II. Factoring and solving equations

A. Factoring polynomials

Examples 1. Factor $3x^2 + 6x$ if possible.

Look for monomial (single-term) factors first; 3 is a factor of both $3x^2$ and $6x$ and so is x. Factor them out to get

$$3x^2 + 6x = 3(x^2 + 2x) = 3x(x+2) .$$

2. Factor $x^2 + x - 6$ if possible.

Here we have no common monomial factors. To get the x^2 term we'll have the form $(x+\_)(x+\_)$. Since

$$(x+A)(x+B) = x^2 + (A+B)x + AB ,$$

we need two numbers A and B whose sum is 1 and whose product is -6. Integer possibilities that will give a product of -6 are

$$-6 \text{ and } 1 , \quad 6 \text{ and } -1 , \quad -3 \text{ and } 2 , \quad 3 \text{ and } -2 .$$

The only pair whose sum is 1 is $\{3 \text{ and } -2\}$, so the factorization is

$$x^2 + x - 6 = (x+3)(x-2) .$$

3. Factor $4x^2 - 3x - 10$ if possible.

Because of the $4x^2$ term the factored form will be either $(4x+A)(x+B)$ or $(2x+A)(2x+B)$. Because of the -10 the integer possibilities for the pair A, B are

10 and -1, -10 and 1, 5 and -2, -5 and 2, plus each of these in reversed order.

Check the various possibilities by trial and error. It may help to write out the expansions

$$(4x+A)(x+B) = 4x^2 + (4B+A)x + AB$$

↕ trying to get -3 here

$$(2x+A)(2x+B) = 4x^2 + (2B+2A)x + AB$$

Trial and error gives the factorization $4x^2 - 3x - 10 = (4x+5)(x-2)$.

4. Difference of two squares. Since $(A+B)(A-B) = A^2 - B^2$, any expression of the form $A^2 - B^2$ can be factored. Note that A and B might be anything at all.

Examples: $9x^2 - 16 = (3x)^2 - 4^2 = (3x+4)(3x-4)$

$x^2 - 2y^2 = x^2 - (\sqrt{2}y)^2 = (x+\sqrt{2}y)(x-\sqrt{2}y)$

For any of the above examples one could also use the

QUADRATIC FORMULA

In the factorization $ax^2 + bx + c = a(x-A)(x-B)$, the numbers A and B are given by

$$A, B = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the "discriminant" $b^2 - 4ac$ is negative, the polynomial cannot be factored over the real numbers (e.g. consider $x^2 + 1$).

In Example 2 above, $a = 1$, $b = 1$, $c = -6$, so

$$A, B = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} = 2, -3 \text{ , so } x^2 + x - 6 = (x-2)(x+3).$$

5. Factor $x^3 + 3x^2 - 4$ if possible.

Useful fact about factoring polynomials

If plugging $x = a$ into a polynomial yields zero, then the polynomial has $(x-a)$ as a factor.

We'll use this fact to try to find factors of $x^3 + 3x^2 - 4$. We look for factors $(x-a)$ by plugging in various possible a's, choosing those that are factors of -4. Try plugging $x = 1, -1, 2, -2, 4, -4$ into $x^3 + 3x^2 - 4$. Find that $x = 1$ gives $1^3 + 3 \cdot 1^2 - 4 = 0$. So $x-1$ is a factor of $x^3 + 3x^2 - 4$. To factor it out, perform long division:

$$\begin{array}{r|l}
 & x^2 + 4x + 4 \\
\hline
x-1 & x^3 + 3x^2 + 0x - 4 \\
 & \underline{x^3 - x^2} \\
 & \quad 4x^2 + 0x - 4 \\
 & \quad \underline{4x^2 - 4x} \\
 & \qquad 4x - 4 \\
 & \qquad \underline{4x - 4} \\
 & \qquad\quad 0
\end{array}$$

Thus

$$x^3 + 3x^2 - 4 = (x-1)(x^2 + 4x + 4).$$

But $x^2 + 4x + 4$ can be factored further as in the examples above; we finally get $x^3 + 3x^2 - 4 = (x-1)(x+2)(x+2) = (x-1)(x+2)^2$.

Exercises II A Factor the following polynomials.

