia to a "cubic-equation off" in 1535.
* Tartaglia destroys Fior

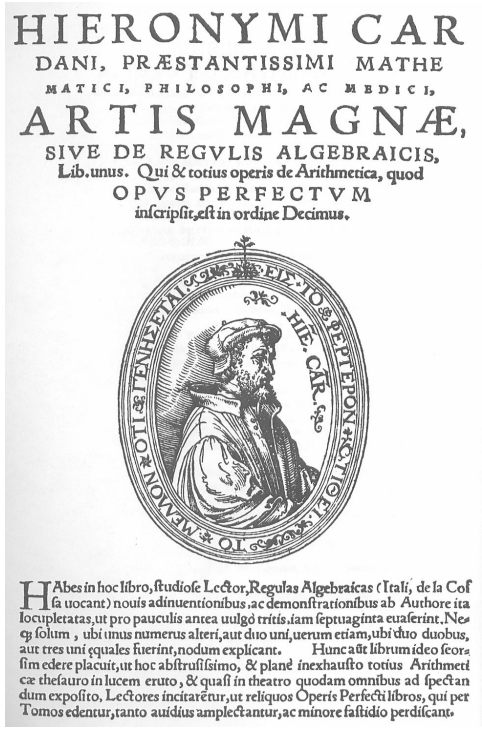
Tartaglia shares his secret

* Gerolamo Cardano persuaded Tartaglia to reveal his cubic secret

"I swear to you, by God's holy Gospels, and as a true man of honor, not only never to publish your discoveries, if you teach me them, but I also promise you, and I pledge my faith as a true Christian, to note them down in code, so that after my death no one will be able to understand them."

Cubic Scandal!

* Cardano learns of del Ferro's original work in solving cubics, and uses this as a loophole to write about solving cubics
* Cardano's student Ferrari uses Tartaglia's techniques to solve the general quartic



HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

HIER. CAR.

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cosa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgo tritis, iam septuaginta euaserint. Nec solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc autē librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarētur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

Complex numbers

* Cardano noticed that some cubic equations with real roots were solved using complex numbers
* This is the reason that complex numbers (and negative numbers) finally became acceptable

What about quintic equations?

* Mathematicians worked for another 250 years to find a quintic formula
* Niels Abel proved in 1824 that a general quintic has no "algebraic" solutions
* This was a big surprise!
* Abel died at 26 due to sledding trip

When do equations have algebraic solutions?

* Evariste Galois determined precisely when an equation had algebraic solutions
* Developed these ideas when he was a teen
* Killed in dual at age 20
* His work is now known as Galois Theory

The two sides of modern algebra

Groups

* One can consider how roots of a polynomial interact with each other.
* This interaction is measured by an object called a group.
* Groups will be the objects we think about most in this class

Rings & Fields

* To determine if roots can be expressed algebraically, one has to study how "complicated" a given root is
* One can measure this "complication" by studying rings and fields
* We'll spend a little time thinking about rings and fields in this class