I. Algebra: exponents, scientific notation, simplifying expressions

---

References for MATH REVIEW
For more practice problems and detailed written explanations, see the following books, both on reserve all year in the Sciences Library.

Algebra and Trigonometry Refresher, by Loren C. Larson
(can be purchased in the bookstore)
Algebra and Trigonometry, by Keedy and Bittinger

A. Exponents: Definitions and rules.

1. Definition $a^1 = a$, $a^2 = a \cdot a$, $a^3 = a \cdot a \cdot a$, $a^n = a \cdot a \cdots a$ (n times)
2. $a^m a^n = a^{m+n}$ (An example showing why: $a^2 a^3 = (a \cdot a)(a \cdot a \cdot a) = a^{2+3} = a^5$)
3. $\frac{a^m}{a^n} = a^{m-n}$ (provided $a \neq 0$) $\left[\frac{a^5}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a^{5-3} = a^2 \right]$
4. Def $a^0 = 1$ (reason: $\frac{a}{a} = 1$ and $\frac{a}{a} = a^{1-1} = a^0$)
   (provided $a \neq 0$)
5. Def $a^{-n} = \frac{1}{a^n}$ $(a \neq 0)$ (reason: $\frac{1}{a^n} = \frac{a^0}{a^n} = a^{0-n} = a^{-n}$)
6. $(ab)^n = a^n b^n$ (An example showing why: $(ab)^3 = ababab = aaabbb = a^3 b^3$)
7. $(a^n)^m = a^{m \cdot n}$ $[(a^2)^3 = a^2 a^2 a^2 = (aa)(aa)(aa) = a^{2 \cdot 3} = a^6]$
8. Note: $(a+b)^n \neq a^n + b^n$!!! (except when $n = 1$ or $a = 0$ or $b = 0$)

e.g. $(1+2)^3 = 3^3 = 27$, while $1^3 + 2^3 = 1 + 8 = 9$.

Correct rules: $(a+b)^2 = a^2 + 2ab + b^2$,

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, etc.; see Section D .

9. Exponentiation precedes multiplication. For example,
$7a^3 = 7\cdot a\cdot a\cdot a$, <u>not</u> $(7a)^3$, which would be $7a\cdot 7a\cdot 7a = 7^3a^3$.

10. $(-a)^n = (-1)^n a^n = \begin{cases} a^n & \text{if } n \text{ even} \\ -a^n & \text{if } n \text{ odd} \end{cases}$

e.g. $(-x)^2 = (-x)(-x) = (-)(-)x^2 = x^2$

$(-x)^3 = (-x)(-x)(-x) = (-)(-)(-)x^3 = -x^3$

Note: $-x^2 = (-)x\cdot x \neq (-x)^2$.

<u>Examples</u> $5^{-1} = \frac{1}{5^1} = \frac{1}{5}$; $10^{-3} = \frac{1}{1000} = 0.001$;

$\frac{1}{x^{-3}} = x^3$; $(x^{-2})^3 = x^{-6}$; $\frac{x^8x^{-4}}{x^2} = \frac{x^4}{x^2} = x^2$;

$(6x^3)^2 = 6^2(x^3)^2 = 36x^6$; $3x^4\cdot 5x^3 = (3\cdot 5)(x^4\cdot x^3) = 15x^7$;

$\frac{x^4y}{x^{-1}y^2} = \frac{x^4}{x^{-1}}\cdot\frac{y}{y^2} = \frac{x^5}{y}$ or x^5y^{-1}

<u>Exercises I A</u> Simplify each expression.

