VIII. Exponential and logarithmic functions

A. Exponential functions

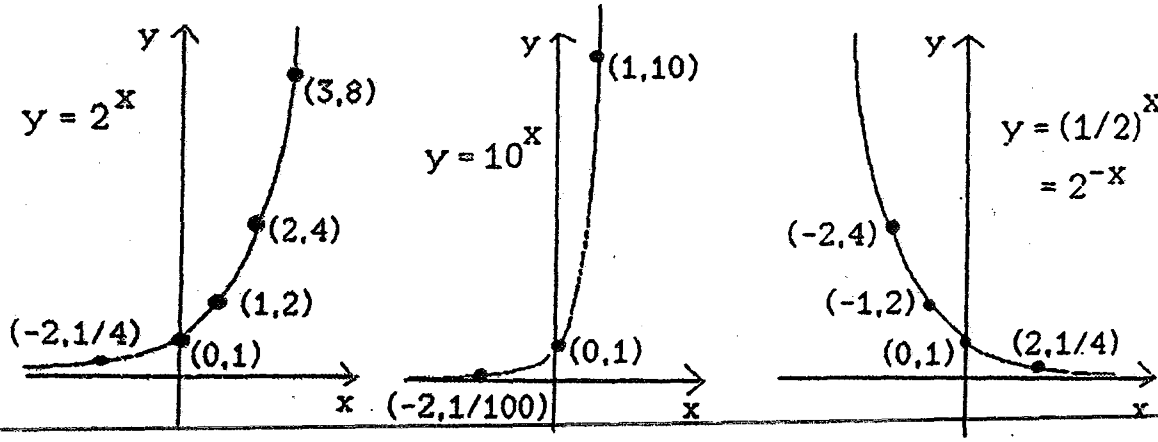
<u>Definition</u>. The function $f(x) = b^x$, where b is a positive constant, is called the <u>exponential function</u> with <u>base</u> b. It is defined for all real numbers x, but see note below.

To graph, we plot a few points and join them with a smooth curve.

<u>Example</u> $f(x) = 2^x$

x	0	1	2	3	-1	-2	-3	1/2	3/2
2^x	1	2	4	8	1/2	1/4	1/8	$\sqrt{2} \approx 1.4$	$(\sqrt{2})^3 \approx 2.7$

(The other graphs shown below were obtained similarly.)



<u>Note</u>: There is no easy way to compute (or even to define) such numbers as 2^π. We can approximate them, however. The number π can be thought of as the limit of the sequence

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots$$

Then we get 2^π as the limit of the sequence

$$2^3,\ 2^{\frac{31}{10}},\ 2^{\frac{314}{100}},\ 2^{\frac{3141}{1000}},\ \ldots$$, which is approximately 8.825 .

<u>Properties</u>

1. $b^x > 0$ for every x.
2. $b^0 = 1$ for every b (so the graph of $f(x) = b^x$ always passes through $(0,1)$).

3. If $b > 1$, then b^x increases without bound as x tends toward infinity and tends toward zero as x tends toward negative infinity.

If $0 < b < 1$, then b^x approaches zero as x tends toward infinity and increases without bound as x approaches negative infinity.

Thus the graph of b^x (for $b \neq 1$) has the x-axis as a horizontal asymptote.

B. Logarithms

Notice from the graphs above that if $b > 0$ but $b \neq 1$ then for each positive number y there is exactly one number x for which $b^x = y$. This number is called the logarithm of y base b or the base-b logarithm of y and is written $\log_b y$. Thus, by definition, $\log_b y$ is the exponent to which we must raise b in order to get y. Saying $x = \log_b y$ is equivalent to saying $b^x = y$. Note that only positive numbers have logarithms!

Examples

$\log_2 8 = 3$ since $2^3 = 8$ $\qquad$ $\log_2 1 = 0$ since $2^0 = 1$

$\log_3 81 = 4$ since $3^4 = 81$ $\qquad$ $\log_2 2 = 1$ since $2^1 = 2$

$\log_2 \frac{1}{4} = -2$ since $2^{-2} = \frac{1}{4}$ $\qquad$ $\log_{16} 8 = \frac{3}{4}$ since $16^{3/4} = 8$

To repeat: $\log_b y = x$ is equivalent to $b^x = y$.

