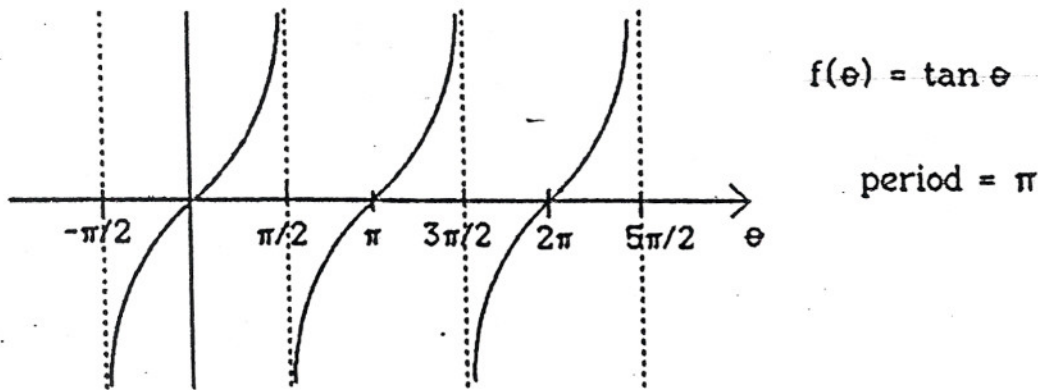
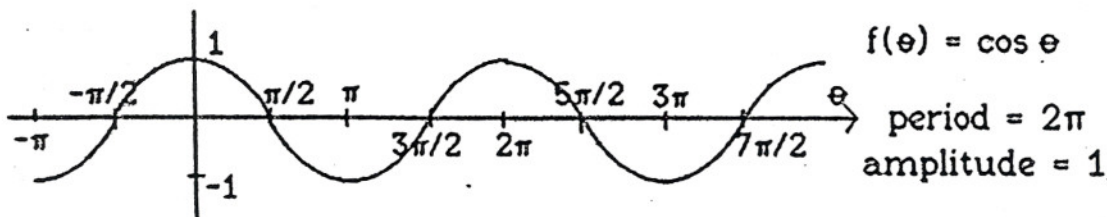
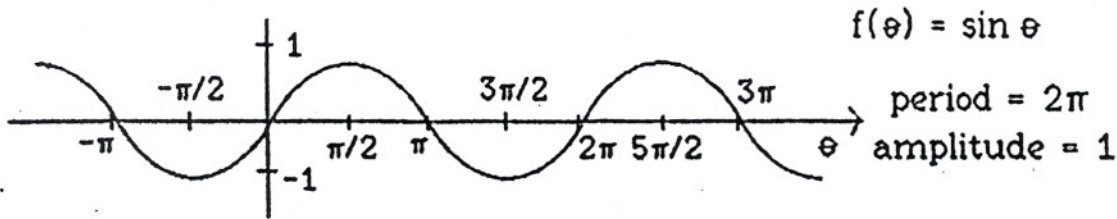