1. $2x^3\cdot 3x^2$
2. $-(2x^3)^4$
3. $(-4x^{-1}z^{-2})^{-2}$
4. $(5x^4y^{-3}z^2)(-2x^2y^4z^{-1})$
5. $\frac{4^{-2} + 2^{-4}}{8^{-1}}$
6. $\frac{x^3y^{-3}}{x^{-1}y^2}$
7. $\frac{(3a^{-1}b^{-2}c^4)^3}{(2a^{-1}b^2c^{-3})^2}$
8. $\frac{5^{-2}x^{-4}y^3}{2^{-3}x^{-5}y}$

B. Radicals (or roots) and fractional exponents

1. Definition

$\sqrt[n]{a}$ denotes $\begin{cases} \text{the positive } n^{th} \text{ root of } a \text{ if } n \text{ is even and } a>0 \\ \\ \text{the } n^{th} \text{ root of } a \text{ if } n \text{ is odd} \end{cases}$

When a is negative and n is even (e.g. $\sqrt[2]{-9}$), $\sqrt[n]{a}$ is undefined within the real number system.

Note: $\sqrt{a}$ is short for $\sqrt[2]{a}$, the positive square root of a .

Examples: $\sqrt[2]{9} = \sqrt{9} = 3$; $\sqrt[3]{8} = 2$; $\sqrt[3]{-8} = -2$; $\sqrt{-9}$ is undefined.

2. Definition $a^{1/n}$ is defined to be $\sqrt[n]{a}$ (where possible)

(Reason: $(a^{1/n})^n = a^{(\frac{1}{n}\cdot n)} = a^1 = a$, so $a^{1/n}$ is the nth root of a)

3. Definition $a^{m/n} = (a^{1/n})^m$ or $(a^m)^{1/n}$ (these have the same value), provided $a^{1/n}$ is defined.

4. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a/b} = \sqrt[n]{a} / \sqrt[n]{b}$

provided $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both defined.

(Note $\sqrt{(-2)(-8)} = \sqrt{16} = 4$ even though $\sqrt{-2}$ and $\sqrt{-8}$ are undefined.)

5. $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$. In particular, $\sqrt{a^2+b^2} \neq a+b$,

e.g. $\sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$, while $3+4=7$.

6. $\sqrt{x^2} = |x|$, the absolute value of x .

Examples

1. $8^{4/3} = (\sqrt[3]{8})^4 = 2^4 = 16$

2. $18^{2/3} = \sqrt[3]{18^2} = \sqrt[3]{(3\cdot3\cdot2)(3\cdot3\cdot2)} = 3\cdot\sqrt[3]{2\cdot3\cdot2} = 3\cdot\sqrt[3]{12}$

3. $\sqrt{\frac{x^4}{9}} = \frac{\sqrt{x^4}}{\sqrt{9}} = \frac{x^2}{3}$ but $\sqrt{\frac{x^2}{9}} = \frac{\sqrt{x^2}}{\sqrt{9}} = \frac{|x|}{3}$

4. $\sqrt{6^2} = 6$; $\sqrt{(-6)^2} = 6$; $\sqrt{12} = \sqrt{4\cdot3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$

5. Simplify $\sqrt[3]{72a^7b^9}$.

$$\sqrt[3]{72a^7b^9} = (72a^7b^9)^{1/3} = (72)^{1/3}a^{7/3}b^{9/3} = 8^{1/3}9^{1/3}a^{7/3}b^3$$
$$= 2\cdot 9^{1/3}a^{7/3}b^3 \quad \text{or} \quad 2a^2b^3\cdot\sqrt[3]{9a} \quad \text{(either of these is OK)}$$

6. Rationalize the denominator of $\frac{5}{\sqrt{3}}$ (eliminate the radical) .

$$\frac{5}{\sqrt{3}} = \frac{5\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{5\sqrt{3}}{3} \quad \text{or} \quad \frac{5}{3}\sqrt{3}$$

7. Simplify $(0.027)^{1/3}$.

$$(0.027)^{1/3} = \left(\frac{27}{1000}\right)^{1/3} = \left(\frac{3^3}{10^3}\right)^{1/3} = \frac{(3^3)^{1/3}}{(10^3)^{1/3}} = \frac{3}{10} \quad \text{or} \quad 0.3$$

<u>Exercises I B</u> Simplify:

