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# Normal/Subnormal and Composition Series and Solvable Groups 7/13/07

Def: A subnormal series ("normal" in some texts) of a group $G$ is a series of subgroups
$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_{r-1} \trianglelefteq G_r = G$.
The quotient groups $G_{i+1}/G_i$ are called factor groups of the series.

Ex: $e \trianglelefteq \langle\sigma\rangle \trianglelefteq \langle\sigma\rangle \trianglelefteq \langle\sigma, \rho^2\rangle \trianglelefteq D_8 \trianglelefteq D_8$
is a subnormal series of $D_8$

Note: ① "repetitions" are allowed, leading to trivial factor groups

② $\langle\sigma\rangle$ is not normal in $G$ (this is ok).
If a subnormal series has $G_i \trianglelefteq G$ for all $G_i$, it is a normal series ("invariant" in some texts)

Recall: A group is simple if its only normal subgroups are $1$ and itself. [Since simple groups cannot be "factored" (as in $G/N$ where $N \trianglelefteq G$) they play a role analogous to that of prime #'s. We are working toward a group theory version of the "unique factorization thm" for #'s]

Def: A composition series (of length $r$) of a group $G$ is a subnormal series with $r$ composition factors $G_{i+1}/G_i$ which are:

① non-trivial (i.e. $G_i \ntrianglelefteq G_{i+1}$)

② simple

Claim: The second condition above is equivalent to saying that $G_i$ is a maximal normal subgp of $G_{i+1}$, so that composition series admit no "refinement". (Proof omitted).

Examples:

1. $e \lhd A_5 \lhd S_5$ is the only composition series of $S_5$, with composition factors $A_5$ and $\mathbb{Z}_2$.
2. $e \lhd \langle \rho^2 \rangle \lhd \langle \rho \rangle \lhd D_8$ and $1 \lhd \langle \sigma \rangle \lhd \langle \sigma, \rho^2 \rangle \lhd D_8$ are two composition series of $D_8$, with comp. factors $\mathbb{Z}_2$, $\mathbb{Z}_2$, and $\mathbb{Z}_2$ in both cases.
3. Three composition series and their factors for $\mathbb{Z}_{12}$:

$$e \underset{\mathbb{Z}_3}{\lhd} \mathbb{Z}_3 \underset{\mathbb{Z}_2}{\lhd} \mathbb{Z}_6 \underset{\mathbb{Z}_2}{\lhd} \mathbb{Z}_{12}$$

$$e \underset{\mathbb{Z}_2}{\lhd} \mathbb{Z}_2 \underset{\mathbb{Z}_2}{\lhd} \mathbb{Z}_4 \underset{\mathbb{Z}_3}{\lhd} \mathbb{Z}_{12}$$

$$e \underset{\mathbb{Z}_2}{\lhd} \mathbb{Z}_2 \underset{\mathbb{Z}_3}{\lhd} \mathbb{Z}_6 \underset{\mathbb{Z}_2}{\lhd} \mathbb{Z}_{12}$$

Prop: All finite groups have a composition series.

Pf: (by induction on the order of $G$)
Let $G$ be a finite group. If $|G| = 1$ (or 2), comp. series $1 \lhd G$. Assume result for all $|G| < n \in \mathbb{Z}_{\geq 0}$.
If $G$ is simple or has a comp series, done.
Otherwise, $G$ has a proper max'l normal subgp $H$ which has a comp. series. Since $G/H$ is simple, $G$ has comp series $1 = H_0 \lhd H_1 \lhd \ldots \lhd H \lhd G$.

Note: Infinite gps need not have composition series ($\mathbb{Z}$, for example, since it has no non-trivial simple normal subgroups).

Def: Two composition series are equivalent if they are the same up to permutation and isomorphism.

In symbols, if $G_0 \lhd G_1 \lhd \ldots \lhd G_s \lhd G$ and $H_0 \lhd H_1 \lhd \ldots \lhd H_r \lhd G$ are equivalent comp. series of $G$, then 1. $s = r$, 2. there is some $\sigma \in S_r$ such that $G_{i+1}/G_i \simeq H_{\sigma(i)+1}/H_{\sigma(i)}$ for every $i \in \{1, \ldots, r-1\}$

Ex: The three comp series given earlier for $\mathbb{Z}_{12}$ are equivalent.