VI. Trigonometry, part 2, plus conic sections.

A. Trigonometric graphs

1. From definitions and data in Part 1 we can graph the trig. functions.

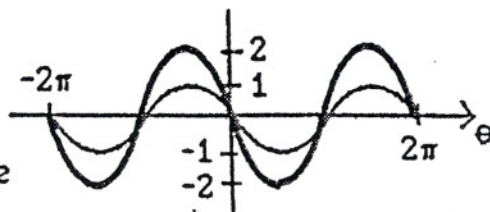


-- $f(\theta) = \sec \theta$ is graphed on the next page, after Example 2 --

2. Variations on the basic graphs.

Example 1

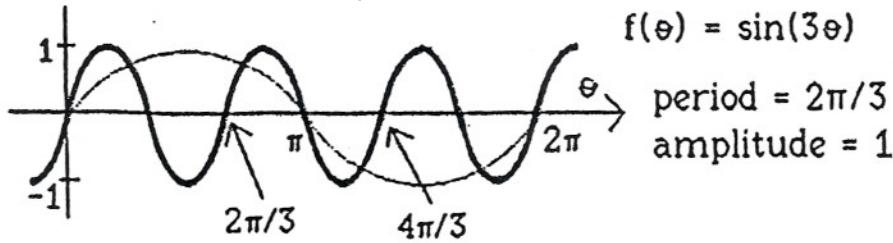
variation
of amplitude



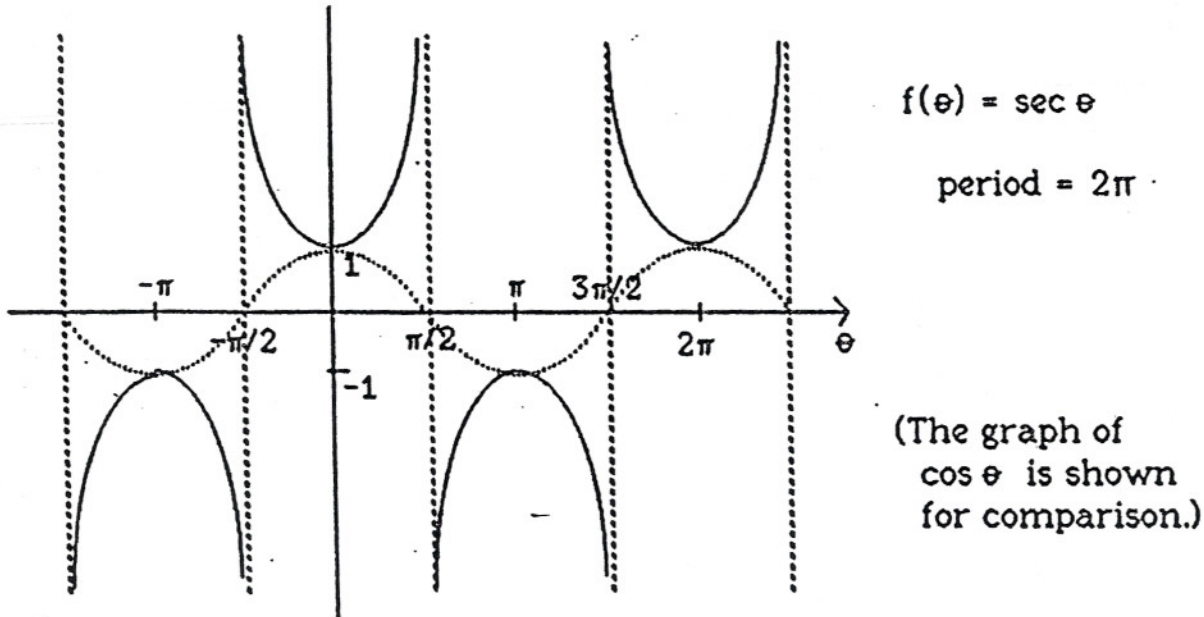
$f(\theta) = 2 \sin \theta$
 period = 2π
 amplitude = 2
 ($\sin \theta$ shown for
comparison)

Example 2

variation of period and frequency



As θ runs from 0 to 2π , 3θ runs from 0 to 6π . Thus the $\sin 3\theta$ curve oscillates 3 times as fast; its period is $(1/3) \cdot 2\pi$. (The graph of $\sin \theta$ is shown for comparison.)



B. Useful identities (Also see V C)

$$\left. \begin{aligned} * \sin^2 \theta + \cos^2 \theta &= 1 \quad (\text{i.e. } (\sin \theta)^2 + (\cos \theta)^2 = 1) \\ \tan^2 \theta + 1 &= \sec^2 \theta ; \quad 1 + \cot^2 \theta = \csc^2 \theta \end{aligned} \right\} \begin{array}{l} \text{Pythagorean} \\ \text{theorem} \end{array}$$

$$* \sin(\theta \pm \zeta) = \sin \theta \cos \zeta \pm \cos \theta \sin \zeta$$

$$* \cos(\theta \pm \zeta) = \cos \theta \cos \zeta \mp \sin \theta \sin \zeta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta ; \quad \cos 2\theta = 1 - 2 \sin^2 \theta ; \quad \cos 2\theta = 2 \cos^2 \theta - 1$$

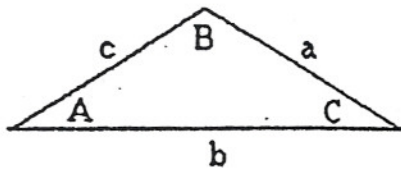
$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} ; \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$\cos \theta = \sin(\theta + \pi/2)$$

$$\sin \theta = \cos(\theta - \pi/2)$$

} Look at
the graphs.

$$\left. \begin{aligned} \cos \theta &= \sin(\pi/2 - \theta) \\ \sin \theta &= \cos(\pi/2 - \theta) \end{aligned} \right\} \begin{array}{l} \text{Think of} \\ \text{right triangles.} \end{array}$$



Law of sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc \cos A$
 (also $b^2 = a^2 + c^2 - 2accosB$, etc.)

