(1) Suppose you want to test the claim that the average Wellesley student gets exactly 7 hours of sleep per night. You survey 20 students, with these results:

Hours of sleep	Number of students
4	2
5	1
5.5	1
6	3
6.5	1
7	5
8	3
9	4

Carry out a hypothesis test at the $\alpha = 0.05$ significance level. What is the p-value of the result?

Solution. Let X be the number of hours of sleep a Wellesley student gets. We're interested in the alternative hypothesis $\mu_X \neq 7$ against the null hypothesis $\mu_X = 7$. Since we have 20 samples, we don't have enough to use a Z-statistic. On the other hand, if we assume that X follows a normal distribution and our samples are drawn iid, then we can use the test statistic

$$T = \frac{\bar{X} - 7}{S_X/\sqrt{20}}.$$

Since this follows a t-distribution with 19 degrees of freedom, and since we're using a two-tailed test at a significance level of $\alpha = 0.05$ our rejection region .05, our rejection region is

$$RR = \{t : |t| > 2.093\}.$$

Now our data gives us

$$\bar{X} = \frac{2 \cdot 4 + 1 \cdot 5 + \cdots + 4 \cdot 9}{20} = 6.9$$
$$S_X^2 = \frac{2 \cdot (4 - 6.9)^2 + 1 \cdot (5 - 6.9)^2 + \cdots + 4 \cdot (9 - 6.9)^2}{19} \approx 2.4368$$
$$S_X \approx 1.561.$$

Hence our test statistics in this case takes the value

$$\frac{6.9 - 7}{1.561/\sqrt{20}} \approx -0.286.$$

Since this is outside the rejection region, we fail to reject the null hypothesis at the given significance level.

The p-value is the largest significance level at which we'd fail to reject the null hypothesis. Based on our t-table, with 19 degrees of freedom, the table doesn't even have a t-value corresponding to this statistic! We'd still fail to reject even at 10%, so the p value is larger than 0.1. □

(2) Is average body temperature the same for women and men? In a sample of $n_1 = 9$ women, the average body temperature was $\bar{Y}_1 = 98.49$, with sample standard deviation $S_1 = 0.549$. In a sample of $n_2 = 9$ men, the average body temperature was $\bar{Y}_2 = 97.86$, with sample standard deviation $S_2 = 0.583$. Based on these sample data, do women have a significantly ($\alpha = 0.05$) warmer average body temperature than men? What is the p-value associated with the hypothesis test?

Solution. We want to compare the alternative hypothesis ($\mu_{Y_1} > \mu_{Y_2}$) to the null hypothesis ($\mu_{Y_1} = \mu_{Y_2}$). Again, we've only got 18 samples in total here, so a Z-statistic is not appropriate. On the other hand, since we're analyzing $\mu_{Y_1} - \mu_{Y_2}$, we can use the statistic

$$T = \frac{\bar{Y}_1 - \bar{Y}_2}{S_p\sqrt{\frac{1}{9} + \frac{1}{9}}}.$$

If we assume that Y_1, Y_2 follow normal distributions with equal variance, and samples are drawn in an iid fashion, then T follows a t-distribution with 16 degrees of freedom. Since we're doing an upper one-tail test, the rejection region is

$$RR = \{t : t > 1.746\}.$$

Most of the relevant data is handed to us; the only data we need to compute is the pooled sample variance. This is given by

$$\begin{aligned} S_p^2 &= \frac{(9-1)S_1^2 + (9-1)S_2^2}{(9-1)+(9-1)} \approx 0.321 \\ S_p &\approx 0.566. \end{aligned}$$

So our statistic is

$$T = \frac{98.49 - 97.86}{0.566\sqrt{\frac{1}{9} + \frac{1}{9}}} \approx 2.36.$$

Since this falls in our rejection region, we reject the null in favor of the alternative hypothesis.

Furthermore, our t-table tells us that this computed statistic value yields a p-value between 2.5% and 1% significance (whose rejection regions correspond to 2.120 and 2.583). □

(3) A group of $n = 5$ students was surveyed twice, once at the start and once at the end of the term. Their reported sleep at the start and end of the term is given in the following chart

Student	Hours of sleep (start of term)	Hours of sleep (end of term)
1	5	4.5
2	7	5
3	6.5	6.5
4	6.5	5.5
5	7.5	7

Carry out a hypothesis test at a 5% significance level to determine if students sleep less at the end of the semester than at the start.

Solution. Let X be the number of hours slept at the start of the semester, and Y the number of hours slept at the end of the semester. Let $D = X - Y$. We want to compare the alternative hypothesis ($\mu_D > 0$) against the null hypothesis ($\mu_D = 0$). Our data gives us

$$D_1 = 0.5 \quad D_2 = 2 \quad D_3 = 0 \quad D_4 = 1 \quad D_5 = 0.5.$$

Since we have 5 data points for D, we'll use a T statistic

$$T = \frac{\bar{D} - 0}{S_{\bar{D}}/\sqrt{5}}$$

(this requires we assume that samples are drawn iid and have an underlying normal distribution). Note that since we are sampling 5 students, our assumptions will give us that $T \sim t(4)$. Hence our rejection region is

$$RR = \{t : t > 2.132\}.$$

Now our data gives us the following

$$\bar{D} = \frac{0.5 + 2 + 0 + 1 + 0.5}{5} = 0.8$$

$$S^2_{\bar{D}} = \frac{1}{4}\left((0.5-0.8)^2 + (2-0.8)^2 + (0-0.8)^2 + (1-0.8)^2 + (0.5-0.8)^2\right) \approx 0.575$$

$$S_{\bar{D}} = 0.758.$$

Hence our computed statistic is

$$T = \frac{0.8}{0.758/\sqrt{5}} \approx 2.359.$$

This value is in our rejection region, so we'll reject the null in favor of the alternative hypothesis. □