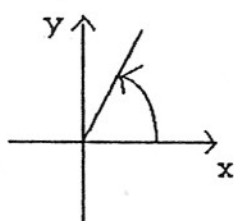


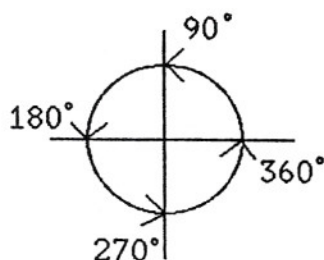
V. Trigonometry, part 1

A. Angle measurement



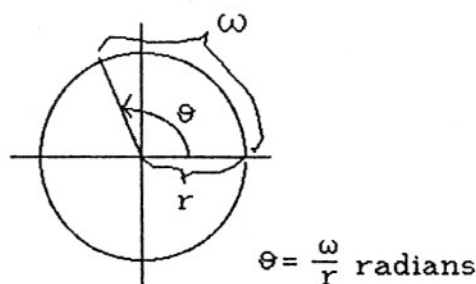
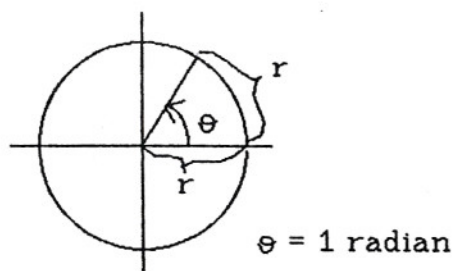
The standard position for angles in the xy -plane is with the initial side on the positive x -axis and the counter-clockwise direction taken to be positive.

Two common units for measuring angles are degrees and radians.



Degrees $360^\circ = 1$ revolution,
so $1^\circ = \frac{1}{360}$ th of a revolution.

Radians. One radian is defined to be the angle subtended at the center by an arc of length r on a circle of radius r . The circumference of a circle of radius r has length $2\pi r$, so r units can be marked off " 2π times". In other words, 1 revolution = 2π radians.



Relation between degrees and radians:

Since 1 revolution = $360^\circ = 2\pi$ radians, $180^\circ = \pi$ radians. This gives the conversion relations:

To convert degrees to radians, multiply by $\frac{\pi \text{ radians}}{180^\circ}$,

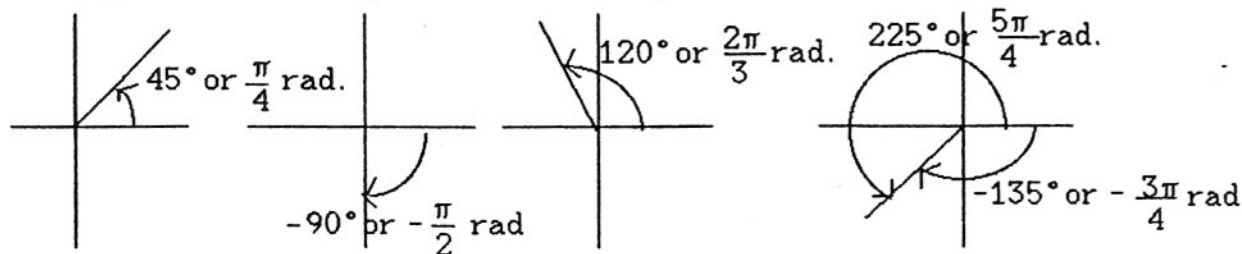
$$\text{e.g. } 45^\circ = 45^\circ \cdot \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{4} \text{ radians.}$$

To convert radians to degrees, multiply by $\frac{180^\circ}{\pi \text{ radians}}$,

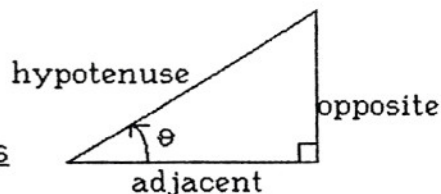
$$\text{e.g. } -\frac{\pi}{3} \text{ radians} = -\frac{\pi}{3} \text{ radians} \cdot \frac{180^\circ}{\pi \text{ radians}} = -60^\circ.$$

In calculus, radian measure is used exclusively (because it simplifies the differentiation formulas for the trig. functions).

Examples: some angles in standard position.



B. Trig. functions for angles in right triangles



1. If θ is an acute angle (more than 0° but less than 90°), the trigonometric functions of θ can be defined as ratios of sides in a right triangle having θ as one angle. The six possible ratios give the six trig. functions.

$$\text{sine of } \theta = \sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}; \quad \text{cosecant of } \theta = \csc(\theta) = \frac{\text{hyp}}{\text{opp}}$$

$$\text{cosine of } \theta = \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}; \quad \text{secant of } \theta = \sec(\theta) = \frac{\text{hyp}}{\text{adj}}$$

$$\text{tangent of } \theta = \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}; \quad \text{cotangent of } \theta = \cot(\theta) = \frac{\text{adj}}{\text{opp}}$$

Notes: We usually write $\sin \theta$ instead of $\sin(\theta)$, etc.