Examples

1. Verify that for all θ , $\frac{\cot \theta}{\csc \theta - 1} = \frac{1 + \sin \theta}{\cos \theta}$.

Sol'n: multiply left side by $\frac{\sin \theta}{\sin \theta}$ to get $\frac{\cos \theta}{1 - \sin \theta}$,

then by $\frac{1 + \sin \theta}{1 + \sin \theta}$ to get $\frac{(1 + \sin \theta) \cos \theta}{1 - \sin^2 \theta}$
 $= \frac{(1 + \sin \theta) \cos \theta}{\cos^2 \theta} = \text{right side.}$

2. Find $\cos \frac{\pi}{12}$.

$$\begin{aligned} \cos \frac{\pi}{12} &= \cos \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1 + \sqrt{3}}{2\sqrt{2}}. \end{aligned}$$

Exercises VIAB

1. Sketch the graph of $f(\theta) = -\cos \theta$.
2. Sketch the graph of $f(\theta) = 2 \sin 4\theta$. What is its period?
3. Verify that $\tan \theta + \sec \theta = \frac{1 + \sin \theta}{\cos \theta}$.

4. Verify: $\frac{1 + \tan^2 \theta}{\csc \theta} = \sec \theta \tan \theta$.

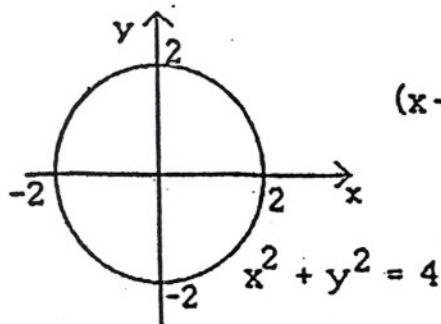
5. Express $\cos^2 2a$ in terms of $\cos 4a$.

6. Find $\sin \frac{5\pi}{12}$ by using trig. identities.

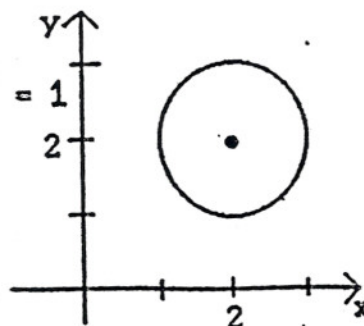
C. Conic sections

1. Circles. (a) centered at origin with radius r : $x^2 + y^2 = r^2$.

(b) centered at (h,k) with radius r : $(x-h)^2 + (y-k)^2 = r^2$.

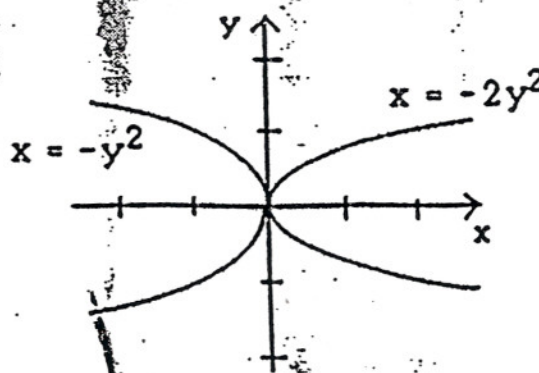
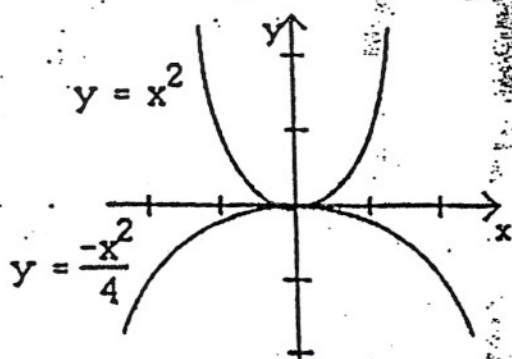


$$(x-2)^2 + (y-2)^2 = 1$$



2. Parabolas.

(a) Vertex at origin: $y = kx^2$ or $x = ky^2$. The larger $|k|$ is, the narrower the parabola will be.

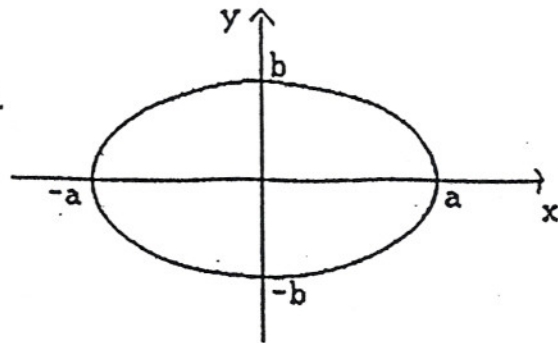


(b) More general parabola will be $Ax^2 + Bx + Cy + D = 0$ or $Ay^2 + By + Cx + D = 0$ with $A, C \neq 0$. By completing the square, these may be rewritten as $y = a(x-b)^2 + c$ or $x = a(y-b)^2 + c$, which are parabolas with vertices at (b,c) or (c,b) , respectively.