1. $x^2 + 8x + 15$
2. $4x^2 - 25$

3. $4y^2 - 13y - 12$

4. $x^3 + 2x^2 - x - 2$

5. $4z^2 + 4z - 8$

6. $a^2 + 3a + 2$

7. Simplify by factoring numerator and denominator: $\dfrac{3x^2 + 3x - 18}{4x^2 - 3x - 10}$

B. Solving equations

1. Linear or first-degree equations: involving x but not x^2 or any other power of x. Collect x-terms on one side, constant terms on the other.

Example

$$\begin{aligned} x + 3 &= 7x - 4 \\ x + (-7x) &= -4 + (-3) \\ -6x &= -7 \\ x &= 7/6 \end{aligned}$$

2. Quadratic equations: involving x^2 but no higher power of x. These are solved by factoring and/or use of the quadratic formula:

The equation $ax^2 + bx + c = 0 \quad (a \neq 0)$

has solutions $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac$ is negative, the equation has no real solutions.

Example Solve $x^2 - 2x - 3 = 0$ for x.

Method 1: Factoring. $x^2 - 2x - 3 = (x-3)(x+1) = 0$. Since a product of two numbers is zero if and only if one of the two numbers is zero, we must have

$x - 3 = 0$ or $x + 1 = 0$. So the solutions are $x = 3, -1$.

Method 2: Quadratic formula. $a = 1$, $b = -2$, $c = -3$.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-3)}}{2(1)} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = 3 \text{ or } -1.$$

3. Other types of equations.

Examples (a) Solve $\dfrac{14}{x+2} - \dfrac{1}{x-4} = 1$. $(x \neq -2,\ x \neq 4)$

Multiply both sides by common denominator $(x+2)(x-4)$ to get $14(x-4) - 1(x+2) = (x+2)(x-4)$.

Expand and simplify. Get a quadratic equation so put all terms on one side.
$$14x - 56 - x - 2 = x^2 - 2x - 8$$
$$x^2 - 15x + 50 = 0$$
Now factor (or use quadratic formula).
$$(x-10)(x-5) = 0\ ,\quad x-10 = 0 \text{ or } x-5 = 0\ ,\quad x = 10 \text{ or } 5\ .$$

(b) Solve $x^3 - 2x^2 - 5x + 6 = 0$.

The idea is much the same as in Example 5 of part A where we used the fact about factoring polynomials. Try $x = 1, -1, 2, -2, 3, -3, 6, -6$. As soon as one of these possibilities satisfies the equation we have a factor. It happens that $x = 1$ is a solution. By long division we get:
$$x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6) = (x-1)(x-3)(x+2) = 0\ ,$$
so $x = 1, 3,$ or -2 .

(c) Solve $\sqrt{x+2} = x$.

Start by squaring both sides, but this may lead to <u>extraneous roots</u> so we'll have to check answers at the end.
$$x + 2 = x^2$$
$$x^2 - x - 2 = (x-2)(x+1) = 0\ , \text{ so } x = 2 \text{ or } -1\ .$$
Check in original equation: $\sqrt{2+2} = 2$, OK; $\sqrt{-1+2} = 1$, not -1 , so reject $x = -1$; only solution is $x = 2$.

<u>Exercises II B</u> Solve the following equations.

1. $\dfrac{2y+3}{7} = \dfrac{3y+1}{3}$

2. $s = \dfrac{1}{2}gt^2$ (solve for g in terms of s and t .)

3. $s = \frac{1}{2}gt^2$ (solve for t in terms of s, g)

4. $x^2 - (x-2)2x = 4$

5. $x^3 - 4x + 3 = 0$

6. $x = \dfrac{4}{4-x}$

7. $\sqrt{8-x^2} = \dfrac{x^2}{\sqrt{8-x^2}}$

C. <u>Solving simultaneous equations</u>

1. Linear systems of equations.

<u>Example</u> Find all values of x and y that satisfy the two equations
$$9x + 2y = 37$$
$$5x + 6y = 45\ .$$

Method 1: Substitution.

Solve one equation for one variable in terms of the other, then substitute into the other equation. For instance, solving first equation for y:

$$2y = 37 - 9x$$
$$y = (37 - 9x)/2$$

Second eq'n: $5x + 6\cdot(37 - 9x)/2 = 45$

$$5x + 111 - 27x = 45$$
$$-22x = -66$$
$$x = -66/-22 = 3$$; plug this into expression for y: $y = (37 - 9(3))/2 = 5$. Solution: $x = 3$, $y = 5$.