1. $16^{3/2}$
2. $(0.008)^{1/3}$
3. $[(-3)(2)]^{1/4}$
4. $(x^{2/3})(x^{1/3})^4$
5. $x^{-3/2} / x^{3/2}$
6. $\sqrt{-25(\sqrt[3]{-64})}$
7. $\frac{(x^{1/3}y^{3/4})^2}{(x^{2/3}y^{1/2})^3}$

Express the following using rational exponents:

8. $\sqrt[4]{x^3}$
9. $\sqrt{\sqrt{x}}$
10. $x\sqrt[3]{x}$

Rationalize the denominator: 11. $\frac{1}{\sqrt{2}}$ 12. $\frac{20}{\sqrt{5}}$

C. <u>Scientific notation</u>

Scientific notation is a uniform way of writing numbers in which each number is written in the form k times 10^n with $1 \le k < 10$ and n an integer.

<u>Examples</u> $5 = 5 \times 10^0$; $25 = 2.5 \times 10^1$;

$93,000,000 = 9.3 \times 10^7$; $0.0032 = 3.2 \times 10^{-4}$;

$$\frac{(17,200,000,000)(0.00000957)}{(0.003)(82,000)} = \frac{(1.72\times10^{10})(9.57\times10^{-6})}{(3.0\times10^{-3})(8.2\times10^{4})}$$

$$= \frac{1.72\cdot9.57}{3.0\cdot8.2}\times\frac{10^{10}\cdot10^{-6}}{10^{-3}\cdot10^{4}} = 0.669\times10^{10-6+3-4}$$
$$= 0.669\times10^{3} = 6.69\times10^{2}$$

Exercises I C

Convert to scientific notation:

1. 58,000,000 2. 0.00000658 3. $\dfrac{(30{,}000)(2{,}700{,}000)}{(0.0001)(0.081)}$

Convert to decimal notation:

4. 5.5×10^{-4} 5. 7.8×10^{6} 6. $\dfrac{(2 \times 10^{-3})(6 \times 10^{2})}{4 \times 10^{3}}$

D. Parentheses and multiplication

First perform the operations within the parentheses and simplify within parentheses where possible. Then carry out the multiplication indicated by the parentheses. (Caution! Be especially careful with minus signs at this stage!) Finally, combine like terms by adding their coefficients.

Examples 1. $-3(x^2 - x + 1) = -3x^2 + (-3)(-x) + (-3)(1) = -3x^2 + 3x - 3$

2. $2(x^2 + 4x + 1) - 3(3x^2 - 2x + 1 - 7x^2 + 2)$

$= 2(x^2 + 4x + 1) - 3(-4x^2 - 2x + 3)$

$= 2x^2 + 8x + 2 + 12x^2 + 6x + (-9) = 14x^2 + 14x - 7$

3. $-2(x + 3(y + 4)) + 2(-2 + 7(x - y))$

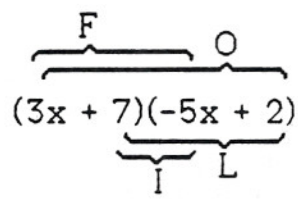
$= -2(x + 3y + 12) + 2(-2 + 7x - 7y)$ (work from

$= -2x - 6y - 24 - 4 + 14x - 14y$ the inside out)

$= 12x - 20y - 28$

4. Expand $(3x + 7)(-5x + 2)$. Use the "FOIL" method:

Firsts	$(3x)(-5x)$	=	$-15x^2$
+ Outers	$(3x)(2)$	=	$6x$
+ Inners	$(7)(-5x)$	=	$-35x$
+ Lasts	$(7)(2)$	=	14
			$-15x^2 - 29x + 14$



For more than two factors, expand two at a time.
Note certain common rules:

$(a+b)(a-b) = a^2 - b^2$ $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^2 = a^2 + 2ab + b^2$ $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$(a-b)^2 = a^2 - 2ab + b^2$

Exercises I D   Perform the indicated operations, then simplify.