3. Ellipses.

(a) Center at origin, x-intercepts $\pm a$, y-intercepts $\pm b$, $a, b > 0$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



major semi-axis = a ,
minor semi-axis = b
since $a > b$.

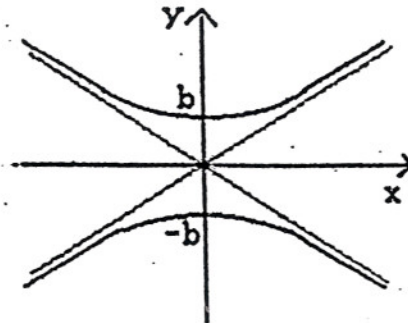
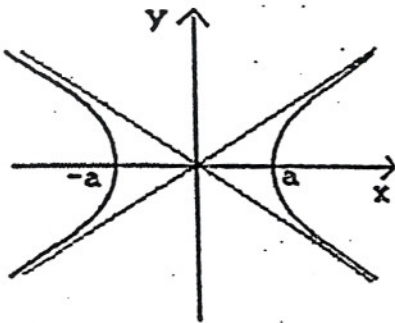
(b) More general ellipse: $Ax^2 + By^2 + Cx + Dy + E = 0$ with $A, B \neq 0$,
 A, B same sign. Rewrite as

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 ; \text{ center at } (h,k), \text{ semi-axes } a \text{ and } b.$$

4. Hyperbolas.

(a) Center at origin, asymptotes $ay = \pm bx$, $a, b > 0$:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

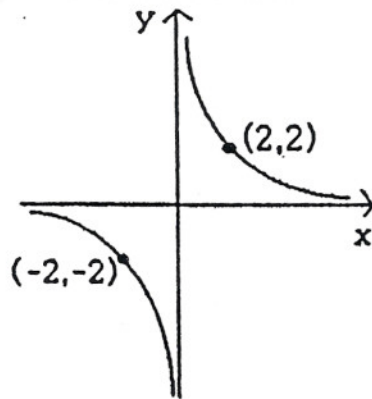


(b) More general hyperbola: $Ax^2 + By^2 + Cx + Dy + E = 0$, $A, B \neq 0$,
 A, B opposite signs.

$$\text{Rewrite as } \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \pm 1, \text{ center at } (h,k).$$

(c) $xy = c$ is also a hyperbola; write as $y = x/c$ to graph;
for example,

$$y = \frac{4}{x}$$



Note: To display the equations of conic sections in a standard, recognizable form as above, it may be necessary to "complete the square". The technique comes from the formula $(x + a)^2 = x^2 + 2ax + a^2$; we can write $x^2 + bx + c = (x + b/2)^2 + c - b^2/4$, getting rid of the x term. (In the exercises, this has already been done.)

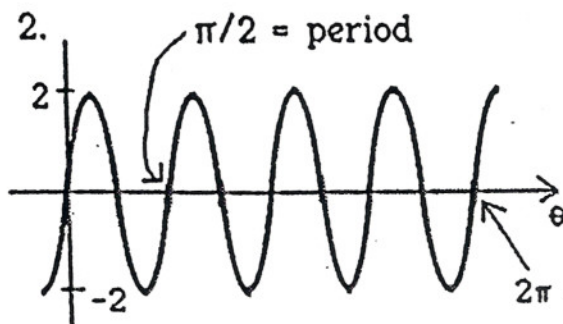
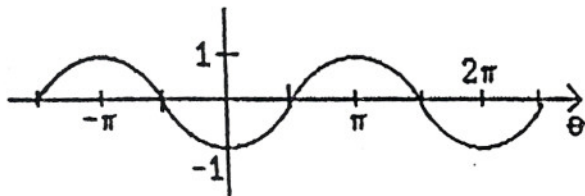
Exercises VI C

Identify and sketch the following conic sections.

1. $8x = -3(y - (1/3))^2 + 4$
2. $(x-1)^2 + (y-2)^2 = 9$
3. $9x^2 + 4y^2 = 36$

Answers for Exercises VI

A: 1.



B: 3. Hint: write the left side in terms of $\sin \theta$ and $\cos \theta$.