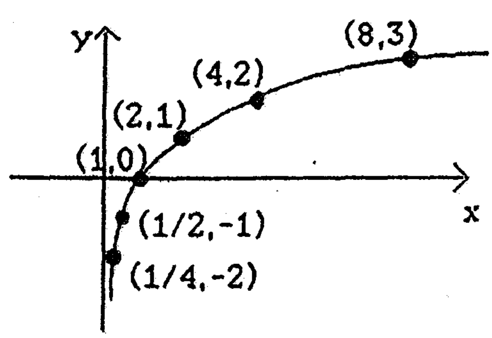
Example Find x if $\log_5 x = 4$.

$$x = 5^4 = 625 \text{ by definition of logs.}$$

To graph a logarithmic function, note that if (c,d) is a point on the graph of b^x, so that $d = b^c$, then (d,c) will be a point on the graph of $\log_b x$, because $c = \log_b d$. So the log. graph is the "inverse" of the exponential graph.

Example

$f(x) = \log_2 x$



Properties For every $b > 0$ ($b \neq 1$):

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b (b^x) = x$, where x is any number or expression.
4. $b^{(\log_b x)} = x$, where x is any positive number or expression.

You may be most familiar with base-10 logarithms,

e.g. $\log_{10}(0.1) = -1$ since $10^{-1} = 0.1$;

$\log_{10} 1000 = 3$ since $10^3 = 1000$.

Exercises VIII AB

Sketch the graphs of the following functions.

1. $f(x) = 3^x$ 2. $g(x) = (1/3)^x$ 3. $h(x) = \log_3 x$
4. Find: (a) $\log_3 27$ (b) $\log_2 \sqrt{2}$ (c) $\log_{10} 100$ (d) $\log_3 (1/27)$
5. Solve for x : (a) $\log_2 x = 3$ (b) $\log_5 (1/x) = 1$ (c) $2^x = 3$
6. Simplify: (a) $\log_{10}(10^x)$ (b) $5^{\log_5 x^2}$

C. Rules of computation for logarithms

Since logarithms are related to exponential functions, each of the rules for exponents gives rise to a corresponding rule for logarithms. Besides the facts that have already been listed, there are the following properties of logs:

$\log_b (xy) = \log_b x + \log_b y$ (this comes from $b^x b^y = b^{x+y}$)

$\log_b \frac{x}{y} = \log_b x - \log_b y$

In particular, $\log_b \frac{1}{y} = -\log_b y$ (since $\log_b 1 = 0$)

$\log_b x^y = y \log_b x$ (this comes from $(b^x)^y = b^{xy}$)

Examples 1. If $a = \log_{10} 2$ and $b = \log_{10} 3$, write $\log_{10} 6$ in terms of a and b .

$$\log_{10} 6 = \log_{10}(2 \cdot 3) = \log_{10} 2 + \log_{10} 3 = a + b .$$

2. With a and b as in (1), write $\log_{10}(0.6)$ in terms of a and b

$$\log_{10} 0.6 = \log_{10}(6/10) = \log_{10} 6 - \log_{10} 10 = a + b - 1 .$$

3. Write $\log_3(x-1) + 2\log_3(x-2) - 3\log_3(x-4)$ as a single logarithm.

$$\log_3(x-1) + 2\log_3(x-2) - 3\log_3(x-4)$$
$$= \log_3(x-1) + \log_3(x-2)^2 - \log_3(x-4)^3$$
$$= \log_3[(x-1)(x-2)^2] - \log_3(x-4)^3 = \log_3 \frac{(x-1)(x-2)^2}{(x-4)^3} .$$

4. Find x if $10^{(\log_{10} x^2 + 3\log_{10} x)} = 2$.

$$\log_{10} x^2 + 3\log_{10} x = \log_{10} x^2 + \log_{10} x^3$$
$$= \log_{10}(x^2 \cdot x^3) = \log_{10} x^5 ,$$

so $10^{(\log_{10} x^2 + 3\log_{10} x)} = 10^{\log_{10} x^5} = x^5 = 2$,

and $x = 2^{1/5} = \sqrt[5]{2}$.