1. $(x + 2y + 7) - (7x - 2y + 1)$
2. $(x^2 + 7x + 5)(2x)$
3. $-(x^2 + 2x(x+1) + 3) + 2(-5 + 2(x+1) - 3x^2)$
4. $3[\sqrt{2} - \sqrt{2}(1 + 4\sqrt{2} - 5\sqrt{3})]$
5. $(3x + 1)(7x - 2) - (3x + 4)(6x - 1) - x(2x - 19)$
6. $(x+2)^3$
7. $(2x + 7y)^2$

E. Arithmetic of fractions

1. Addition   To add fractions with the same denominator, add their numerators and retain the original denominator. To add fractions with unlike denominators, first find the "lowest common denominator", then rewrite each fraction so all have the "LCD" as denominator, then add. Simplify the result if possible by cancelling any factors common to numerator and denominator.

Examples   1. $\frac{3}{4} + \frac{1}{4} - \frac{7}{4} - \frac{5}{4} = \frac{3+1-7-5}{4} = \frac{-8}{4} = -2$

2. $\frac{4}{7} + \frac{2}{3} = \frac{4\cdot 3}{7\cdot 3} + \frac{2\cdot 7}{3\cdot 7} = \frac{12+14}{7\cdot 3} = \frac{26}{21}$

$7, 3$:
LCD $= 7\cdot 3 = 21$

3. $\frac{4}{7} + \frac{2}{3} - \frac{2}{9} + \frac{5}{6}$

$7, 3, 9 = 3^2, 6 = 2\cdot 3$:
LCD $= 7\cdot 3^2\cdot 2 = 126$

$$= \frac{4\cdot 18}{7\cdot 18} + \frac{2\cdot 42}{3\cdot 42} - \frac{2\cdot 14}{9\cdot 14} + \frac{5\cdot 21}{6\cdot 21} = \frac{72+84-28+105}{126} = \frac{233}{126}$$

4. $\frac{4x+1}{x^2} + \frac{3}{x+1} = \frac{(4x+1)(x+1)}{x^2(x+1)} + \frac{3x^2}{(x+1)x^2} = \frac{4x^2 + 4x + x + 1 + 3x^2}{x^2(x+1)}$

$= \frac{7x^2 + 5x + 1}{x^2(x+1)}$ or $\frac{7x^2 + 5x + 1}{x^3 + x^2}$

2. Multiplication Multiply numerators, multiply denominators, and simplify if possible. Do any possible cancellation of common factors before actually performing the multiplications.

Examples 1. $\frac{2}{9} \cdot \frac{3}{8} = \frac{2 \cdot 3}{9 \cdot 8} = \frac{2 \cdot 3}{3 \cdot 3 \cdot 2 \cdot 4} = \frac{1}{3 \cdot 4} = \frac{1}{12}$

2. $\frac{x+3}{y-4} \cdot \frac{x^3}{y+5} = \frac{(x+3)x^3}{(y-4)(y+5)}$ or $\frac{x^4 + 3x^3}{y^2 + y - 20}$

3. Division Dividing by a fraction is the same as multiplying by its reciprocal.

Example $\frac{x-2}{x+1} \Big/ \frac{x+5}{x-3} = \frac{x-2}{x+1} \cdot \frac{x-3}{x+5} = \frac{(x-2)(x-3)}{(x+1)(x+5)}$

Note: $\frac{\frac{a}{b}}{c} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$

4. Compound fractions (fractions within fractions)
Method 1 : Add fractions in numerator. Add fractions in denominator. Then divide as above.
Method 2 : Multiply numerator and denominator by their LCD .

Example

Method 1 : $\frac{x + \frac{1}{5}}{x - \frac{1}{3}} = \frac{\frac{5x}{5} + \frac{1}{5}}{\frac{3x}{3} - \frac{1}{3}} = \frac{\frac{5x+1}{5}}{\frac{3x-1}{3}} = \frac{5x+1}{5} \cdot \frac{3}{3x-1} = \frac{15x+3}{15x-5}$

Method 2 : $\frac{x + \frac{1}{5}}{x - \frac{1}{3}} = \frac{(x + \frac{1}{5})(15)}{(x - \frac{1}{3})(15)} = \frac{15x + 3}{15x - 5}$

Exercises I E Perform the indicated operations.