Method 2: Elimination.

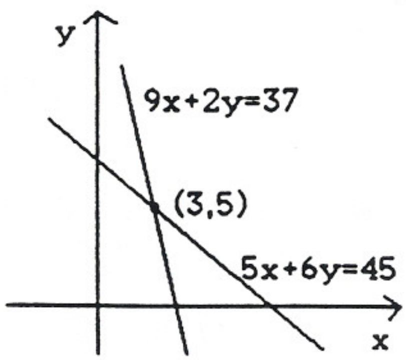
Multiply the equations by appropriate constants so that when the equations are added one variable will be eliminated. For instance, to eliminate y, multiply both sides of first equation by -3:

$-3\cdot$ first eq'n:	$-27x - 6y = -111$	
second eq'n:	$5x + 6y = 45$	
Add:	$-22x = -66$	so $x = 3$.

Now sub. $x = 3$ into one of the original equations, e.g. the second:

$5(3) + 6y = 45$ so $y = 5$.

Note What we've done geometrically in this example is to find $(3,5)$ as the point of intersection of the lines $9x + 2y = 37$ and $5x + 6y = 45$.



2. Systems of nonlinear equations.

Example Find the point(s) of intersection of the curves $y = 3 - x^2$ and $y = 3 - 2x$.

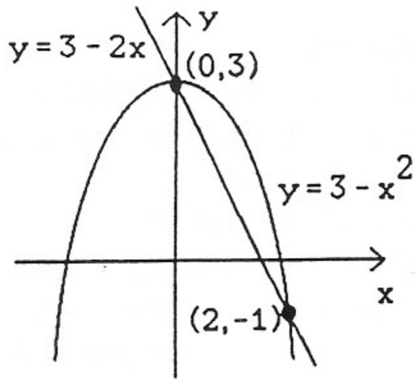
$$\left.\begin{aligned} y &= 3 - x^2 \\ y &= 3 - 2x \end{aligned}\right\} \text{ Set equal to get } 3 - x^2 = 3 - 2x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x = 0 \text{ or } 2.$$

Now sub. $x = 0$, $x = 2$ into one of the given equations ($y = 3 - 2x$ is easier) to get y's :

$x = 0$ $\quad$ $x = 2$

$y = 3 - 2(0) = 3$ $\quad$ $y = 3 - 2(2) = -1$

The points of intersection are (0,3) and (2,-1) .



Exercises II C $\quad$ Solve the following systems of equations.

1. $-x + y = -1$
   $x + y = 3$
2. $3x + y = 10$
   $4x + 5y = -16$
3. $y = 3 - x^2$
   $y = -2x$
4. $3x + 5y = 11$
   $8x - 12y = -15$
5. $x = 5 - y^2$
   $x = y^2 - 3$

Answers to Exercises II

A:

1. $(x+5)(x+3)$
2. $(2x-5)(2x+5)$
3. $(4y+3)(y-4)$
4. $(x+1)(x-1)(x+2)$
5. $4(z+2)(z-1)$
6. $(a+2)(a+1)$
7. $\dfrac{3(x+3)(x-2)}{(4x+5)(x-2)} = \dfrac{3(x+3)}{4x+5}$ $\quad$ $(x \neq 2,\ x \neq -5/4)$

B:

1. $x = \dfrac{2}{15}$
2. $g = \dfrac{2s}{t^2}$
3. $t = \pm\sqrt{\dfrac{2s}{g}}$
4. $-x^2 + 4x - 4 = -(x-2)^2 = 0$
   $x = 2$
5. $(x-1)(x^2+x-3) = 0$
   $x = 1, (-1 \pm \sqrt{13})/2$
6. $x(4-x) = 4$
   $-x^2 + 4x - 4 = 0$
   $x = 2$ (see no. 4)
7. $(8-y^2) - y^2 = 0$
   $2y^2 - 8 = 2(y-2)(y+2) = 0$
   $y = 2, -2$

C:

1. $x = 2,\ y = 1$
2. $x = 6,\ y = -8$
3. $x = 3,\ y = -6$ or $x = -1,\ y = 2$
4. $x = 57/76,\ y = 133/76$
5. $x = 1,\ y = 2$ or $x = 1,\ y = -2$