Cotangent of θ is sometimes abbreviated $\text{ctn}(\theta)$.

Remember to think "sine of θ ", not "sine times θ " !!

2. Reciprocal relations: $\csc \theta = \frac{1}{\sin \theta}$ and $\sin \theta = \frac{1}{\csc \theta}$

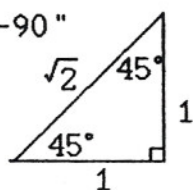
$\sec \theta = \frac{1}{\cos \theta}$ and $\cos \theta = \frac{1}{\sec \theta}$

$\cot \theta = \frac{1}{\tan \theta}$ and $\tan \theta = \frac{1}{\cot \theta}$

Also note that $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\cot \theta = \frac{\cos \theta}{\sin \theta}$.

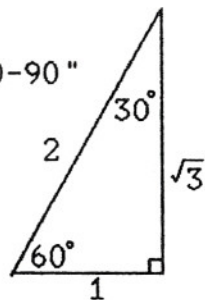
3. The most common triangles:

"45-45-90"



$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}; \quad \tan 45^\circ = 1.$$

"30-60-90"

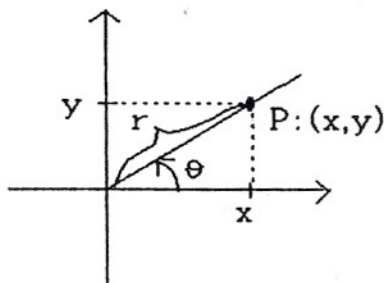


$$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$$

4. For other angles in right triangles, consult trig. tables or use a calculator. (Take care, however -- be sure you and your calculator agree on whether you're punching in degree measure or radian measure!)

C. Trig. functions for general angles



1. Place the angle θ in standard position. Mark a point $P:(x,y)$ on its terminal side. Let r be the distance from the origin to P . Then the six trig. functions of θ are defined by:

$$\sin \theta = \frac{Y}{r}; \quad \cos \theta = \frac{X}{r}; \quad \tan \theta = \frac{Y}{X};$$

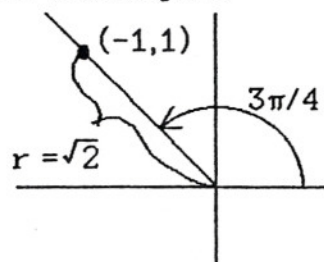
$$\csc \theta = \frac{r}{Y}; \quad \sec \theta = \frac{r}{X}; \quad \cot \theta = \frac{X}{Y}.$$

provided the ratio in question is defined (does not have zero in the denominator).

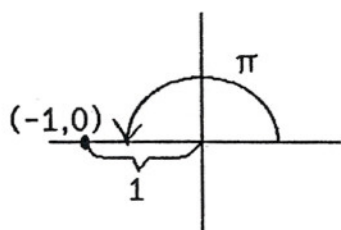
Notes: When θ is an acute angle, this definition gives the same results as the definition in Section B.

If P is chosen so that $r = 1$, the formulas simplify; $\cos \theta$ and $\sin \theta$ are the x and y coordinates of the appropriate point on the unit circle $x^2 + y^2 = 1$.

2. Examples



$$\begin{aligned} \cos \frac{3\pi}{4} &= \frac{-1}{\sqrt{2}} \text{ or } \frac{-\sqrt{2}}{2} & \tan \frac{3\pi}{4} &= \frac{+1}{-1} = -1 \\ \sin \frac{3\pi}{4} &= \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2} \end{aligned}$$



$$\begin{aligned} \cos \pi &= \frac{-1}{1} = -1 & \tan \pi &= \frac{0}{-1} = 0 \\ \sin \pi &= \frac{0}{1} = 0 & \text{but } \csc \pi, & \text{ which would be } \frac{1}{0}, \\ & & \text{is } & \text{undefined.} \end{aligned}$$

3. Trig. functions at the popular angles

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef'd	0	undef'd

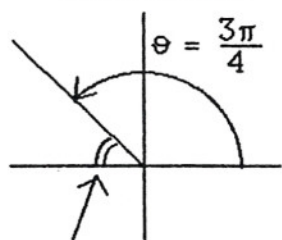
4. Reference angles. If θ is not a quadrantal angle (the terminal side is not an axis), then

sin +	all +
cos, tan -	
tan +	cos +
sin, cos -	sin, tan -

Trig. fn. of $\theta = \pm$ same trig. fn. of reference angle for θ ,
and the choice of \pm is determined by the chart at left. The reference angle for θ

is the smallest unsigned angle between the terminal side of θ and the x-axis.