1. $(\frac{2}{3} - \frac{1}{4}) + (\frac{1}{2} - \frac{1}{6})$

2. $\frac{b}{c} + \frac{a}{d}$

3. $\frac{1}{x^2} + \frac{2}{xy} - \frac{3}{y}$

4. $\frac{a}{a-b} + \frac{b}{b-a}$

5. $(\frac{2}{3} + \frac{1}{4})(4+3)$

6. $(\frac{2}{3} + \frac{4x}{5}) / (\frac{8}{3} - \frac{2x}{5})$

7. $\dfrac{a - \frac{a}{b}}{b - \frac{b}{a}}$

8. $(x^{-1} + y^{-1})^{-1}$

9. $2x^{-1} + (2x)^{-1}$

Answers to Exercises I

A:

1. $6x^5$
2. $-16x^{12}$
3. $x^2z^4/16$
4. $-10x^6yz$
5. $(\frac{1}{16} + \frac{1}{16}) / \frac{1}{8} = 1$
6. $\frac{x^4}{y^5}$ or x^4y^{-5}
7. $\frac{27c^{18}}{4ab^{10}}$ or $\frac{27}{4}a^{-1}b^{-10}c^{18}$
8. $\frac{8xy^2}{25}$ or $\frac{8}{25}xy^2$

B:

1. 64
2. 0.2 or 2/10 or 1/5
3. undefined
4. $x^{2/3}x^{4/3} = x^2$
5. x^{-3} or $1/x^3$
6. $\sqrt{-25(-4)} = \sqrt{100} = 10$
7. $\frac{x^{2/3}y^{3/2}}{x^2y^{3/2}} = \frac{1}{x^{4/3}}$ or $x^{-4/3}$
8. $x^{3/4}$
9. $(x^{1/2})^{1/2} = x^{1/4}$
10. $x^1x^{1/3} = x^{4/3}$
11. $\sqrt{2}/2$ or $\frac{1}{2}\sqrt{2}$
12. $4\sqrt{5}$

C:

1. 5.8×10^7
2. 6.58×10^{-6}

3. $\dfrac{(3 \times 10^{4})(2.7 \times 10^{6})}{(1 \times 10^{-4})(8.1 \times 10^{-2})} = \dfrac{8.1 \times 10^{10}}{8.1 \times 10^{-6}} = 1 \times 10^{16}$

4. 0.00055

5. 7,800,000

6. $\dfrac{12 \times 10^{-1}}{4 \times 10^{3}} = 3 \times 10^{-4} = 0.0003$

D: 1. $-6x + 4y + 6$ 2. $2x^3 + 14x^2 + 10x$

3. $-9x^2 + 2x - 9$

4. $2\sqrt{2} - 24 + 15\sqrt{6}$

5. $x^2 - x - 2$

6. $x^3 + 6x^2 + 12x + 8$

7. $4x^2 + 28xy + 49y^2$

E: 1. $\dfrac{3}{4}$

2. $\dfrac{bd + ac}{cd}$

3. $\dfrac{y + 2x - 3x^2}{x^2 y}$

4. 1 (because $\dfrac{b}{b - a} = -\dfrac{b}{a - b}$)

5. $\dfrac{77}{12}$

6. $\dfrac{5 + 6x}{20 - 3x}$

7. $\dfrac{a^2 b - a^2}{ab^2 - b^2}$

8. $\dfrac{1}{\frac{1}{x} + \frac{1}{y}} = \dfrac{1}{\frac{y + x}{xy}} = \dfrac{xy}{y + x}$

9. $\dfrac{2}{x} + \dfrac{1}{2x} = \dfrac{5}{2x}$