Exercises VIII C

1. Write as a single logarithm.
   (a) $\log_b(x+1) + \log_b(x-2) + 2\log_b(x-3)$
   (b) $\frac{1}{2}\log_b(x+1) - \frac{1}{2}\log_b(x-1)$

2. Let $a = \log_{10} 2$, $b = \log_{10} 3$, $c = \log_{10} 5$. Write the following in terms of a, b, and c:

   (a) $\log_{10} 360$ (b) $\log_{10} \frac{54}{25}$

3. Write using sums and differences of logs and only first powers of x.

   (a) $\log_b \frac{x+1}{x+2}$ (b) $\log_b \frac{(x-1)^2(2x+1)^3}{\sqrt[3]{(4x-1)^2}}$

4. Solve for x: (a) $\log_2 \sqrt{3x+1} = 1$ (b) $3^{-2\log_3 x} = 1/3$

D. <u>The natural logarithm</u>

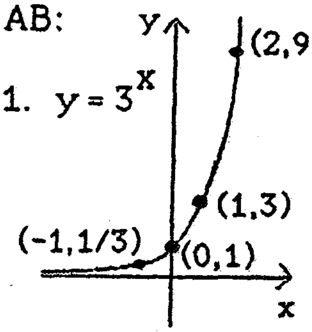
There is a special number, e, equal to approximately 2.71828, which occurs frequently in mathematics and the sciences. The logarithm using e as base turns out to be most important. This logarithm is called the <u>natural logarithm</u> and one often writes $\ln$ instead of $\log_e$. Thus

$y = \ln x$ means $y = \log_e x$ which means $e^y = x$. Sometimes instead of e^x one writes $\exp(x)$. This is called the natural exponential function. All the usual properties of exponents and logarithms hold for the functions $\exp(x)$ and $\ln x$.

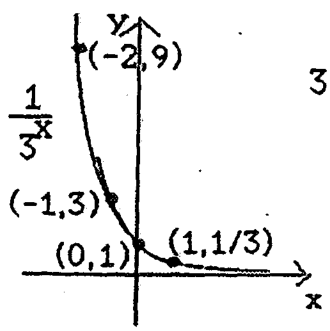
<u>Answers to Exercises VIII</u>

AB:

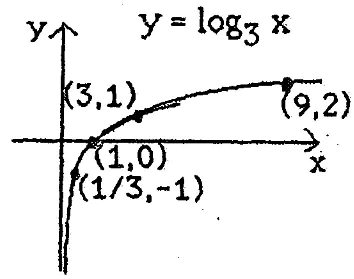
1. $y = 3^x$



2. $y = \frac{1}{3^x}$



3.



4.(a) 3 (b) 1/2 (c) 2 (d) -3

5.(a) $x = 2^3 = 8$ (b) $x = 5$ (c) $x = \log_2 3$

6.(a) x (b) x^2

C: 1.(a) $\log_b (x+1)(x-2)(x-3)^2$ (b) $\log_b \sqrt{\frac{x+1}{x-1}}$

2.(a) $\log_{10} 360 = \log_{10}(2^2 \cdot 3^2 \cdot 10) = 2\log_{10} 2 + 2\log_{10} 3 + \log_{10} 1$
$= 2a + 2b + 1$

(b) $\log_{10}(54/25) = \log_{10} 54 - \log_{10} 25 = \log_{10}(2 \cdot 3^3) - \log_{10} 5^2$
$= a + 3b - 2c$

3.(a) $\log_b (x+1) - \log_b (x+2)$

(b) $2\log_b (x-1) + 3\log_b (2x+1) - (2/3)\log_b (4x-1)$

4.(a) $\frac{1}{2}\log_2 (3x+1) = 1$, $2^{\log_2 (3x+1)} = 3x+1 = 2^2$, so $x = 1$.

(b) $3^{\log_3 \frac{1}{x^2}} = \frac{1}{x^2} = \frac{1}{3}$, so $x = \pm\sqrt{3}$