Jordan-Hölder Thm: If a group $G$ has a composition series, then any two composition series of $G$ are equivalent. (Pf: omitted)

Note: ① This is somewhat analogous to "unique factorization" of #'s since simple groups are analogous to primes in that they cannot be "factored" $G/N$. Knowing the composition factors of a group is helpful in understanding the group. This served as motivation to classify all finite simple gps (completed 1980). [∃ 18 infinite families of simple groups and 26 sporadic simple gps.]

② Two nonisomorphic groups may have equivalent composition factors ($D_8$, $\mathbb{Z}_8$ both have $\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2$).

<u>Def</u>: A group $G$ is <u>solvable</u> if it has a (sub)normal series $1=G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{r-1} \trianglelefteq G_r = G$ whose factor groups $G_{i+1}/G_i$ are all abelian.

<u>Fact</u>: For <u>finite</u> groups, this is equivalent to saying $G$ has a composition series whose factors are all cyclic gps of prime order. [Not so for infinite - again $\mathbb{Z}$ is solvable $1 \trianglelefteq \mathbb{Z}$ but has no comp series.]

<u>Note</u>: Think of solvable groups as the "opposite" of simple - they break down into many normal subgroups. The only simple solvable gps have prime order.

<u>Ex</u>: ① All abelian gps (including all cyclic gps) are solvable trivially
② Non-abelian solvable gps: (i) $D_{2n}$ since $1 \underset{\mathbb{Z}_n}{\trianglelefteq} \langle \rho \rangle \underset{\mathbb{Z}_2}{\trianglelefteq} D_{2n}$
(ii) $S_3$ since $1 \underset{\mathbb{Z}_3}{\trianglelefteq} A_3 \cong \mathbb{Z}_3 \underset{\mathbb{Z}_2}{\trianglelefteq} S_3$ (iii) $S_4$ since $1 \trianglelefteq V_4 \underset{\mathbb{Z}_3}{\trianglelefteq} A_4 \trianglelefteq S_4$

<u>CounterEx</u>: $S_n$ is <u>not</u> solvable for $n \geq 5$ (also not simple) since $1 \trianglelefteq A_n \underset{\mathbb{Z}_2}{\trianglelefteq} S_n$
$A_n$ not abelian, and $A_n$ is simple
[Note: all gps of order < 60 are solvable]

<u>Prop</u>: If $G$ is solvable and $H \leq G$, then $H$ is solvable.

<u>Pf</u>: Let $G$ be solvable with subnormal series $1=G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_r = G$ s.t. $G_{i+1}/G$ all abelian.
Let $H$ be a subgroup of $G$ and let $H_i = S \cap G_i$ for all $i=0,\ldots,r$
Consider the subgp chain $1 = S_0 \leq S_1 \leq \ldots \leq S_r = S$. For any $i$, from the Diamond Isom. thm, since $G_i \trianglelefteq G_{i+1}$ and $S_{i+1} \leq G_{i+1}$, we have $S_{i+1}G_i$ a subgp, $G_i \trianglelefteq S_{i+1}G_i$, and $S_{i+1} \cap G_i \trianglelefteq S_{i+1}$ with $S_{i+1}/S_{i+1}\cap G_i \cong S_{i+1}G_i/G_i$ i.e. $S_{i+1}/S_i \cong S_{i+1}G_i/G_i$

But $S_{i+1}G_i \leq G_{i+1}$ so $S_{i+1}/S_i$ isomorphic to a subgp of $G_{i+1}/G_i$ and therefore is abelian. □

<u>Aside</u>: There are many other ways in which solvability is "inheritable"

One of the most important results about which gps are sol'ble:
<u>Feit - Thompson Thm</u> (early 1960's - pf 255 pages long)
All finite gps of odd order are solvable.
<u>Note</u>: This was important in classification of finite simple gps.

<u>Thm (Galois)</u>: A polynomial is solvable by radicals (i.e. factorable over $\mathbb{Q}$ via radical extensions) iff its Galois group is solvable.

<u>Note</u>: $S_n$ not solvable for $n \geq 4$ is important in proving that there are poly's of every degree $\geq 5$ which are not solvable by radicals, and $\therefore$ there is no "quintic formula" akin to the Quadratic Formula. [Ex: $x^5 - 6x - 3 \in \mathbb{Q}[x]$ has 3 real, 2 cplx roots and $\therefore$ Galois group has elts of order 5 and 2 and is $\therefore$ all of $S_5$ which is not solvable.] This does not preclude the existence of solvable quintics, of course.