Solutions to odd numbered problems can be found in the text. Here we include solutions to the other problems.

(1) Find the global maximum and minimum for the function $f(x,y) = xy + y$ on the set $x^2 + y^2 \leq 4$.

Solution. The maximum value of f occurs at

$$(x,y) = \left(\frac{-1+\sqrt{33}}{4}, \sqrt{\frac{15+\sqrt{33}}{8}}\right) \approx (1.186, 1.610),$$

where the function is approximately 3.519. The minimum value of f occurs at

$$(x,y) = \left(\frac{-1+\sqrt{33}}{4}, -\sqrt{\frac{15+\sqrt{33}}{8}}\right) \approx (1.186, 1.610),$$

where the function is approximately -3.519. Both of these extreme values occur on the boundary and can be found using Lagrange multipliers. There is one critical point in the interior, but it is not a global extreme. □

(2) Find the global maximum and minimum for the function $f(x,y,z) = 3xz + 6y$ on the set $x^2 + 2y^2 + z^2 \leq 6$.

Solution. The maximum value of f occurs when $(x,y) = (\sqrt{2}, 1, \sqrt{2})$ or $(x,y) = (-\sqrt{2}, 1, -\sqrt{2})$, at which point the function takes the value 12. The minimum value of f occurs when $(x,y) = (-\sqrt{2}, -1, \sqrt{2})$ or $(x,y) = (\sqrt{2}, -1, -\sqrt{2})$, at which point the function takes the value -12. There are no critical points in the interior. □

(15.8.26) *Solution.* The constraint is $x + y + z = p$, and we're trying to maximize

$$A(x,y,z) = \sqrt{s\,(s-x)\,(s-y)\,(s-z)}.$$

Note that inputs which correspond to extreme values of $A(x,y,z)$ are precisely the same inputs which give extreme values for

$$f(x,y,z) = s(s-x)(s-y)(s-z) = s^4 - s^3(x+y+z) + s^2(xy+xz+yz) - sxyz.$$

Hence we'll try to optimize f instead. Using Lagrange multipliers, note that $\nabla g = \langle 1,1,1\rangle$, so we are trying to solve

$$\langle f_x, f_y, f_z\rangle = \nabla f = \lambda \nabla g = \lambda\langle 1,1,1\rangle,$$

where (using the chain rule and the fact that s is a function of x, y and z)

$$\begin{aligned}
f_x &= 4s^3\frac{1}{2} - 3s^2\frac{1}{2}(x+y+z) - s^3 + 2s\frac{1}{2}(xy+xz+yz) + s^2(y+z) - xyz\frac{1}{2} - syz\\
f_y &= 4s^3\frac{1}{2} - 3s^2\frac{1}{2}(x+y+z) - s^3 + 2s\frac{1}{2}(xy+xz+yz) + s^2(x+z) - xyz\frac{1}{2} - sxz\\
f_z &= 4s^3\frac{1}{2} - 3s^2\frac{1}{2}(x+y+z) - s^3 + 2s\frac{1}{2}(xy+xz+yz) + s^2(y+x) - xyz\frac{1}{2} - sxy.
\end{aligned}$$

Since we are trying to solve $\langle f_x, f_y, f_z\rangle = \langle \lambda,\lambda,\lambda\rangle$, we have $f_x = f_y = f_z = \lambda$. Since $f_x = f_y$ we get $0 = f_x - f_y$, and plugging the computed partials above into this equation, we have

$$0 = s^2(y-x) - s(yz - xz) = s^2(y-x) - sz(y-x) = s(y-x)(s-z).$$

Likewise the equations $0 = f_x - f_z$ and $0 = f_y - f_z$ give

$$0 = s^2(z-x) - s(yz-xy) = s(z-x)(s-y)$$
$$0 = s^2(z-y) - s(xz-xy) = s(z-y)(s-x).$$

In order for all three equations to be 0, one of the following situations must occur:

(a) $s = 0$
(b) $y = x$ and $z = x$ (this would force $y = z$, and so the factor of $z - y$ in f_z would make $f_z = 0$)
(c) $y = x$ and $s = y$ (this would force $s = x$, and so the factor of $s - x$ in f_z would make $f_z = 0$)
(d) $s = z$ and $z = x$ (this would force $s = x$, and so the factor of $s - x$ in f_z would make $f_z = 0$)
(e) $s = z$ and $s = y$ (this would force $z = y$, and so the factor of $z - y$ in f_z would make $f_z = 0$).

Notice that (a) doesn't make sense in the context of this problem: s is half the perimeter of the triangle. If $s = 0$ then the perimeter is 0, and hence all the sides of the triangle have length 0. This wouldn't make much of a triangle.

Case (b) is precisely the case where all lengths have the same length (i.e., when the triangle is equilateral).

Now if we are in case (c), then note that $y = s = \frac{x+y+z}{2} = \frac{2y+z}{2} = y + \frac{z}{2}$. But this means that $z = 0$, and so one of the sides of the triangle has length 0. Again: this isn't much of a triangle.

Cases (d) and (e) degenerate in much the same way as we found in case (c). Hence f is maximized when $x = y = z$, and according to the constraint this means $x = y = z = \frac{p}{3}$. □