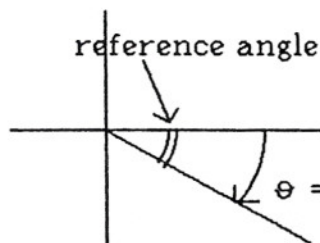
Examples:



reference angle = $\frac{\pi}{4}$

$$\sin \frac{3\pi}{4} = +\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}; \quad \cos \frac{3\pi}{4} = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}};$$

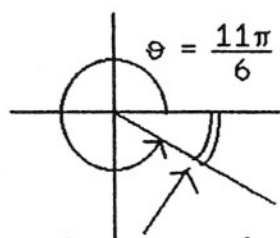
$$\tan \frac{3\pi}{4} = -\tan \frac{\pi}{4} = -1.$$



reference angle = $\frac{\pi}{6}$

$$\sin -\frac{\pi}{6} = -\sin \frac{\pi}{6} = -\frac{1}{2}; \quad \cos -\frac{\pi}{6} = +\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2};$$

$$\tan -\frac{\pi}{6} = -\tan \frac{\pi}{6} = -\frac{1}{\sqrt{3}}.$$



reference angle = $\frac{\pi}{6}$

The trig functions of $\frac{11\pi}{6}$ are the same as those of $-\frac{\pi}{6}$, computed above.

This method is based on the following identities, which come from the definitions of the trig. functions:

$$\sin(-\theta) = -\sin \theta; \quad \cos(-\theta) = \cos \theta;$$

$$\sin(\theta + \pi) = -\sin \theta; \quad \cos(\theta + \pi) = -\cos \theta.$$

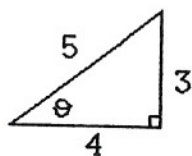
Also, note that the trig. functions are periodic: they repeat every 2π radians: $\sin(\theta + 2\pi) = \sin \theta$; $\cos(\theta + 2\pi) = \cos \theta$, etc.

See the next handout, "Trigonometry, part 2, plus conic sections", for other trig. identities.

Exercises V ABC

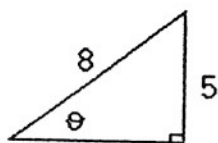
- Convert to radians. (a) 15° (b) 450°
- Convert to degrees. (a) $5\pi/6$ radians (b) $-5\pi/2$ radians
- For each of the following angles, find $\sin \theta$, $\cos \theta$, $\tan \theta$. Use reference angles where appropriate.
(a) $5\pi/4$ (b) $-11\pi/6$ (c) 3π (d) -60°
- Find $\csc(4\pi/3)$, $\sec(4\pi/3)$, $\cot(4\pi/3)$.

5. Given:



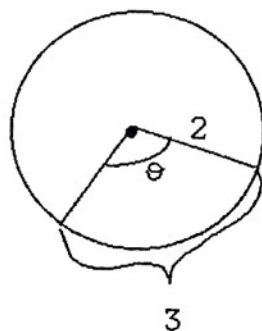
- Find (a) $\sin \theta$
(b) $\cos \theta$
(c) $\tan \theta$

6. Given:

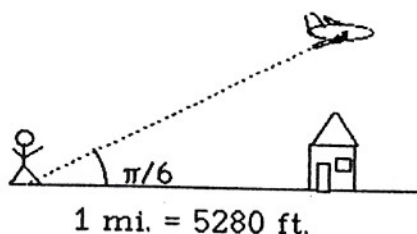


- Find (a) the length of the third side
(b) $\sin \theta$
(c) $\cos \theta$
(d) $\tan \theta$

7. In the diagram at right, find the radian measure of θ .



8.



An airplane is flying over a house 1 mile from the spot where you're standing. At that instant, the plane's angle of elevation from your viewpoint is $\pi/6$ radians. Find the plane's altitude (in feet).

Answers to Exercises V

1.(a) $\pi/12$ (b) $5\pi/2$

2.(a) 150° (b) -450°

3.(a) $-1/\sqrt{2}$, $-1/\sqrt{2}$, 1 (b) $1/2$, $\sqrt{3}/2$, $1/\sqrt{3}$
(c) 0, -1, 0 (d) $-\sqrt{3}/2$, $1/2$, $-\sqrt{3}$

4. $-2/\sqrt{3}$, -2, $1/\sqrt{3}$

5.(a) $3/5$ (b) $4/5$ (c) $3/4$

6.(a) $(\text{side})^2 + 5^2 = 8^2$ so $(\text{side}) = \sqrt{64-25} = \sqrt{39}$

(b) $5/8$ (c) $\sqrt{39}/8$ (d) $5/\sqrt{39}$ or $5\sqrt{39}/39$

7. By ratios: θ is to angle of whole circle as 3 is to circumference,

$$\text{or } \frac{\theta}{2\pi} = \frac{3}{2\pi \cdot 2}, \text{ so } \theta = \frac{3}{2}.$$

8. $\tan \frac{\pi}{6} = \frac{\text{alt.}}{1 \text{ mi.}}$, so $\text{alt.} = \frac{\sqrt{3}}{3} \cdot 5280 \text{ ft.} = 1760\sqrt{3} \text{ ft.}$